Zonotopic state estimation and fault detection for systems with time-invariant uncertainties

Zhenhua Wang, Wentao Tang, Qinghua Zhang, Vicenc Puig, Yi Shen

Zonotopic state estimation and fault detection for systems with time-invariant uncertainties ⋆

Zhenhua Wang * **Wentao Tang** * **Qinghua Zhang** ** **Vicenç Puig** *** **Yi Shen** *

* *School of Astronautics, Harbin Institute of Technology, Harbin, 150001 P. R. China (e-mail: zhenhua.wang@hit.edu.cn).*
** *Inria/IFSTTAR, Campus de Beaulieu, 35042, Rennes Cedex, France. (e-mail: qinghua.zhang@inria.fr)*
*** *Institut de Robòtica i Informàtica Industrial, CSIC-UPC, Universitat Politècnica de Catalunya-BarcelonaTech (UPC), C/. Llorens i Artigas 4-6, 08028, Barcelona, Spain (e-mail: vpuig@iri.upc.edu)*

Abstract: This paper proposes a robust guaranteed state estimation method with application to fault detection by combining H_∞ observer design with zonotopic analysis for discrete-time systems with both time-variant and time-invariant uncertainties. In order to improve the estimation accuracy, based on the H_∞ technique, the observer design is achieved by solving a linear matrix inequality. The main contribution of this method lies in that the time invariance of some uncertainties is considered to reduce the conservatism of interval estimation. Finally, a numerical example is provided to demonstrate the effectiveness of the proposed method.

Keywords: Set-membership, time-invariant uncertainty, zonotopes, H_∞ design.

1. INTRODUCTION

State estimation is widely studied and practiced in control theory and applications. Due to ubiquitous uncertainties in system models and measurements, state estimation does not converge to true states. To handle uncertainties, many methods have been proposed, such as the Kalman filter addressing stochastic noises (Kalman, 1963), the Kalman filter incorporating unknown inputs (Keller and Darouach, 1999) and H_∞ observer designs (Wang, Huang, and Unbehauen, 2001; Darouach, Boutat-Baddas, and Zerrougui, 1999). Recently, state estimation with guaranteed error bounds, based on set-membership computations, has received a lot of attention, since error bounds are useful in some applications, such as fault diagnosis (Xu, Puig, Ocampo-Martinez, Stoican, and Olaru, 2014; Xu, Puig, Ocampo-Martinez, Olaru, and Stoican, 2015) and reachability analysis (Althoff, Stursberg, and Buss, 2010). There exist several set-membership methods in the literature, e.g. the zonotope-based method (Puig, Cugueró, and Quevedo, 2001; Combastel, 2003; Alamo, Bravo, and Camacho, 2005), ellipsoid-based method (Maksarov and Norton, 1996; Kurzhanski and Varaiya, 2000; Chernousko, 2005) and interval observer(Bernard and Gouzé, 2004; Efimov, Perruquetti, Raïssi, and Zolghadri, 2004). Among these existing set-membership methods, the zonotope-based method has recently received much attention, and some criteria such as P-radius (Le, Stoica, Alamo, Camacho, and D, 2013) and F-radius (Combastel, 2015) have been proposed to improve the estimation accuracy. Although the zonotope-based method is effective in set-membership estimation, like the other aforementioned methods, it has a certain conservatism due to the wrapping effect. More accurate estimation can be obtained by using higher dimensional zonotopes, but at the cost of increasing computation burden. Therefore, it remains an important issue to establish a trade-off between the estimation accuracy and the computational burden in set-membership methods.

This paper addresses the particular case where the considered systems are subject to *time-invariant* uncertainties, in addition to time varying uncertainties as usually considered in set-membership based state estimators. Typical examples of time-invariant uncertainties are biases in a system model, such as actuator bias and sensor bias. Most existing methods based on set-membership computations *do not* consider time-invariant uncertainties. Of course, it is possible to treat *time-invariant* uncertainties as *time varying* uncertainties in order to apply existing methods, but ignoring the time-invariant nature of the uncertainties increases the conservatism in set-membership computations.

