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Samia Haraketi[*], Ezedine Haouala[†] and Antoine Lejay[‡]

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Abstract

We construct $\mathbb{S}_{\mathbb{H}}$, a lift of the Sierpinski gasket to the Heisenberg group, the invariant set of a horizontal iterated functions system. As we have a post critically finite self-similar set, using analytic approach, we define a local regular Dirichlet form. By using the theory of Dirichlet forms, we have a diffusion process and a Laplacian which we have defines as being the limit of discrete Laplacians on a sequence of finite graphs which approximate this set.

1 Intoduction

Fractals were originally considered as pathological examples in analysis: the Koch curve is an example of a compact curve with infinite length, the Cantor set is an uncountable set with zero Lebesgue measure. Mandelbrot has the revolutionary idea to introduce them as a new class of mathematical objects representing nature. Fractal geometry and analysis quickly rose. Studies on fractals has developed from both probabilistic and analysis points of view. The pioneering work was the construction of the Brownian motion on the Sierpinski gasket by Kusuoka [22] and Goldstein [11]. The Sierpinski gasket $\mathbb{S}$, introduced by Sierpinski in [42], is a self-similar set of $\mathbb{R}^2$ it is a connected compact subset with Lebesgue measure zero.

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Barlow and Perkins [3] proved that the Browmian motion on $\mathbb{S}$ is a diffusion process characterized by local isotropy and homogeneity properties. They show that the process has a continuous symmetric transition density $p_t(x, y)$ with respect to an appropriate Hausdorff measure. They obtain estimates on $p_t(x, y)$. Hambly and Lyons [13] construct a Lévy stochastic area $A_{s,t}$ for Brownian motion X_t on the Sierpinski gasket. They prove that the pair $(X_t, A_{s,t})$ have finite p-variation for all $p > \log 5/\log 2$.

The analytic "Japanese" approach — see for example [17, 18, 23, 24] — developed by Kusuoka, Kigami, and others, provides an alternative approach to the probabilistic by using discrete approximations.

Recall the fact that Laplacian $\Delta = \mathrm{d}^2/\mathrm{d}x^2$ on $\mathbb{R}$ is expressed as a scaling of difference operators, that is

$$(\Delta f)(x) = \lim_{h \longrightarrow 0} h^{-2}(f(x+h) - f(x-h) - 2f(x)).$$

By analogy, can we define a kind of "Laplacian" on fractal to define and study the properties of a process on them. As shown in [15], for the post critically finite self-similar sets (for short p.c.f. self-similar sets) the essential idea is to define discrete Laplacians on a sequence of finite graphs which approximate the fractal. A Dirichlet form is then constructed under a certain kind of re-normalization of this discrete Laplacians and so we have an associated Laplacian.

In [15], Kigami introduced also the concept of harmonic structure on p.c.f. self-similar sets and gives an explicit definition of Dirichlet forms associated with regular harmonic structure and defined harmonic functions on them. In [2], Barlow gives the construction of diffusions on some classes of regular self-similar sets using both approachs.

In recent years, focus has been set on some degenerate situations, where the methods used for the elliptic case do not apply. One of the simplest example of such a situation is the Heisenberg group $\mathbb{H}$. The sub-Laplacian $\Delta_{\mathcal{H}}$ on $\mathbb{H}$ is hypoelliptic in the Hörmander sense. As a consequence the heat semigroup $(P_t)_{t\geq 0} = (e^{t\Delta_{\mathcal{H}}})_{t\geq 0}$ obtained by solving the heat equation associated to $\Delta_{\mathcal{H}}$ admits a smooth density with respect to the Lebesgue measure on $\mathbb{R}^3$. It is remarkable that the Markov process associated to this semigroup is the couple formed by a Brownian motion on $\mathbb{R}^2$ and its area, and for every fixed $t > 0$ and $x \in \mathbb{H}$, the probability distribution $P_t(\cdot)(x)$ is Gaussian on $\mathbb{H}$.

In this paper and in order to construct a Markov process on the Sierpinski gasket defined as a couple of the Borwnian motion and its area, we lift the Sierpinski gasket on the Heisenberg group.

First, we define the fractal $\mathbb{S}_{\mathbb{H}}$ to be the invariant set for some contractions $\{F_i\}_{i\in I}$ obtained by lifting the ones that give rise to $\mathbb{S}$. This set is a connected compact subset of $\mathbb{H}$ with dimension of Hausdorff and of similarity equal to $\log 3/\log 2$.

Second, we define a Laplacian Δ on $\mathbb{S}_{\mathbb{H}}$ as a scaling limit of discrete Laplacians $\{H_m\}_{m\geq 0}$ on the finite graphs $\{V_m\}_{m\geq 0}$ approximating $\mathbb{S}_{\mathbb{H}}$. It is defined as

$$(\Delta f)(p) = \lim_{m\longrightarrow\infty} \alpha_p^m (H_m f)(p)$$

for some $\alpha_p > 0$ that depends of p.

The sequence $\{(V_m, H_m)\}_{m\geq 0}$ is invariant under a kind of re-normalization whose scaling factor has to be identified. For the Sierpinski gasket $\mathbb{S}$ this scaling factor is $5/3$. For the principal horizontal lift $\mathbb{S}_{\mathbb{H}}$, this factor is 2. The values of α_p are changed as well.

The underlying networks associated to $\mathbb{S}_{\mathbb{H}}$ differ from the one associated to $\mathbb{S}$. This is due to the fact that two points which are identified after the first application of two contractions differ when the lifted version of these contractions are lifted. A symmetry is then broken. Therefore, we had to find a correct formulation for the first two levels of the Laplacian (from which the other are deduced).

Third, a regular and local Dirichlet form $\mathcal{E}$ is associated to the Laplacian Δ on $\mathbb{S}_{\mathbb{H}}$. This ensures the existence of a Hunt μ-symmetric process $X = (X_t, t \geq 0, \mathbb{P}^x, x \in \mathbb{S}_{\mathbb{H}})$ on $L^2(\mathbb{S}_{\mathbb{H}}, \mu)$. In addition, X is a diffusion. Although this diffusion may be called a Brownian motion on $\mathbb{S}_{\mathbb{H}}$, its projection on $\mathbb{R}^2$ differs from a Brownian motion on $\mathbb{S}$, as it emerges as the limit of networks with a different topology. This construction differs from the one of Hambly and Lyons.

This paper is organized as follows: in Section 2, we provide some results on the Heisenberg group. We introduce post critically finite self-similar sets. We define $\mathbb{S}_{\mathbb{H}}$ the principal lift of the Sierpinski gasket in the Heisenberg group. In Section 3, we introduce the notion of effective resistance. In section 4, we state the main results concerning the definition of a harmonic structure on $\mathbb{S}_{\mathbb{H}}$. We then define a Laplacian Δ_μ. We give a Gauss-Green's formula relating the Laplacian to the corresponding Dirichlet form. In Section 5, we define a diffusion X_t on $\mathbb{S}_{\mathbb{H}}$ using the theory of Dirichlet forms.

2 Iterated functions system and horizontal fractals in $\mathbb{H}$

2.1 The Heisenberg group

We present some classical results on the Heisenberg group [12, 4]. We denote by $\mathbb{H}$ the Heisenberg group, the simplest non-trivial example of a sub-Riemannian manifold. Namely, $\mathbb{H} \cong \mathbb{R}^3$ with the group law $\boxplus$ defined by

$$(x, y, z) \boxplus (x', y', z') = \left(x + x', y + y', z + z' + \frac{1}{2}(xy' - yx') \right).$$

Clearly $(\mathbb{H}, \boxplus)$ is a non-commutative group with 0 as neutral element and the inverse of an element $h = (x, y, z)$ is $h^{-1} = (-x, -y, -z)$. We also denote, for any real number λ, by δ_λ the dilatation in $\mathbb{H}$ defined by

$$\delta_\lambda(x, y, z) = (\lambda x, \lambda y, \lambda^2 z).$$

Note that, for $\lambda, \mu \in \mathbb{R}$ and $h, h' \in \mathbb{H}$,

$$\delta_\lambda(h \boxplus h') = (\delta_\lambda h) \boxplus (\delta_\lambda h') \quad \text{and} \quad \delta_\lambda \delta_\mu h = \delta_{\lambda\mu} h.$$

One can easily see that the bracket

$$[h, h'] = h \boxplus h' - h' \boxplus h = \left(0, 0, \frac{1}{2}(xy' - yx') \right)$$

for $h = (x, y, z)$, $h' = (x', y', z') \in \mathbb{H}$, satisfies the Jacobi identity. Therefore, the space $(\mathbb{H}, [\cdot, \cdot])$ is a Lie algebra. On the other hand,

$$\|h\|_\infty = \max(|x|, |y|, |z|), \quad \text{for } h = (x, y, z) \in \mathbb{H} \tag{2.1}$$

is a norm on $\mathbb{H}$. The maps $(h, h') \mapsto h \boxplus h'$ and $h \mapsto -h$ are continuous which ensures that $(\mathbb{H}, \boxplus)$ is a Lie group. Moreover,

$$|h| = \max\left(|x|, |y|, \frac{1}{2}\sqrt{|z|} \right), \quad \text{for } h = (x, y, z) \in \mathbb{H} \tag{2.2}$$

defines a homogeneous norm that is $|\delta_\lambda h| = |\lambda| \cdot |h|$, for any $\lambda \in \mathbb{R}$, $h \in \mathbb{H}$.