Few studies have been reported on state estimation for systems subject to both time-invariant and time varying uncertainties. In Combastel (2011), discrete-time linear systems with both time-varying bounded inputs and time-invariant bounded parametric uncertainties are considered

⋆ This work was supported by National Natural Science Foundation of China (Grant No. 61403104, 61773145), State Key Laboratory of Robotics (Grant No. 2014-O04), and the Fundamental Research Funds for the Central Universities (HIT.KLOF.2015.076).

by using the so-called parameterized families of zonotopes. However, the proposed method in Combastel (2011) can only obtain the out-approximation of the reachable state set at a finite given time. Pourasghar et al. (2017) considers a large class of linear dynamic systems subject to time-invariant uncertainties. The proposed solution in this reported work requires a monotonicity property of the designed interval observer. But the conditions for achieving such a design are not always easy to satisfy.

In this paper, we propose a guaranteed state estimation method for discrete-time state-space systems with both time-varying and time-invariant uncertainties. The proposed method consists of two steps: First, an optimized Luenbeger observer is designed for a single state trajectory estimation, then its state estimation error is bounded by set-membership computations based on zonotopes. The Luenbeger observer and the zonotope based error bound computation then constitute a set-membership estimation estimator. For the class of considered systems, sufficient conditions for achieving the design of such an estimator are explicitly formulated. Next, the proposed method is applied to fault detection for systems with bounded disturbances and time-invariant uncertainties. *The main contribution* of this paper is an algorithm for error bound computation taking into account time invariant uncertainties. By incorporating the information that some uncertainties are time invariant, the conservatism of zonotopic state estimation is significantly reduced.

The structure of the rest part of the paper is as follows: In Section 2, background on zonotopic analysis is introduced. Section 3 presents the problem formulation. The robust zonotopic estimation is presented in Section 4. Section 5 introduces the application of the proposed method in fault detection. Simulation results are provided in Section 6 to demonstrate the effectiveness of the proposed method. Finally, Section 7 presents the conclusion of this paper.

2. PRELIMINARIES

Definition 1. An s-order zonotope $\mathcal{Z}$ is the affine transformation of a hypercube $\mathbb{B}^s = [-1, +1]^s$, i.e.

$$\mathcal{Z} = p \oplus H\mathbb{B}^s = \{p + Hz, z \in \mathbb{B}^s\}$$

where $p \in \mathbb{R}^n$ is the center of $\mathcal{Z}$ and $H \in \mathbb{R}^{n\times s}$, which defines the shape of $\mathcal{Z}$, is called the generator matrix of $\mathcal{Z}$. Herein and throughout this paper, we use $\oplus$ to denote the Minkowski sum.

For the sake of simplicity and readability, we also use $\mathcal{Z} = \langle p,\, H\rangle$ to denote a zonotope.

In the sequel, the following properties of zonotopes will be used:

$$\langle p_1, H_1\rangle \oplus \langle p_2, H_2\rangle = \langle p_1 + p_2, [H_1\ \ H_2]\rangle \tag{1a}$$

$$L \odot \langle p, H\rangle = \langle Lp, LH\rangle \tag{1b}$$

$$\langle p, H\rangle \subseteq \langle p, \bar{H}\rangle \tag{1c}$$

where p, p_1, $p_2 \in \mathbb{R}^n$ and H, H_1, $H_2 \in \mathbb{R}^{n\times s}$ denote the centers and generator matrices of zonotopes, respectively, $L \in \mathbb{R}^{l\times n}$ is a matrix with appropriate dimensions, $\odot$ denotes linear map, and $\bar{H} \in \mathbb{R}^{n\times n}$ is a diagonal matrix as follows:

$$\bar{H} = \begin{bmatrix} \sum_{j=1}^{s} |H_{1,j}| & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sum_{j=1}^{s} |H_{n,j}| \end{bmatrix},$$

and $\langle p, \bar{H}\rangle$ is also a box.

In the applications of zonotopes, the reduction operator will be used to bound a high dimensional zonotope by a lower one, which can be described by the following property(Alamo et al., 2005) and is originally proposed by Combastel (2003).