Let $h = (x, y, z)$ be a point of $\mathbb{H}$; denote by $T_h\mathbb{H}$ the tangent plane at h and $\partial_x, \partial_y, \partial_z$ the basis of $T_h\mathbb{H}$. Let $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3$ be the left-invariant vector fields that goes through 0 and that coincide respectively with ∂_x, ∂_y and ∂_z at this point so that

$$\mathcal{Y}_1 = \partial_x - \frac{1}{2} y \partial_z, \ \mathcal{Y}_2 = \partial_y + \frac{1}{2} x \partial_z \text{ and } \mathcal{Y}_3 = \partial_z.$$

The Lie brackets satisfy $[\mathcal{Y}_1, \mathcal{Y}_2] = \mathcal{Y}_3$ and $[\mathcal{Y}_i, \mathcal{Y}_j] = 0$ in the other cases. The tangent space at any point of $\mathbb{H}$ is then equipped with a scalar product $\langle \cdot, \cdot \rangle$ such that $\langle \mathcal{Y}_i, \mathcal{Y}_j \rangle = \delta_{ij}$ for $i, j = 1, 2, 3$ so that $\{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3\}$ forms an orthonormal frame. The sub-bundle $\mathcal{H}$ of the tangent bundle of $\mathbb{H}$ generated by $\mathcal{Y}_1, \mathcal{Y}_2$ is called *horizontal* and a path γ on $\mathbb{H}$ is called *horizontal* if $\dot{\gamma}(t) \in \mathcal{H}_{\gamma(t)}\mathbb{H}$ for any t. The *Carnot length* of $\gamma : [0, 1] \to \mathbb{H}$ is

$$L_{\mathrm{C}}(\gamma) = \int_0^1 \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_{\gamma(t)}} \, \mathrm{d}t.$$

The *Carnot-Carathéodory metric* on $\mathbb{H}$ is

$$d_{\mathrm{CC}}(h, h') = \inf \left\{ L_{\mathrm{C}}(\gamma) \,\middle|\, \begin{array}{l} \gamma : [0, 1] \to \mathbb{H} \text{ horizontal} \\ \gamma(0) = h \text{ and } \gamma(1) = h' \end{array} \right\}. \tag{2.3}$$

Left translations, rotations on the Euclidean plane and involution are isometries of $\mathbb{H}$ with respect to the Carnot-Carathéodory metric.

Another metric is the gauge metric, the *Heisenberg metric* (or Korányi metric) defined by

$$d_{\mathbb{H}}(p, q) = |p^{-1} \boxplus q|_{\mathbb{H}}, \qquad p, q \in \mathbb{H}$$

where $| \cdot |_{\mathbb{H}}$ denotes the *Heisenberg norm* given by

$$|(a, b, c)|_{\mathbb{H}} = ((a^2 + b^2)^2 + c^2)^{\frac{1}{4}}, \ \forall (a, b, c) \in \mathbb{H}.$$

Each of the metric d_{CC} and $d_{\mathbb{H}}$ are homogeneous (*i.e.* for each $\lambda > 0, p, q \in \mathbb{H}$, $d(\delta_\lambda p, \delta_\lambda q) = \lambda d(p, q)$). Moreover,

$$\frac{1}{\sqrt{\pi}} d_{\mathbb{H}}(p, q) \leqslant d_{\mathrm{CC}}(p, q) \leqslant d_{\mathbb{H}}(p, q), \text{ for any } p, q \in \mathbb{H}.$$

Besides using these metrics on $\mathbb{H}$, the so-called *resistance metric* R, which is restricted to fractals, is well adapted to the study of diffusions. This resistence metric gives the same topology as the one of the Euclidean metric. More details are given in Section 3.

Remark 2.1. On the sub-Riemmannian manifold $(\mathbb{H}, \mathcal{H}, \langle, \cdot, \cdot \rangle)$ the second order differential operator

$$\Delta_{\mathcal{H}} = \frac{1}{2} \mathcal{Y}_1^2 + \frac{1}{2} \mathcal{Y}_2^2$$

defined on $C^\infty(\mathbb{H})$ is called the natural sub-Laplacian operator on the Heisenberg group $\mathbb{H}$. The process associated to $\Delta_{\mathcal{H}}$ is $\left(B_t^1, B_t^2, \frac{1}{2} \int_0^t B_s^1 \, \mathrm{d}B_s^2 - \frac{1}{2} \int_0^t B_s^2 \, \mathrm{d}B_s^1 \right)_{t \geq 0}$ for a 2-dimensional Bronwnian motion (B^1, B^2).

2.2 The horizontal lift

An *iterated function system* (IFS for short) on a complete metric space E is a finite collection of contraction mappings $f_i : E \to E$ where i runs through a finite set I. As other fractals, the Sierpinski gasket is entirely described by the IFS as the set which is unvariant under $\{f_i\}_{i\in I}$. In a first time, we show how to transform the $f_i : \mathbb{R}^2 \to \mathbb{R}^2$ as contractions $F_i : \mathbb{H} \to \mathbb{H}$ from which the "lift" of $\mathbb{S}$ will be constructed.

Definition 2.1. Given a map $f : \mathbb{R}^2 \to \mathbb{R}^2$, a map $F : \mathbb{H} \to \mathbb{H}$ is a *lift* of f if $\pi \circ F = f \circ \pi$, where $\pi : \mathbb{H} \to \mathbb{R}^2$ denotes the projection map defined, for $h = (x, y, z) \in \mathbb{H}$, by $\pi(h) = (x, y)$.

The following result is Theorem 1.6 in [4].

Theorem 2.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be r-Lipschitz. Then there exist many lifts $F : \mathbb{H} \to \mathbb{H}$ of f. In addition, let us set*

$$F(x, t) = (f(x), \lambda t + h_0(x))$$

where $h_0 : \mathbb{R}^2 \to \mathbb{R}$ is a map with $\nabla h_0 = \dfrac{1}{2}(\lambda J - \mathrm{D}f^* \cdot \mathbf{J}f)$, f^* denotes the pullback, $\lambda = \det \mathrm{D}f$ is the Jacobian determinant of f and $\mathbf{J} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $\mathbf{J}(x_1, x_2) = (-x_2, x_1)$. Then F is a Lipschitz map with respect to $d_{\mathbb{H}}$. Any other Lipschitz lift G of f is of the form $G(x, t) = F(x, t + \tau)$ for some constant τ.

Remark 2.2. As we use a different group law as the one on [4], the formula for F involves different constants.

We can now conclude that the affine map f on $\mathbb{R}^2$ defined by $f(x) = Ax + b$, where A is a 2-square matrix of real coefficients and x, b are vectors in $\mathbb{R}^2$, can be lifted as an affine map F on $\mathbb{H}$ of form

$$F(x, t) = \left(Ax + b, \det(A)t - \frac{1}{2}\langle Ax, \mathbf{J}b\rangle_{\mathbb{R}^2} + \tau\right), \tag{2.4}$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}^2}$ is the canonical inner product on $\mathbb{R}^2$ and τ is any real number.

The *principal horizontal lift* of IFS $\mathcal{I}$ to $\mathbb{H}$ is defined to be the IFS $\mathcal{I}_{\mathbb{H}}$ on $\mathbb{H}$ for which all of the fixed points have third coordinate zero.

2.3 The Sierpinski gasket

In this section, we show how to construct the Sierpinski gasket $\mathbb{S}$ using the IFS through a recursive procedure.

Let J_0 be the vertices of a closed convex equilateral triangle of unit side in $\mathbb{R}^2$, that is $J_0 = \{a_1, a_2, a_3\}$ where $a_1 = (0,0)$, $a_2 = (1,0)$ and $a_3 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. For $i = 1,2,3$, define

$$f_i : \mathbb{R}^2 \to \mathbb{R}^2 \text{ with } f_i(x) = \frac{1}{2}(x + a_i),\ x \in \mathbb{R}^2.$$

Then $\mathcal{I} = \{f_1, f_2, f_3\}$ is an IFS on $\mathbb{R}^2$. There exists a unique nonempty compact set $\mathbb{S}$ invariant for the IFS $\mathcal{I}$, that is $\mathbb{S} = \bigcup_{i=1}^3 f_i(\mathbb{S})$. The set $\mathbb{S}$ is called the 2-dimensional *Sierpinski gasket* (see Figure 1). Moreover, set

$$J_n = \bigcup_{w \in \{1,2,3\}^n} f_w(\{a_1, a_2, a_3\})$$

and

$$B_n = \bigcup_{w \in \{1,2,3\}^n} \{(f_w(a_1), f_w(a_2)), (f_w(a_1), f_w(a_3)), (f_w(a_2), f_w(a_3))\},$$

where $f_w = f_{w_1} \circ f_{w_2} \circ \cdots \circ f_{w_n}$ for $w = w_1 w_2 \cdots w_n \in \{1,2,3\}^n$. In particular $J_0 = \{a_1, a_2, a_3\}$ and $B_0 = \{(a_1, a_2), (a_1, a_3), (a_2, a_3)\}$.

Each pair of the sequence $(J_n, B_n)_{n \geq 1}$ is a graph where J_n is the set of vertices and B_n is the set of edges.

As $\mathbb{S} = \overline{\bigcup_{n \geq 0} J_n}$, we can consider $(J_n, B_n)_{n \geq 0}$ as a sequence approximating graphs of the Sierpinski gasket (See Figure 2).

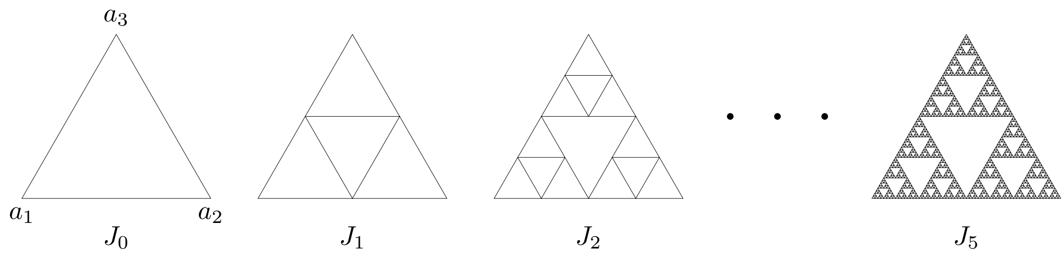


Figure 1: Construction of Sierpinski gasket $\mathbb{S}$.

Another approach to the construction of the Sierpinski gasket (see Figure 2), adopted by Lindstrøm [32], and most of mathematical physics literature, is to look at fractal subsets of $\mathbb{R}^d$ obtained by generalization of the Cantor set. That is, if A_0 is a closed, convex triangle of unit side, let A_1 the set obtained from A_0 by deleting the open convex triangle whose vertices are the midpoints of the edges of A_0. Thus A_1 consists of 3 closed convex triangles with side 1/2. Repeating this procedure one obtains successively $A_2, A_3, \ldots$. The Sierpinski gasket is $\mathbb{S} = \bigcap_{n=0}^{\infty} A_n$.

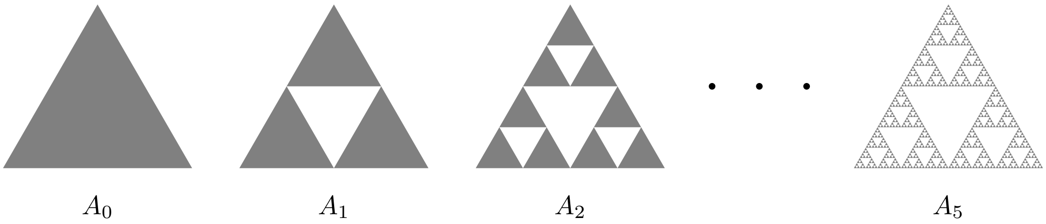


Figure 2: Another construction of $\mathbb{S}$.

2.4 Horizontal lift of the gasket to the Heisenberg group

Proposition 2.2. *Consider the IFS $\mathcal{I} = \{f_1, f_2, f_3\}$ given in Section 2.3, the principal horizontal Lipschitz lift $\mathcal{I}_{\mathbb{H}} = \{F_1, F_2, F_3\}$ of $\mathcal{I}$ to $\mathbb{H}$ is given by*

$$F_1(q) = \delta_{\frac{1}{2}}(p_1 \boxplus q), \quad F_2(q) = \delta_{\frac{1}{2}}(p_2 \boxplus q), \quad F_3(q) = \delta_{\frac{1}{2}}(p_3 \boxplus q), \tag{2.5}$$

where $p_1 = (0,0,0)$, $p_2 = (1,0,0)$, $p_3 = (\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$ and $q \in \mathbb{H}$.

Proof. Using Theorem 2.1 a Lipschitz lift $\mathcal{I}_{\mathbb{H}} = \{F_1, F_2, F_3\}$ of $\mathcal{I}$ for which p_i is a fixed point of F_i is given by

$$F_i(x,t) = \left(A_i x + b_i, \det(A_i)t - \frac{1}{2}\langle A_i(x - a_i), \mathbf{J}b_i\rangle_{\mathbb{R}^2}\right), \quad i = 1,2,3,$$

where A_i is 1/2 times identity and $b_i = 1/2 \cdot a_i$. A simple calculation shows that $F_i(q) = \delta_{\frac{1}{2}}(p_i \boxplus q)$ and $F_i(p_i) = p_i$ for all $i = 1,2,3$ so $\mathcal{I}_{\mathbb{H}}$ is the principal lift of $\mathcal{I}$. □

Definition 2.2. The invariant set for $\mathcal{I}_{\mathbb{H}}$ is the *principal horizontal lift of the Sierpinski gasket to the Heisenberg group* that we will denote by $\mathbb{S}_{\mathbb{H}}$ (see Figure 3).

By construction and according to [4, Theorem 1.9], the principal horizontal lift $\mathbb{S}_{\mathbb{H}}$ exists. Besides, $\pi(\mathbb{S}_{\mathbb{H}}) = \mathbb{S}$, that is the projection of $\mathbb{S}_{\mathbb{H}}$ onto $\mathbb{R}^2$ is $\mathbb{S}$.

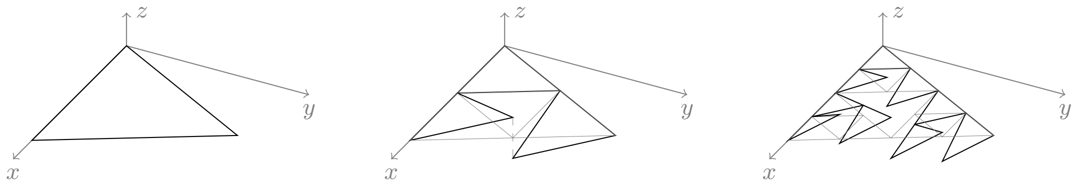


Figure 3: The principal lift $\mathbb{S}_{\mathbb{H}}$ of the Sierpinski gasket $\mathbb{S}$.

2.5 Construction of $\mathbb{S}_{\mathbb{H}}$ as a p.c.f. self-similar set

We now use the IFS $\{F_i\}$ as a system of coordinates for the points in $\mathbb{S}_{\mathbb{H}}$.

Set $I = \{1, 2, 3\}$. For $m \geq 1$, denote by $W_m = I^m$ the space of words of length m, by $W = \bigcup_{m \in \mathbb{N}} W_m$ the space of words of finite length and by $W_* = I^{\mathbb{N}}$ the space of words of infinite length that is

$$W_* = \{\omega = \omega_1 \omega_2 \omega_3 \ldots : \omega_i \in \{1, 2, 3\}, i \in \mathbb{N}\}.$$

Fix $0 < r < 1$ and let $d_r(\omega, \tau) = r^{m(\omega,\tau)}$, $\omega, \tau \in W_*$, with $m(\omega, \tau) = \min\{n : \omega_n \neq \tau_n\}$. Then d_r defines a distance on W_* and (W_*, d_r) becomes a compact metric space.

For $m \in \mathbb{N}$ and $\omega = \omega_1 \ldots \omega_m \in W_m$, let $F_\omega = F_{\omega_1} \circ F_{\omega_2} \circ \cdots \circ F_{\omega_m}$.

Let us define

$$V_0 = \{p_1, p_2, p_3\},\ V_m = \bigcup_{\omega \in W_m} F_\omega(V_0) \text{ for } m \geq 1 \text{ and } V_* = \bigcup_{m=0}^{\infty} V_m. \tag{2.6}$$

Corollary 2.3. *The principal horizontal lift $\mathbb{S}_{\mathbb{H}}$ of the Sierpinski gasket to $\mathbb{H}$ is the closure $\overline{V_*}$ of V_*.*

By Corollary 4.15 in [4], $\mathbb{S}_{\mathbb{H}}$ is a compact connected subset of $\mathbb{H}$. According to Theorem 1.2.3 in [15], there exists a continuous surjection $s : W_* \to \mathbb{S}_{\mathbb{H}}$ defined by $s(\omega) = q$ where, for $\omega = \omega_1\omega_2 \cdots \in W_*$, $\{q\} = \cap_{m\in\mathbb{N}} F_{\omega_1\ldots\omega_m}(\mathbb{S}_{\mathbb{H}})$. The 3-uple $(\mathbb{S}_{\mathbb{H}}, I, \{F_i\}_{i\in I})$ is called a *self-similar structure.*

Note also that for $\omega \neq \tau \in W_*$, $s(\omega) = s(\tau)$ if and only if for any $m \in \mathbb{N}$, $s(\sigma^m(\omega)) = s(\sigma^m(\tau))$ where $\sigma : W_* \to W_*$ is the shift operator [15, Proposition 1.2.5].

Definition 2.3. Let $(K, I, \{F_i\}_{i\in I})$ be a self-similar structure. We define the *critical set* $\mathcal{C} \subset I^{\mathbb{N}}$ and the *post critical* set $\mathcal{P} \subset I^{\mathbb{N}}$ by

$$\mathcal{C} = s^{-1}\left(\bigcup_{\substack{i,j\in I \\ i\neq j}} \Big(F_i(K) \bigcap F_j(K)\Big)\right) \text{ and } \mathcal{P} = \bigcup_{n\geqslant 1} \sigma^n(\mathcal{C}).$$

Example 2.1. For the Sierpinski gasket $\mathbb{S}$, the IFS is $\{f_1, f_2, f_3\}$ with

$$f_i(x) = \frac{1}{2}(x+a_i),\ i = 1,2,3,\ x \in \mathbb{R}^2,\ a_1 = (0,0),\ a_2 = (1,0) \text{ and } a_3 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right).$$

For $n \geq 1$ and $\omega \in I^n$,

$$\begin{aligned} f_{\overline{1}}(a_1) = a_1,\ f_{\overline{2}}(a_2) = a_2,\ f_{\overline{3}}(a_3) = a_3, \\ f_{1\overline{2}}(a_2) = f_{2\overline{1}}(a_1) = \frac{a_1 + a_2}{2} = b_{12} = \left(\frac{1}{2}, 0\right), \\ f_{1\overline{3}}(a_3) = f_{3\overline{1}}(a_1) = \frac{a_1 + a_3}{2} = b_{13} = \left(\frac{1}{4}, \frac{\sqrt{3}}{4}\right), \\ f_{2\overline{3}}(a_3) = f_{3\overline{2}}(a_2) = \frac{a_2 + a_3}{2} = b_{23} = \left(\frac{3}{4}, \frac{\sqrt{3}}{4}\right). \end{aligned}$$

The self-similar structure $(\mathbb{S}, I, \{f_i\}_{i \in I})$ has for critical set $\mathcal{C} = \left\{1\overline{2}, 2\overline{1}, 1\overline{3}, 3\overline{1}, 2\overline{3}, 3\overline{2}\right\}$ and for post-critical set $\mathcal{P} = \left\{\overline{1}, \overline{2}, \overline{3}\right\}$ [16, Example 3.1.3]. The element $1\overline{2}$ in $\mathcal{C}$ is identified through s with the point $b_{12} = f_1(\mathbb{S}) \cap f_2(\mathbb{S})$, ...