Property 1. Given the zonotope $\mathcal{Z} = p \oplus H\mathbb{B}^s \subset \mathbb{R}^n$, reorder the columns of matrix H in decreasing Euclidean norm and denote the obtained matrix as $\hat{H}$. Then $\mathcal{Z} \subseteq p \oplus \mathscr{R}_q(H)\mathbb{B}^q$ $(n \le q \le s)$, where $\mathscr{R}_q(\cdot)$ is the reduction operator that can reduce the order of $\mathcal{Z}$ from s to q that is a chosen reduced dimension. And $\mathscr{R}_q(H) = [H_T\ Q]$, where H_T consists of the first $q - n$ columns of $\hat{H}$ and Q is a diagonal matrix with

$$Q_{i,i} = \sum_{j=q-n+1}^{s} |\hat{H}_{i,j}|, \quad i = 1, \ldots, n$$

For a symmetric matrix $P \in \mathbb{R}^{n\times n}$, $P \succ 0 (P \prec 0)$ denotes that P is positive(negative) definite. In the following, the comparison operator $\le$ and the absolute value operator $|\cdot|$ are understood elementwise. In a symmetric block matrix, we use $*$ to represent a term that can be induced by symmetry.

3. PROBLEM FORMULATION

In this paper, we consider the following discrete-time system

$$\begin{cases} x_{k+1} = Ax_k + Bu_k + G\theta + Ew_k \\ y_k = Cx_k + Fv_k \end{cases} \tag{2}$$

where $x_k \in \mathbb{R}^{n_x}$, $u_k \in \mathbb{R}^{n_u}$, and $y_k \in \mathbb{R}^{n_y}$ denote state, control input, measurement output vectors, respectively, $w_k \in \mathbb{R}^{n_w}$ is the process disturbance, $v_k \in \mathbb{R}^{n_v}$ denotes the measurement noise, and $\theta \in \mathbb{R}^{n_\theta}$ is a *time-invariant* disturbance, namely, an unknown but constant vector. A, B, C, G, E and F are known matrices with appropriate dimensions. θ can be used to represent the actuator bias when $G = B$.

We assume that x_0, w_k, v_k and θ are unknown but bounded, i.e.

$$|x_0 - p_0| \le \overline{x}, \quad |w_k| \le \overline{w}, \quad |v_k| \le \overline{v}, \quad |\theta| \le \overline{\theta} \tag{3}$$

where $p_0 \in \mathbb{R}^{n_x}$, $\overline{x} \in \mathbb{R}^{n_x}$, $\overline{w} \in \mathbb{R}^{n_w}$, $\overline{v} \in \mathbb{R}^{n_v}$ and $\overline{\theta} \in \mathbb{R}^{n_\theta}$ are known vectors. According to the definition of zonotopes, (3) can be reformulated as

$$x_0 \in \langle p_0,\, H_0\rangle,\ w_k \in \langle 0,\, W\rangle,\ v_k \in \langle 0,\, V\rangle,\ \theta \in \langle 0,\, \Theta\rangle \tag{4}$$

where $H_0 = \mathrm{diag}(\overline{x})$, $W = \mathrm{diag}(\overline{w})$, $V = \mathrm{diag}(\overline{v})$ and $\Theta = \mathrm{diag}(\overline{\theta})$.

The aim of this paper is to design a robust guaranteed state estimator for system (2). In this paper, we integrate robust observer design with zonotopic analysis to achieve guaranteed state estimation. The estimator to be designed is based on the observer for system (2) with the following form:

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + L(y_k - C\hat{x}_k) \tag{5}$$

where $\hat{x}_k \in \mathbb{R}^{n_x}$ is the state estimation and $L \in \mathbb{R}^{n_x \times n_y}$ is the gain matrix to be designed. In order to increase the estimation accuracy, we first use H_∞ technique to design L such that the estimation error is robust to uncertainties. Then, based on the obtained observer, guaranteed state estimation is achieved via zonotopic analysis. Moreover, the proposed method is applied to fault detection for systems with bounded disturbances and time-invariant uncertainties.

4. ROBUST ZONOTOPIC STATE ESTIMATION

In this section, we aim to obtain robust state estimation described by zonotope sets, which are consistent with the initial state set, measurements and uncertainties.

4.1 Robust observer design based on H_∞ technique

In order to attenuate the effects of uncertainties and thus to increase the estimation accuracy, we use the well-known H_∞ technique to design the observer gain L for observer (5). To this end, we first analyze the error dynamics of observer (5).