For the IFS $\{F_i\}_{i=1,2,3}$ and the vertices p_1, p_2, p_3 of the equilateral triangle of side one in $\mathbb{H}$ we can easily see that $F_{\overline{i}}(p_i) = p_i$ for $i = 1,2,3$ where $\overline{i} = iii\ldots \in W_*$. Set (See figure 4)

$$\begin{aligned} q_{12} = F_1(p_2) = F_{1\overline{2}}(p_2) = F_{2\overline{1}}(p_1) = F_2(p_1) = q_{21} = \left(\frac{1}{2}, 0, 0\right), \\ q_{13} = F_1(p_3) = F_{1\overline{3}}(p_3) = F_{3\overline{1}}(p_1) = F_3(p_1) = q_{31} = \left(\frac{1}{4}, \frac{\sqrt{3}}{4}, 0\right), \\ q_{23} = F_2(p_3) = F_{2\overline{3}}(p_3) = \left(\frac{3}{4}, \frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{16}\right), \\ \text{and } q_{32} = F_3(p_2) = F_{3\overline{2}}(p_2) = \left(\frac{3}{4}, \frac{\sqrt{3}}{4}, -\frac{\sqrt{3}}{16}\right). \end{aligned}$$

As $p_2 \boxplus p_3 \neq p_3 \boxplus p_2$ since the $\boxplus$ operation is non-commutative, the critical set of $\mathbb{S}_{\mathbb{H}}$ differs from the one of $\mathbb{S}$ which we identified in Example 2.1.

Proposition 2.4. *The critical set of the self-similar structure* $(\mathbb{S}_{\mathbb{H}}, I, \{F_i\}_{i \in I})$ *is* $\mathcal{C} = \left\{1\overline{2}, 2\overline{1}, 1\overline{3}, 3\overline{1}\right\}$*, and its post-critical set is* $\mathcal{P} = \left\{\overline{1}, \overline{2}, \overline{3}\right\}$.

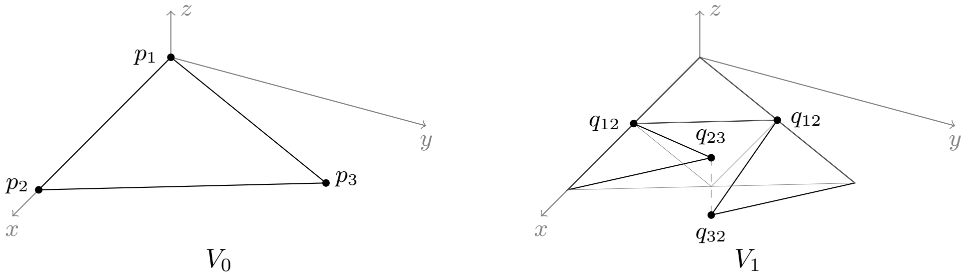


Figure 4: The sets V_0 and V_1.

2.6 Similarity and Hausdorff dimension of $\mathbb{S}_{\mathbb{H}}$

As shown in Section 2.3, the Sierpinski gasket is constructed from a sequence of isocele triangle J_0 by applying the IFS recursively.

Using the distance $d_{\mathbb{H}}$ instead of the Euclidean distance

$$d_{\mathbb{H}}(p_1, p_2) = d_{\mathbb{H}}(p_1, p_3) = 1 \text{ while } d_{\mathbb{H}}(p_2, p_3) = \left(\frac{19}{16}\right)^{\frac{1}{4}} > 1.$$

For each $m \geq 1$, $\omega \in W_m$, $p, q \in V_0$,

$$d_{\mathbb{H}}(F_\omega(p), F_\omega(q)) = |(F_w(q))^{-1} \boxplus F_w(p)|_{\mathbb{H}} = \left(\frac{1}{2}\right)^m d_{\mathbb{H}}(p, q).$$

In particular,

$$\text{if } (p, q) \in \{(p_1, p_2), (p_1, p_3)\} \text{ then } d_{\mathbb{H}}(F_\omega(p), F_\omega(q)) = \left(\frac{1}{2}\right)^m,$$
$$\text{if } (p, q) = (p_2, p_3) \text{ then } d_{\mathbb{H}}(F_\omega(p), F_\omega(q)) = \left(\frac{1}{2}\right)^m \left(\frac{19}{16}\right)^{\frac{1}{4}}.$$

Does this affects the dimension of the fractal?

The *similarity dimension* of a finite set of linear contractions $\{f_1, f_2, \ldots, f_n\}$ is the unique value s such that $\sum_{i=1}^n \|f_i\|^s = 1$ where $\|f_i\| = r_i$ is the Lipschitz constant of f_i.

Example 2.2. The planar Sierpinski gasket $\mathbb{S}$ is the invariant set of $\mathcal{F} = \{f_1, f_2, f_3\}$ with $\|f_i\| = \frac{1}{2}$ for $i = 1, 2, 3$. Therefore, the similarity dimension of $\mathcal{F}$ (or $\mathbb{S}$) is $\frac{\log 3}{\log 2}$.

Let (X, d) be a metric space, B a bounded set of X and $s > 0$. The s-dimensional Hausdorff measure of B is $\mathcal{H}_d^s(B) = \lim_{\delta \searrow 0} \inf \sum_n \operatorname{diam}(B_i)$, where $\operatorname{diam}(A) =$

$\sup_{x,y\in A} d(x,y)$ for $A \subset X$ and the infinum is taken over the set of the δ-covers $B_1, B_2, \dots$ of B, and the *Hausdorff dimension* of B is $\sup\{s : \mathcal{H}_d^s(B) = 0\} = \inf\{s : \mathcal{H}_d^s(B) = \infty\}$.

Proposition 2.7 in [5] gives a relationship between the Hausdorff and the similarity dimension of the invariant set of an IFS.

Proposition 2.5. *Let $\mathcal{F} = \{f_1, f_2, \dots, f_M\}$ be a self-similar Affine IFS in $\mathbb{R}^n$ which satisfies the open set condition, i.e., there exists a bounded non-empty open set $O \subset \mathbb{R}^n$ such that $\cup_{i=1}^{i=M} f_i(O) \subset O$ and $f_i(O) \cap f_j(O) = \varnothing$ for $i \neq j$. Let K be the invariant set of $\mathcal{F}$. Let $\mathcal{A}$ denotes the collection of conformal matrices which arise as the linear parts of elements of $\mathcal{F}$ (counted with multiplicity). Then the Hausdorff dimension of K is equal to the similarity dimension s of $\mathcal{A}$. Moreover, $0 < \mathcal{H}_d^s(K) < \infty$.*

Using Proposition 3.14 in [4] which asserts that the open set condition passes to horizontal lifts, we have the following result by Corollary 3.8 in [5].

Proposition 2.6. *Let $\mathcal{F}$ be a self-similar planar Affine IFS which satisfies the open set condition and let $\mathcal{F}_{\mathbb{H}}$ be the horizontal lift of $\mathcal{F}$. Let $\mathcal{A}_f$ be the set of contractions obtained from $\mathcal{F}$. Then the Hausdorff dimension of $\mathbb{S}_{\mathbb{H}}$ with respect to $d_{\mathbb{H}}$ or the Euclidean metric is equal to the similarity dimension of $\mathcal{A}_f$.*

We now combine Proposition 2.5 and Corollary 2.6 to identify the similarity and Hausdorff dimensions of $\mathbb{S}_{\mathbb{H}}$.

Corollary 2.7. *The Hausdorff dimension of $\mathbb{S}_{\mathbb{H}}$ with respect to $d_{\mathbb{H}}$ or the Euclidean metric is equal to $\frac{\log 3}{\log 2}$, the one of $\mathbb{S}$.*

3 Effective resistance

Our diffusion process is constructed as a limit of random walks on nested discreted sets seen as an electrical network. The probability transition are computed through the concept of "resistance".

For a finite set V, let $\ell(V) = \{f \mid f : V \to \mathbb{R}\}$ equipped with the standard inner product.

Definition 3.1. A symmetric operator $H : \ell(V) \longrightarrow \ell(V)$ is called a *Laplacian on V* if it statisfies:
(i) H is non-positive definite,
(ii) $Hu = 0$ if and only if u is constant on V,

(iii) $H_{pq} \geq 0$ for all $p \neq q \in V$, where $H_{pq} = H\mathbb{1}_{\{q\}} \cdot \mathbb{1}_{\{p\}}$.
We denote by $\mathcal{LA}(V)$ collection of Laplacians on V.

Remark 3.1. Let $V = \{a, b, c\}$ and H a symmetric operator on $\ell(V)$. Due to (iii), we identfiy H with the matrix $\{H_{pq}\}_{p,q\in V}$.

For a symmetric linear map $H : \ell(V) \longrightarrow \ell(V)$, we define a symmetric bilinear form $\mathcal{E}_H$ by $\mathcal{E}_H(u, v) = -{}^t uHv$ for all $u, v \in \ell(V)$.

Definition 3.2. The pair (V, H) is called a *resistance network* (r-network for short) if $\mathcal{E}_H(u, v) \geq 0$ and the equality holds if and only if u is constant on V.

The pair (V, H) is a r-network if and only if $H \in \mathcal{LA}(V)$.

Remark 3.2. From probabilistic point of view, a r-network (V, H) corresponds to a random walk on V with transition probability from $p \in V$ to $q \in V$ given by

$$P(p, q) = \begin{cases} -H_{pq}/H_{pp} & \text{if } p \neq q, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 3.3. Let (V_1, H_1) and (V_2, H_2) be two r-networks where H_2 is decomposed as $H_2 = \begin{bmatrix} T & {}^tJ \\ J & X \end{bmatrix}$ for matrices T, J and X identified with linear operators $T : \ell(V_1) \to \ell(V_1)$, $J : \ell(V_1) \to \ell(V_2 \setminus V_1)$, $X : \ell(V_2 \setminus V_1) \longrightarrow \ell(V_2 \setminus V_1)$. Then we write $(V_1, H_1) \leq (V_2, H_2)$ if and only if $V_1 \subseteq V_2$ and $P_{V_2,V_1}(H_2) = H_1$ where

$$P_{V_2,V_1} = T - {}^tJX^{-1}J. \tag{3.1}$$

The linear operator P_{V_2,V_1} — which belongs to $\mathcal{LA}(V_1)$ — may be thought as the restriction of H_2 onto H_1. When $(V_2, H_2) \leq (V_1, H_1)$,

$$\mathcal{E}_{H_1}(v, v) = \min\{\mathcal{E}_{H_2}(u, u), v \in \ell(V_2), u_{|V_1} = v\}.$$

Definition 3.4. Let each $m \geq 0$, let V_m be a finite set and $H_m \in \mathcal{LA}(V_m)$. Then $\{(V_m, H_m)\}_{m\geq 0}$ is called a *compatible sequence* if $(V_m, H_m) \leq (V_{m+1}, H_{m+1})$ for all $m \geq 0$.

Definition 3.5. Let V a finite set and $H \in \mathcal{LA}(V)$. For $p \neq q \in V$, define

$$\begin{aligned} R_H(p, q) &= (\min\{\mathcal{E}_H(u, u) : u \in \ell(V),\ u(p) = 1,\ u(q) = 0\})^{-1} \\ &= \max\left\{\frac{|u(p) - u(q)|^2}{\mathcal{E}(u, u)} : u \in \ell(V),\ \mathcal{E}(u, u) \neq 0\right\}. \end{aligned}$$

We define also $R_H(p, p) = 0$ for all $p \in V$. The quantity $R_H(p, q)$ is the *effective resistance* between p and q with respect to H.

The notion of effective resistance allows to define a metric on V. Note that if

$$(V_1, H_1) \leq (V_2, H_2)$$

then

$$R_{H_1}(p, q) = R_{H_2}(p, q), \ \forall (p, q) \in V_1 \times V_1.$$

Furthermore, this property is passed on by passage in the limit. Assume that we have a sequence of compatible r-networks $S = \{(V_m, H_m)\}_{m \geq 0}$. Then we define a non-negative symmetric form on $V_\star = \bigcup_{m \geq 0} V_m$ by

$$\mathcal{F} = \{u, u \in \ell(V_*), \lim_{m \longrightarrow \infty} \mathcal{E}(u|_{V_m}, u|_{V_m}) < \infty\},$$
$$\mathcal{E}(u, v) = \lim_{m \longrightarrow \infty} \mathcal{E}(u|_{V_m}, u|_{V_m}) \text{ for } u, v \in \mathcal{F}.$$

The *effective resistance* associated to S is defined by

$$R(p, q) = R_{H_m}(p, q),$$

where m is chosen so that $p, q \in V_m$. The quantity R is a metric on $V_\star$.

Let $u \in \mathcal{F}$ and $p, q \in V_\star$. As $|u(p) - u(q)|^2 \leq R(p, q)\mathcal{E}(u, u)$, we define (Ω, R) to be the completion of the metric space $(V_\star, R)$. Thus, $u \in \mathcal{F}$ has a natural extension to a uniformly continuous function on (Ω, R).

Finally we have a complete metric space (Ω, R) and a quadratic form $(\mathcal{E}, \mathcal{F})$.

Can we identify such a set Ω with $\mathbb{S}_{\mathbb{H}}$? The answer is yes if we can construct a sequence $\{(V_m, H_m)\}_{m \geq 0}$ such that the corresponding space Ω is the horizontal lift $\mathbb{S}_{\mathbb{H}}$. The sequence $(V_m)_{m \geq 0}$ is constructed from the knowledge of the IFS $\{F_1, F_2, F_3\}$ and of the post-critical set $\mathcal{P}$.

4 Harmonic structure and Dirichlet form on $\mathbb{S}_{\mathbb{H}}$

We have now to identify the sequence of r-networks which is suitable for the construction of $\mathbb{S}_{\mathbb{H}}$. According to Definition 3.3 in [16], the nested sequence $\{V_m\}_{m \geq 0}$ is given by (2.6). The compatible sequence $\{(V_m, H_m)\}_{m \geq 0}$ will be constructed from the sole knowledge of a suitable Laplacian H_0 on V_0, as well as a vector r.

4.1 Regular harmonic structures

Let $\{(V_m, H_m)\}_{m \geq 0}$ be a sequence of r-networks where V_0 has N elements. The elements of V_0 are seen as the "boundary" of the network.

Let r be a vector of $\mathbb{R}^N$ with $r_i > 0$ for $i = 1, \ldots, N$. This vector describes how to scale the resistances in the network.

For $\omega = \omega_1\omega_2\ldots\omega_m \in W_m$, we define $R_\omega : \ell(V_m) \to \ell(V_0)$ by $R_\omega u = u \circ F_\omega$. We also set $r_\omega = r_{\omega_1} r_{\omega_2} \ldots r_{\omega_m}$.

For any $m \geq 1$ and functions u, v on V_m, define

$$H_m = \sum_{\omega \in W_m} \frac{1}{r_\omega} {}^t R_\omega D R_\omega \text{ with } D = H_0 \tag{4.1}$$

and

$$\mathcal{E}^{(m)}(u,v) = \sum_{\omega \in W_m} \frac{1}{r_\omega} \mathcal{E}^{(0)}(R_\omega u, R_\omega v).$$

Definition 4.1. The couple (D, r) is called a *harmonic structure* if and only if $\{(V_m, H_m)\}_{m \geqslant 0}$ constructed by (4.1) is a compatible sequence of r-networks. Besides, (D, r) is *regular* if $0 < r_i < 1$ for all $i = 1, 2, 3$.

The condition for harmonic structure is simplified by Proposition 3.1.3 in [17].

Proposition 4.1. *The couple (D, r) is a harmonic structure if and only if $(V_0, D) \leqslant (V_1, H_1)$.*

4.2 A regular harmonic structure for $\mathbb{S}_{\mathbb{H}}$

The nested sequence $\{V_m\}_{m \geq 0}$ is already given by (2.6). For the principal horizontal lift of the Sierpinski gasket,

$$V_0 = \{p_1, p_2, p_3\} \text{ and } V_1 = \{p_1, p_2, p_3, q_{12}, q_{13}, q_{23}, q_{32}\},$$

where the coordinates of the p_i and q_{ij}, $i, j \in \{1, 2, 3\}$ are given in Section 2.5. In particular, the set V_1 has one more point than the corresponding set for the Sierpinski gasket since $q_{23} \neq q_{32}$.

For $m \geq 0$, the Laplacian H_m is described by connecting the points of V_m with resistances. The sequence $\{(V_m, H_m)\}_{m \geq 0}$ has to be compatible. Combining Definition 4.1 and Proposition 4.1, two strategies may be considered. The first one consists in guessing H_0 and H_1 so that $(V_0, H_0) \leq (V_1, H_1)$ and then to identify the vector r. The second one consists in guessing the vector r such that for H_1 given by (4.1) with $m = 1$, the projection P_{V_2,V_1} given by (3.1) is equal to H_0.

Here, we consider using the first strategy. The principal horizontal lift of the Sierpinski gasket $\mathbb{S}_{\mathbb{H}}$ differs from $\mathbb{S}$ on two points: while $f_2(\mathbb{S}) \cap f_3(\mathbb{S}) = \{b_{23}\}$,

$F_2(\mathbb{S}_{\mathbb{H}}) \cap F_3(\mathbb{S}_{\mathbb{H}}) = \emptyset$. It is easily seen that $\mathbb{S}_{\mathbb{H}} \setminus V_0$ contains two connected components, while $\mathbb{S} \setminus V_0$ contains only one. In view of Theorem 3.2.11 in [17], it is natural to assume that $D_{p_2,p_3} = 0$, which means that p_2 and p_3 are not connected.

Let V^0 be the graph with vertices V_0 with edges (p_1, p_2) and (p_1, p_3). This means that p_2 and p_3 are not neighbors in V^0.

We start with the symmetric linear operator $D : \ell(V_0) \to \ell(V_0)$ defined by

$$Df(p_i) = \sum_{(p,q)\in V^0} [f(q) - f(p_i)], \quad i = 1, 2, 3,$$

with matrix representation

$$D = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

We set for any functions u, v on V_0, $\mathcal{E}^{(0)}(u, v) = -{}^t uDv$.

Remark 4.1. For the Sierpinski gasket $\mathbb{S}$, a natural choice is

$$D_{\mathbb{S}} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

since (p_2, p_3) belongs to V^0. Other choices are possible [17, Examples 2.1.4 and Exercice 3.1], still with a connection between p_2 and p_3.