Define the estimation error as $e_k = x_k - \hat{x}_k$. By subtracting (5) from (2), we obtain the following error dynamics:

$$e_{k+1} = (A - LC)e_k + G\theta + Ew_k - LFv_k \tag{6}$$

that can be rewritten as

$$e_{k+1} = (A - LC)e_k + Dd_k \tag{7}$$

where

$$D = [G \ E \ -LF]\,,\ d_k = \begin{bmatrix} \theta \\ w_k \\ v_k \end{bmatrix}.$$

We denote the transfer function from d_k to e_k as $G_{ed}(z)$. The H_∞ norm of $G_{ed}(z)$, i.e. its maximum singular value, is denoted as $\|G_{ed}(z)\|_\infty$.

Based on (7), we propose the following theorem to design a robust observer, which is based on H_∞ technique.

Theorem 1. Given a scalar $\gamma > 0$, the error system (7) is stable and satisfies the following performance index

$$\|G_{ed}(z)\|_\infty < \gamma \tag{8}$$

if there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n_x \times n_x}$ ($P \succ 0$) and a matrix $W \in \mathbb{R}^{n_x \times n_y}$ such that

$$\begin{bmatrix} -P + I & \star & \star & \star & \star \\ 0 & -\gamma^2 I & \star & \star & \star \\ 0 & 0 & -\gamma^2 I & \star & \star \\ 0 & 0 & 0 & -\gamma^2 I & \star \\ PA - WC & PG & PE & -WF & -P \end{bmatrix} \prec 0, \tag{9}$$

Moreover, if the LMI in (9) is solvable, the gain matrix L can be determined by $L = P^{-1}W$.

Proof. Applying the well-known bounded real lemma to error dynamics (7), we know that (7) is stable and satisfies the H_∞ performance index in (8), if and only if there exists a symmetric positive definite matrix $P \succ 0$ and a matrix L such that

$$\begin{bmatrix} (A - LC)^T P (A - LC) - P + I & \star \\ D^T P (A - LC) & D^T P D - \gamma^2 I \end{bmatrix} \prec 0. \tag{10}$$

Note that (10) is not a standard LMI, we use Schur complement lemma to convert (10) as

$$\begin{bmatrix} -P + I & \star & \star \\ 0 & -\gamma^2 I & \star \\ P(A - LC) & PD & -P \end{bmatrix} \prec 0. \tag{11}$$

By substituting $D = [G \ E \ -LF]$ into (11), we obtain

$$\begin{bmatrix} -P + I & \star & \star & \star & \star \\ 0 & -\gamma^2 I & \star & \star & \star \\ 0 & 0 & -\gamma^2 I & \star & \star \\ 0 & 0 & 0 & -\gamma^2 I & \star \\ P(A - LC) & PG & PE & -PLF & -P \end{bmatrix} \prec 0. \tag{12}$$

Letting $W = PL$, (12) becomes (9). □

Remark 1. To optimize the estimation accuracy, the minimal γ can be obtained by solving the following optimization problem:

$$\begin{aligned} \min \quad & \gamma^2, \\ s.t. \quad & (9). \end{aligned} \tag{13}$$

and the corresponding observer gain can be obtained by $L = P^{-1}W$.

4.2 Guaranteed state estimation based on zonotopes

Based on the observer (5) designed via H_∞ technique, we can obtain guaranteed state estimation described by zonotopes sets, which can enclose all the admissible values of state. We note that the Zonotopic Kalman Filter(ZKF) (Combastel, 2015) is based on a similar observer structure with that of (5). The ZKF for system (2) can be described by the following lemma.