We now exhibit a regular harmonic structure for $\mathbb{S}_{\mathbb{H}}$. Let us consider the linear operator $L_1 : \ell(V_1) \to \ell(V_1)$ defined by

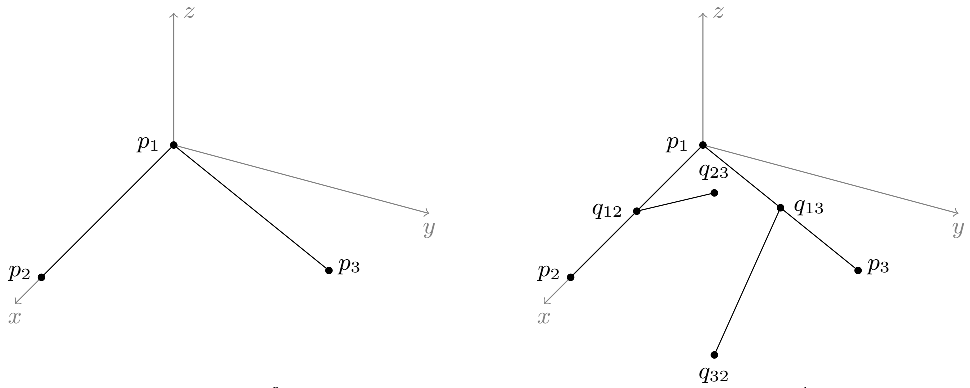
$$(L_1 f)(p) = \sum_{q\in V_{1,p}} (f(q) - f(p)),$$

where for $p \in V_1$,

$$V_{1,p} = \left\{ q \in V_1 : \begin{array}{l} q \neq p, \text{ there exists } i \in \{1, 2, 3\} \text{ with } p, q \in F_i(V_0) \\ \text{and } D_{F_i^{-1}(p) F_i^{-1}(q)} \neq 0 \end{array} \right\}.$$

A simple computation shows that

$$V_{1,p_1} = \{q_{12}, q_{13}\}, \ V_{1,p_2} = \{q_{12}\}, \ V_{1,p_3} = \{q_{13}\},$$
$$V_{1,q_{12}} = \{p_1, p_2, q_{23}\}, \ V_{1,q_{13}} = \{p_1, p_3, q_{32}\}, \ V_{1,q_{23}} = \{q_{12}\} \text{ and } V_{1,q_{32}} = \{q_{13}\}.$$



V_0 and V^0 $\qquad$ V_1 and V^1

Figure 5: The graphs (V_0, V^0) and (V_1, V^1).

The graph V^1 giving the connections between the points of $V_1 = \cup_{p \in V_1} \{(p,q) \in V_{1,p}\}$ are represented in Figure 5.

The matrix representation of L_1 is given by (the order of the points is $p_1, p_2, p_3, q_{12}, q_{13}, q_{23}, q_{32}$)

$$L_1 = \begin{bmatrix} -2 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & -3 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & -3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix}.$$

Let f be a function on V_1 such that

$$f(p_1) = a,\ f(p_2) = b,\ f(p_3) = c \text{ and } L_1 f = 0.$$

Considering that V_0 is the boundary of V_1 and solving the linear equations $(L_1 f)(q) = 0$ for $q \in V_1 \setminus V_0$,

$$f(q_{12}) = f(q_{23}) = \frac{a+b}{2},\ f(q_{13}) = f(q_{32}) = \frac{a+c}{2}.$$

As F_i is one-to-one between V_0 and $F_i(V_0)$ for $i \in \{1,2,3\}$, we identify $\ell(V_0)$ with $\ell(F_i(V_0))$ through F_i. We define the linear maps $A_i : \ell(V_0) \longrightarrow \ell(F_i(V_0))$ by

$$f_{|F_i(V_0)} = A_i(f_{|V_0}).$$

For $i = 1,2,3$, the linear maps A_i have matrix representations

$$A_1 = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad A_2 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} \text{ and } A_3 = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

The matrices A_1, A_2 and A_3 have a common second eigenvalue which is $\frac{1}{2}$. Using Example 3.2.6 and Appendix A.1 in [17], we set

$$r = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right).$$

This choice of r corresponds to the eigenfunctions given in Figure 6. The meaning of the eigenfunction u_i of A_i is to find u_i on V_0 that is extended to V_1 in a way such that $L_1 u_i = 0$ and $u_i(F_i(p)) = r_i u_i(p)$, *e.g.* $u_1(q_{13}) = r_1 u_1(p_3)$ and $u_1(q_{1,2}) = r_1 u(p_2)$. This means that u_i has a self-similar property. We may check that it is the case for the eigenfunctions presented in Figure 6.

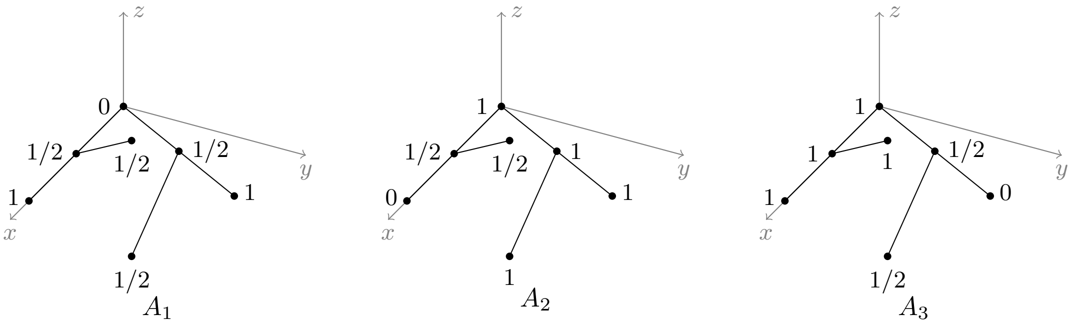


Figure 6: The values of the eigenfunctions of A_1, A_2 and A_3 at p_1, p_2 and p_3 and their harmonic extensions on V_1.

Remark 4.2. For the Sierpinski gasket $\mathbb{S}$ and $D_{\mathbb{S}}$ given in Remark 4.1, $r = (3/5, 3/5, 3/5)$ so that the scaling is different. This is due to the fact that p_2 and p_3 are connected.

Finally with the initial Laplacian D on V_0 and the vector r, we construct a sequence of self-similar Laplacians H_m on V_m by

$$H_m = \sum_{\omega \in W_m} \frac{1}{r_\omega} {}^t R_\omega D R_\omega.$$

To prove that (D, r) is a harmonic structure, we calculate H_1 in order to check that $(V_0, D) \leq (V_1, H_1)$.

A computation shows that $H_1 = 2 \sum_{i=1,2,3} {}^t R_i D R_i = 2L_1$, which we write

$$H_1 = \begin{bmatrix} T & {}^t J \\ J & X \end{bmatrix}, \tag{4.2}$$

where

$$T = 2\begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad J = 2\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } X = 2\begin{bmatrix} -3 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}.$$

Now denote by $X : \ell\,(V_1 \setminus V_0) \to \ell\,(V_1 \setminus V_0)$ and $J : \ell(V_0) \to \ell(V_1 \setminus V_0)$ the linear operators whose matrices are respectively X and J.

It is easily checked that $T - {}^tJX^{-1}J = D$, meaning that $P_{V_1,V_0} = D$, where P_{V_1,V_0} is defined by (3.1). This proves that $(V_1, H_1) \leq (V_0, H_0)$. From this formula, using Proposition 4.1, (D, r) is a harmonic structure on $\mathbb{S}_{\mathbb{H}}$. As $0 < r_i < 1$, the harmonic structure (D, r) is regular.

Using [17, Theorem 3.3.4], the following proposition holds.

Proposition 4.2. *The following properties hold for* $\mathbb{S}_{\mathbb{H}}$*:*

(i) $\Omega = \mathbb{S}_{\mathbb{H}}$.
(ii) (Ω, R) *is compact.*
(iii) (Ω, R) *is bounded.*
(iv) $\forall u \in \mathcal{F}, sup_{p\in\Omega}|u(p)| < \infty$.
(v) R *is a metric on* $\mathbb{S}_{\mathbb{H}}$ *compatible with the initial metric.*

For each function u on V_*, $\mathcal{E}^{(m)}(u|_{V_m}, u|_{V_m})$ is non-decreasing in m, and $\mathcal{E}(u, u) = \lim_{m\to\infty} \mathcal{E}^{(m)}(u|_{V_m}, u|_{V_m})$ exists (possibly infinite). Let $\mathcal{F}(\mathbb{S}_{\mathbb{H}}) = \{u : \mathcal{E}(u, u) < \infty\}$. Using [15, Sections 2.2 and 3.1], every $u \in \mathcal{F}(\mathbb{S}_{\mathbb{H}})$ is continuous on V_* and therefore can be extended to a continuous function on $\mathbb{S}_{\mathbb{H}}$. In other words, $\mathcal{F}(\mathbb{S}_{\mathbb{H}}) \subset C(\mathbb{S}_{\mathbb{H}})$, which in fact follows easily from

$$\text{osc}_{\mathbb{S}_{\mathbb{H}}}(u) \leq C_*\mathcal{E}(u, u), \ \forall u \in \mathcal{F}(\mathbb{S}_{\mathbb{H}}),$$

where $C_* > 0$ is a universal constant [17, Lemma 2.3.9 and Theorem 3.3.4].

Since $\mathcal{F}(\mathbb{S}_{\mathbb{H}}) \subset C(\mathbb{S}_{\mathbb{H}})$, the empty set $\emptyset$ is the only subset of $\mathbb{S}_{\mathbb{H}}$ with zero capacity. As a consequence of the definition of $\mathcal{E}^{(m)}$, it is seen that $\mathcal{E}^{(m+1)}(u, v) = \sum_{i=1,2,3} \frac{1}{2}\mathcal{E}^{(m)}(u \circ F_i, v \circ F_i), m \in \mathbb{N}$, which implies the following self-similar property of $\mathcal{E}$:

$$\mathcal{E}(u, v) = \sum_{i=1,2,3} \frac{1}{2}\mathcal{E}(u \circ F_i, v \circ F_i).$$

4.3 Harmonic functions on $\mathbb{S}_{\mathbb{H}}$

In this section, we characterize the space of harmonic functions on $\mathbb{S}_{\mathbb{H}}$.

Definition 4.2. Let $C(\mathbb{S}_{\mathbb{H}})$ the set of real-valued functions on $\mathbb{S}_{\mathbb{H}}$. A function $f \in C(\mathbb{S}_{\mathbb{H}})$ is a *harmonic function* if $(H_m f)(p) = 0$ for every $m \geq 1$ and every $p \in V_m \setminus V_0$.