Lemma 1. For system (2), if $x_k \in \langle c_k, H_k \rangle$, then $x_{k+1} \in \langle c_{k+1}, H_{k+1} \rangle$ with

$$\begin{aligned} c_{k+1} &= Ac_k + Bu_k + L_k(y_k - Cc_k) \\ H_{k+1} &= \left[(A - LC)\bar{H}_k,\ G,\ E,\ -LF \right], \quad \bar{H}_k = \mathscr{R}_q(H_k). \end{aligned} \tag{14}$$

Moreover, L_k can be obtained by

$$L_k = A\bar{H}_k \bar{H}_k^T C^T (C\bar{H}_k \bar{H}_k^T C^T + FF^T)^{-1} \tag{15}$$

Proof. Lemma 1 can be easily proved by applying Proposition 1 and Theorem 5 in Combastel (2015) to system (2). Hence, readers can refer to this reference for more details about the proof. □

We note that the time invariance of some uncertainties is not considered in (14), which will introduce conservatism. Moreover, the observer gain L_k is computed by (15) on line, which may cause heavy computational burden since the dimensions of $\bar{H}_k$ may be considerably large. In order to reduce conservatism, we consider the time invariance of θ via the reachability analysis of the estimation error and then obtain more accurate guaranteed state estimation than ZKF. To this end, we first introduce the following useful lemma.

Lemma 2. For a discrete-time linear system as follows

$$x_{k+1} = Ax_k + Ew_k \tag{16}$$

where $x_0 \in \mathcal{X}_0 = \langle p_0,\, H_0\rangle$ and $w_k \in \mathcal{W} = \langle 0,\, W\rangle$, the state x_k belongs to zonotope $\hat{\mathcal{X}}_k = \langle p_k,\, \hat{H}_k\rangle$ with

$$p_{k+1} = Ap_k, \tag{17}$$

$$\hat{H}_{k+1} = \left[A\mathscr{R}_q(\hat{H}_k)\ \ EW\right] \tag{18}$$

Proof. Assuming $x_k \in \mathcal{X}_k$, based on (16), $\mathcal{X}_{k+1}$ is updated as follows:

$$\mathcal{X}_{k+1} = A \odot \mathcal{X}_k \oplus E \odot \mathcal{W} \tag{19}$$

Denote $\mathcal{X}_k = \langle p_k,\, H_k\rangle$. Using the properties of zonotope in (1), (19) can be rewritten as:

$$p_{k+1} = Ap_k,$$
$$H_{k+1} = \left[AH_k\ \ EW\right].$$

According to Property 1, we have $\langle p_k,\, H_k\rangle \subseteq \langle p_k,\, \mathscr{R}_q(\hat{H}_k)\rangle$ and it follows that $\langle p_{k+1},\, H_{k+1}\rangle \subseteq \langle p_{k+1},\, \hat{H}_{k+1}\rangle$. Therefore, we have $\mathcal{X}_k \subseteq \hat{\mathcal{X}}_k$. $\square$

Based on Lemma 2, we propose the following theorem to achieve guaranteed state estimation for system (2), where the time invariance of θ is considered.

Theorem 2. For system (2), under the condition (4), x_k can be bounded by a zonotope $\hat{\mathcal{X}}_k$ as follows:

$$\hat{\mathcal{X}}_k = \langle \hat{x}_k,\, \hat{H}_k\rangle \tag{20}$$

where $\hat{x}_k$ is obtained by (5) with $\hat{x}_0 = p_0$ and $\hat{H}_k$ has the following form:

$$\hat{H}_k = \left[\bar{H}_k\ \ \tilde{H}_k\right]. \tag{21}$$

with $\bar{H}_k$ and $\tilde{H}_k$ obtained from

$$\bar{H}_k = \sum_{i=0}^{k-1}(A-LC)^i G\Theta \tag{22}$$

and

$$\tilde{H}_{k+1} = \left[(A-LC)\mathscr{R}_q(\tilde{H}_k)\ \ EW\ \ -LFV\right] \tag{23}$$

and $\tilde{H}_0 = H_0$.

Proof. Since $e_k = x_k - \hat{x}_k$, it follows that

$$x_k = \hat{x}_k + e_k$$

Since $\hat{x}_k$ can be obtained from (5), the guaranteed state estimation can be achieved via estimating the reachable set of e_k. In the following, we first obtain the guaranteed estimation of e_k, and then give that of x_k.

Motivated by Wang et al. (2017), we split the error dynamics in (6) into two subsystems as follows:

$$\bar{e}_{k+1} = (A-LC)\bar{e}_k + G\theta \tag{24}$$

and

$$\tilde{e}_{k+1} = (A-LC)\tilde{e}_k + Ew_k - LFv_k \tag{25}$$

where $e_k = \bar{e}_k + \tilde{e}_k$ with $\bar{e}_0 = 0$ and $\tilde{e}_0 = e_0$. It is obvious that $\bar{e}_k$ is only affected by θ and $\tilde{e}_k$ is driven by w_k and v_k.