Corollary 4.3. *For $f \in \ell(V_1)$,*

$$Df_{|V_0} = (H_1 f)_{|V_0} + B(H_1 f)_{|V_1 \setminus V_0} \text{ with } B = {}^t J X^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}. \tag{4.3}$$

If f is a harmonic function on V_1, given the values of f on V_0, the values of f on V_1 are recovered by

$$f_{|V_1 \setminus V_0} = {}^t B f_{|V_0}.$$

Proof. For $f \in \ell(V_1)$, let $f_0 = f_{|V_0}, f_1 = f_{|V_1 \setminus V_0}$. Using (4.2),

$$(H_1 f)_{|V_1 \setminus V_0} = J f_0 + X f_1 \text{ and } (H_1 f)_{|V_0} = T f_0 + {}^t J f_1. \tag{4.4}$$

Therefore,

$$\begin{aligned} Df_{|V_0} &= T f_0 - {}^t J X^{-1} J f_0 \\ &= (T f_0 + {}^t J f_1) - {}^t J X^{-1} (J f_0 + X f_1) \\ &= (H_1 f)_{|V_0} + B(H_1 f)_{|V_1 \setminus V_0} \end{aligned}$$

with $B = -{}^t J X^{-1}$.

When f is harmonic on V_1, $(H_1 f)_{|V_1 \setminus V_0} = 0$. With (4.4) and since X is symmetric,

$$J f_0 + X f_1 = 0 \text{ so that } f_1 = -X^{-1} J f_0 = {}^t B f_0.$$

This concludes the proof. □

For any function u on V_0, there exists a unique $h \in \mathcal{F}(\mathbb{S}_{\mathbb{H}})$ such that $h|_{V_0} = u$ and $\mathcal{E}(h, h) = \min\{\mathcal{E}(v, v) : v \in \mathcal{F}(\mathbb{S}_{\mathbb{H}}) \text{ and } v|_{V_0} = u\}$ [15, Corollary 3.2.15]. The function h is called the *harmonic function* in $\mathbb{S}_{\mathbb{H}}$ with boundary value u, and denoted by $h = Hu$. For the harmonic function $h = Hu$ with boundary value u, we have

$$\mathcal{E}(h, h) = \mathcal{E}^{(0)}(u, u) = -{}^t u D u.$$

The values of h on V_* are given by

$$h \circ F_\omega = A_\omega u \text{ for all } \omega \in W_m, m \in \mathbb{N}, \tag{4.5}$$

where $A_\omega = A_{\omega_m} A_{\omega_{m-1}} \cdots A_{\omega_1}$. In particular, the values of h on $V_m \setminus V_0$ are obtained by the following formula:

$$\begin{bmatrix} h\left(F_\omega(q_{12})\right) \\ h\left(F_\omega(q_{13})\right) \\ h\left(F_\omega(q_{23})\right) \\ h\left(F_\omega(q_{32})\right) \end{bmatrix} = {}^tB \begin{bmatrix} h\left(F_\omega(p_1)\right) \\ h\left(F_\omega(p_2)\right) \\ h\left(F_\omega(p_3)\right) \end{bmatrix} \tag{4.6}$$

where B is given in (4.3).

The connection between the Dirichlet form $(\mathcal{E}, \mathcal{F}(\mathbb{S}_\mathbb{H}))$ on $\mathbb{S}_\mathbb{H}$ and the Dirichlet form $(\mathcal{E}^0, \mathcal{F}(\mathbb{S}))$ on $\mathbb{S}$ is given by the following result.

Proposition 4.4. *The* 3*-dimensional space* $\mathcal{H}(\mathbb{S}_\mathbb{H})$ *of harmonic functions in* $\mathbb{S}_\mathbb{H}$ *is the direct sum*

$$\mathcal{H}(\mathbb{S}_\mathbb{H}) = \mathcal{H}_0 \oplus \mathcal{H}_0^\perp \tag{4.7}$$

where orthogonality is w.r.t. $\mathcal{E}_1$ *and* $\mathcal{H}_0$ *is isomorphic to a* 2*-dimensional subspace of* $\mathcal{H}(\mathbb{S})$*.*

Proof. The functions $u_1 = \mathbb{1}_{\{p_1\}}$, $u_2 = \mathbb{1}_{\{p_2,p_3\}}$, $u_3 = \mathbb{1}_{\{p_2\}} - \mathbb{1}_{\{p_3\}}$ define a basis of $\ell(V_0)$ so that $\{h_i = Hu_i\}_{i=1,2,3}$ is a basis of $\mathcal{H}(\mathbb{S}_\mathbb{H})$. Observe that $h_i(q_{23}) = h_i(q_{32})$ and $h_i(p_{12}) = h_i(p_{13})$ for $i = 1, 2$. Let π be the projection $\pi : \mathbb{S}_\mathbb{H} \to \mathbb{S}$, we claim that for any $\omega \in W_*$, $h_i(F_\omega(q_{23})) = h_i(F_\omega(q_{32}))$ and $h_i(F_\omega(p_{12})) = h_i(_\omega(p_{13}))$ for $i = 1, 2$. $q \in V_* \subset \mathbb{S}_\mathbb{H}$ such that $\pi^{-1}(\pi(q)) = \{q, q'\}$ we have also $h_i(q) = h_i(q')$. Indeed, such points are of the form $q = F_\omega(q_{23})$ and $q' = F_\omega(q_{32})$ for a $\omega \in W_*$. By induction on the length $|\omega|$ of ω, suppose that the claim is true for $|\omega| \leq m$, $m \in \mathbb{N}$. By virtue of (4.5) we get

$$h_i(F_{j\omega}(q_{23})) = (A_{j\omega} u_i)|_{q_{23}} = (A_j h_i(F_\omega))|_{q_{23}}, \quad \text{for } i = 1, 2 \text{ and } j = 1, 2, 3.$$

The induction implies that for $k = 1, 2$, $h_i(F_{\omega k}(p_1)) = \alpha_i$ and $h_i(F_{\omega k}(p_2)) = h_i(F_{\omega k}(p_3)) = \beta_i$ simple calculation gives

$$h_i \circ F_{1\omega k}(p_2) = A_1 h_i(F_{\omega k})(p_2) = \alpha_i + \beta_i = A_1(h_i(F_{\omega k})(p_3) = h_i \circ F_{1\omega k}(p_3) \text{ and}$$
$$h_i \circ F_{2\omega}(q_{23}) = A_2 h_i(F_\omega)(q_{23}) = 2\beta_i = A_3 h_i(F_{3\omega})(q_{32}) = h_i \circ F_{1\omega}(q_{32}).$$

For the harmonic function h_3, proceeding in the same manner, it is easily seen by induction on the length of $\omega \in W_*$ that

$$h_3 \circ F_1(F_\omega(p_1) = 0 \text{ and } h_3 \circ F_1(F_\omega(p_2) = -h_3 \circ F_1(F_\omega(p_3) \text{ for any } \omega \in W_*,$$
$$h_3 \circ F_2(q) = -h_3 \circ F_3(q) \text{ for any } q \in V_*.$$

Thus concludes the proof. □

4.4 Laplacian on $\mathbb{S}_{\mathbb{H}}$

As asserted in the introduction, we construct a Laplacian on $\mathbb{S}_{\mathbb{H}}$ using a renormalization argument. In particular, we identify the renormalization constant α_p.

Let μ be the *normalized Hausdorff measure* on $\mathbb{S}_{\mathbb{H}}$, that is, μ is the unique Borel probability measure on $\mathbb{S}_{\mathbb{H}}$ such that $\mu(F_\omega(\mathbb{S}_{\mathbb{H}})) = 3^{-m}$ for all $\omega \in W_m$, $m \in \mathbb{N}$ [15, Definition 1.10 and Theorem 1.11]. The choice of $\frac{1}{3}$ is from [2, Remark after Definition 5.27].

Lemma 4.5. *The Dirichlet form $(\mathcal{E}, \mathcal{F}(\mathbb{S}_{\mathbb{H}}))$ is a regular on $L^2(\mathbb{S}_{\mathbb{H}}; \mu)$.*

Standard semigroup theory [6, 13] allows us to associate a non-positive self-ajoint operator Δ_μ to $(\mathcal{E}, \mathcal{F}(\mathbb{S}_{\mathbb{H}}))$. The operator Δ_μ can be obtained directly in the following way. Define

$$\Psi(x, y) = \sum_{p,q \in V_1 \setminus V_0} G_{pq} \mathbb{1}_{\{p\}}(x) \mathbb{1}_{\{q\}}(y) \text{ for } x, y \in \mathbb{S}_{\mathbb{H}},$$

where

$$G = -X^{-1} = \begin{bmatrix} \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} \end{bmatrix}.$$

The function Ψ takes non zero values on V_1. More precisely,

$$\Psi(x, y) = \begin{cases} \dfrac{1}{4} & \text{if } (x, y) \in (V_1 \setminus V_0)^2 \setminus \{(q_{23}, q_{23}); (q_{32}, q_{32})\}, \\ \dfrac{3}{4} & \text{if } (x, y) \in \{(q_{23}, q_{23}); (q_{32}, q_{32})\}. \end{cases}$$

Remark 4.3. For $\mathbb{S}$, G is a 3×3 matrix.

For any $\omega \in W_k$, $k \geq 0$,

$$\Psi_\omega(x, y) = \begin{cases} \Psi(F_\omega^{-1}(x), F_\omega^{-1}(y)) & \text{if } (x, y) \in F_\omega(\mathbb{S}_{\mathbb{H}}), \\ 0 & \text{otherwise.} \end{cases}$$

The function Ψ_ω is non-negative and continuous on $\mathbb{S}_{\mathbb{H}} \times \mathbb{S}_{\mathbb{H}}$.

For $m \geq 0$ and $p \in V_m$ define $\Psi_{m,p}$ to be the unique function on $\mathcal{F}$ that attains the following minimum

$$\min \left\{ \mathcal{E}(u, v), u \in \mathcal{F}, u_{|V_m} = \mathbb{1}_{\{p\}} \right\}.$$

Then $\Psi_{m,p} \circ F_\omega$ is a harmonic function on $\mathbb{S}_{\mathbb{H}}$. Consider $\int_{F_\omega(\mathbb{S}_{\mathbb{H}})} \psi_{m,p} d\mu$. By symmetry,

$$\int_{F_\omega(\mathbb{S}_{\mathbb{H}})} \psi_{m,F_\omega(A_1)}\, \mathrm{d}\mu = \int_{F_\omega(\mathbb{S}_{\mathbb{H}})} \psi_{m,F_\omega(A_2)}\, \mathrm{d}\mu = \int_{F_\omega(\mathbb{S}_{\mathbb{H}})} \psi_{m,F_\omega(A_3)}\, \mathrm{d}\mu.$$

As $\psi_{m,F_\omega(A_1)} + \psi_{m,F_\omega(A_2)} + \psi_{m,F_\omega(A_3)} \equiv 1$ on $\mathbb{S}_{\mathbb{H}}$,

$$\int_{F_\omega(\mathbb{S}_{\mathbb{H}})} \psi_{m,F_\omega(A_i)}\, \mathrm{d}\mu = \frac{1}{3} \int_{F_\omega(\mathbb{S}_{\mathbb{H}})} \mathrm{d}\mu = \left(\frac{1}{3}\right)^{m+1}.$$

Observe now that

$$\#\left\{\omega : \omega \in W_m, p \in F_\omega(\mathbb{S}_{\mathbb{H}})\right\} = \begin{cases} 2 & \text{if } p \in \left\{F_\omega(q_{12}), F_\omega(q_{13})\right\}, \\ 1 & \text{otherwise} \end{cases}$$

so that

$$\begin{aligned} \mu_{m,p} = \int_{\mathbb{S}_{\mathbb{H}}} \psi_{m,p}\, \mathrm{d}\mu &= \sum_{\omega \in W_m, p \in F_\omega(\mathbb{S}_{\mathbb{H}})} \int_{F_\omega(\mathbb{S}_{\mathbb{H}})} \psi_{m,p}\, \mathrm{d}\mu \\ &= \begin{cases} 2(\frac{1}{3})^{m+1} & \text{if } p \in \left\{F_\omega(q_{12}), F_\omega(q_{13})\right\}, \\ (\frac{1}{3})^{m+1} & \text{otherwise.} \end{cases} \end{aligned}$$

We now come to the definition of the Δ_μ associated to $(\mathcal{E}, \mathcal{F}, \mu)$. Let D_μ the set of functions $u \in C(\mathbb{S}_{\mathbb{H}})$ such that there exists $f \in C(\mathbb{S}_{\mathbb{H}})$ for which

$$\lim_{m\to\infty} \max_{p \in V_m \setminus V_0} \left| \mu_{m,p}^{-1} (H_m u)(p) - f(p) \right| = 0.$$

Then we set $\Delta_\mu u = f$ and D_μ is the domain of Δ_μ.

We also define the Neumann derivative at a point $p \in V_0$ by

$$(\,\mathrm{d}f)_p = \lim_{m\to\infty} -(H_m f)(p),\ \forall f \in D_\mu.$$

Its existence is ensured by [15, Lemma 6.3].

Theorem 3.11 in [16] relates the Laplacian (Δ_μ, D_μ) with the Dirichlet form $(\mathcal{E}, \mathcal{F})$.

Proposition 4.6 (Gauss-Green's formula)**.** *It holds that $D_\mu \subset \mathcal{F}$. Besides, for $u \in \mathcal{F}$ and $v \in D_\mu$,*

$$\mathcal{E}(u, v) = \sum_{p \in V_0} u(p)(\,\mathrm{d}v)_p - \int_{\mathbb{S}_{\mathbb{H}}} u \Delta_\mu \,\mathrm{d}\mu.$$

By [16, Theorem 3.12], we get the following proposition for Green's function $g(\cdot,\cdot)$ associated to $(\mathcal{E},\mathcal{F},\mu)$.

Proposition 4.7. *There exists a non-negative continuous function $g : \mathbb{S}_{\mathbb{H}} \times \mathbb{S}_{\mathbb{H}} \to \mathbb{R}$ with $g(x,y) = g(y,x)$ for all $x,y \in \mathbb{S}_{\mathbb{H}}$ that satisfies $\mathcal{E}(g^x, f) = f(x)$ for all $f \in \mathcal{F}$ with $f|_{V_0} = 0$, where $g^x(y) = g(x,y)$. Also for given $\phi \in C(\mathbb{S}_{\mathbb{H}})$, there exists a unique $f \in \mathcal{D}_\mu$ which satisfies*

$$\begin{cases} \Delta_\mu f = \phi, \\ f_{|V_0} = 0. \end{cases}$$

Furthermore, f is given by $f(x) = -\int_{\mathbb{S}_{\mathbb{H}}} g(x,y)\phi(y)\,\mathrm{d}\mu(y)$.

For $m \geq 1$ define $g_m(x,y) = \sum_{k=0}^{m-1} \sum_{\omega \in W_k} r_\omega \Psi_\omega(x,y)$. Let $H_m = \sum_{\omega \in W_m} \frac{1}{r_\omega} {}^tR_\omega D R_\omega$ and write its matrix representation in the form $H_m = \begin{bmatrix} T_m & {}^tJ_m \\ J_m & X_m \end{bmatrix}$. Then X_m is invertible by [17, Lemma 3.5.1]. Besides, if we write $G_m = -X_m^{-1}$ then

$$g_m(x,y) = \sum_{p,q \in V_m \setminus V_0} r_\omega (G_m)_{pq} \Psi_p^m(x) \Psi_q^m(y).$$

Note that $g_m(p,q) = (G_m)_{pq}$ for $p,q \in V_m$. The explicit form of g is given the formula

$$g(x,y) = \lim_{m\to\infty} g_m(x,y) = \sum_{\omega \in W_*} r_\omega \Psi_\omega(x,y) \text{ for } x,y \in \mathbb{S}_{\mathbb{H}}.$$

5 Diffusion on $\mathbb{S}_{\mathbb{H}}$

The theory of Dirichlet forms and Markov processes [10, Chapter 7] associate a diffusion process to $(\mathcal{E},\mathcal{F})$.

Proposition 5.1. *There exists a standard Hunt process $(\Omega, \mathcal{F}, (X_t)_{t\geq 0}, (\theta_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in \mathbb{S}_{\mathbb{H}} \cup \{\dagger\}})$ with state space $\mathbb{S}_{\mathbb{H}}$, where $\dagger$ is an isolated point adjoined to $\mathbb{S}_{\mathbb{H}}$, and $\theta_t : \Omega \to \Omega$, $t \geq 0$ are the shift operators.*

The Hunt process X is a diffusion on $\mathbb{S}_{\mathbb{H}}$, called (the reflected) *Brownian motion* on $\mathbb{S}_{\mathbb{H}}$. The associated Markov semigroup will be denoted by P_t. Let $\mathcal{P}(\mathbb{S}_{\mathbb{H}})$ be the family of all Borel probability measures on $\mathbb{S}_{\mathbb{H}}$. For each $\lambda \in \mathcal{P}(\mathbb{S}_{\mathbb{H}})$, the probability measure $\mathbb{P}_\lambda$ on Ω is defined by

$$\mathbb{P}_\lambda(E) = \int \mathbb{P}_x(E)\lambda(dx), \ \forall E \in \mathcal{F}.$$

The expectation with respect to $\mathbb{P}_\lambda$ is denoted by $\mathbb{E}_\lambda$. We denote by $\{\mathcal{F}_t\}_{t\geq 0}$ the minimal admissible filtration determined by $\{X_t\}_{t\geq 0}$, that is, $\mathcal{F}_t = \cap_{\lambda\in\mathcal{F}(\mathbb{S}_{\mathbb{H}})}\mathcal{F}_t^\lambda$, $t \geq 0$, where $\mathcal{F}_t^\lambda$ is the $\mathbb{P}_\lambda$-completion of $\sigma(X_r : r \leq t)$ in $\mathcal{F}$.

Let $\mathcal{F}(\mathbb{S}\setminus V_0) = \{u \in \mathcal{F}(\mathbb{S}_{\mathbb{H}}) : u|_{V_0} = 0\}$. The restriction of $\mathcal{E}$ on $\mathcal{F}(\mathbb{S}_{\mathbb{H}}\setminus V_0)$ is also a Dirichlet form with $\mathcal{F}(\mathbb{S}_{\mathbb{H}}\setminus V_0)$ as its Dirichlet space. Dirichlet spaces $\mathcal{F}(\mathbb{S}_{\mathbb{H}})$ and $\mathcal{F}(\mathbb{S}_{\mathbb{H}}\setminus V_0)$ correspond to the Neumann boundary conditions and the Dirichlet boundary conditions respectively [15, Theorem 3.7.9]. Let σ_{V_0} be the hitting time $\sigma_{V_0} = \inf\{t > 0 : X_t \in V_0\}$.

We define the *killed Brownian motion* $\{X_t^0\}$ by killing $\{X_t\}$ on hitting V_0, that is, $X_t^0 = X_t$ if $t < \sigma_{V_0}$, and $X_t^0 = \dagger$ if $t \geq \sigma_{V_0}$ where $\dagger$ is a cemetery point added to $\mathbb{S}_{\mathbb{H}}$. Then $\{X_t^0\}$ is a μ-symmetric Hunt process on $\mathbb{S}_{\mathbb{H}}\setminus V_0$ with $(\mathcal{E}, \mathcal{F}(\mathbb{S}_{\mathbb{H}}\setminus V_0))$ as its associated Dirichlet form, and the associated semigroup, denoted by $\{P_t^0\}$, is given by $P_t^0(x, E) = \mathcal{P}_x(X_t \in E, t < \sigma_{V_0})$ for all $x \in \mathbb{S}_{\mathbb{H}}\setminus V_0$, $E \in B(\mathbb{S}_{\mathbb{H}}\setminus V_0)$.

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