By using (24) iteratively, we obtain

$$\bar{e}_k = \sum_{i=0}^{k-1}(A-LC)^i G\theta \tag{26}$$

Note that $\theta \in \langle 0,\, \Theta\rangle$, (26) follows that

$$\bar{e}_k \in \bar{\mathcal{E}}_k = \langle 0,\, \bar{H}_k\rangle \tag{27}$$

where $\bar{H}_k$ is given by (22).

On the other hand, applying Lemma 2 to (25) yields

$$\tilde{e}_k \in \tilde{\mathcal{E}}_k = \langle 0,\, \tilde{H}_k\rangle \tag{28}$$

where $\tilde{H}_k$ is given by (23).

Since $e_k = \bar{e}_k + \tilde{e}_k$, we obtain

$$e_k \in \hat{\mathcal{E}}_k = \bar{\mathcal{E}}_k \oplus \tilde{\mathcal{E}}_k = \langle 0,\, \hat{H}_k\rangle \tag{29}$$

where $\hat{H}_k = \left[\bar{H}_k\ \ \tilde{H}_k\right]$.

Finally, according to $x_k = \hat{x}_k + e_k$, we have

$$x_k \in \hat{\mathcal{X}}_k = \hat{x}_k \oplus \hat{\mathcal{E}}_k = \langle \hat{x}_k,\, \hat{H}_k\rangle, \tag{30}$$

which ends the proof. $\square$

Remark 2. From (1c), we know that a zonotope can be bounded by a box. Then, we can obtain the interval estimation of the state including the upper bound $\overline{x}_k$ and lower bound $\underline{x}_k$, which can be obtained from

$$\begin{cases} \overline{x}_k(i) = \hat{x}_k(i) + \sum_{j=1}^{s}|\hat{H}_k(i,j)|, & i = 1,\ldots,n \\ \underline{x}_k(i) = \hat{x}_k(i) - \sum_{j=1}^{s}|\hat{H}_k(i,j)|, & i = 1,\ldots,n \end{cases} \tag{31}$$

where s is the column number of $\hat{H}_k$.

5. ZONOTOPIC FAULT DETECTION

When system (2) is affected by additive actuator faults, it can be formulated as follows

$$\begin{cases} x_{k+1} = Ax_k + Bu_k + G\theta + Ew_k + Bf_k \\ y_k = Cx_k + Fv_k \end{cases} \tag{32}$$

where $f_k \in \mathbb{R}^{n_u}$ represents the actuator fault vector.

The state estimations obtained by ZKF and the proposed method are both based on (2) without consideration of actuator fault.

When no fault occurs, the state estimation based on zonotopes can guarantee to enclose the admissible state values, i.e. $x_k \in \hat{\mathcal{X}}_k$. Since $y_k = Cx_k + Fv_k$, it follows that

$$y_k \in C\hat{\mathcal{X}}_k \oplus F\langle 0,\, V\rangle \tag{33}$$

According to the properties of zonotopes, (1), we have

$$y_k \in \hat{\mathcal{Y}}_k = \langle p_k^y,\, H_k^y\rangle \tag{34}$$

where

$$\begin{cases} p_k^y = Cp_k \\ H_k^y = \left[C\hat{H}_k\ \ F\right] \end{cases} \tag{35}$$

However, when a fault occurs and the magnitude of it is big enough to be detected, y_k will fall outside $\hat{\mathcal{Y}}_k$. Therefore, we propose a zonotopic fault detection strategy:

If $y_k \notin \hat{\mathcal{Y}}_k$, it can be inferred that a fault has occurred.

An interval estimation enclosing $\hat{\mathcal{Y}}_k$ can be obtained similar to (31) for the simplicity of fault detection.

Remark 3. The zonotopic fault detection can also be achieved based on the state estimation obtained by ZK-F. However, the fault detection based on the proposed method can achieve higher fault detection rate (FDR) than that based on zonotopes, since the time-invariance of θ is considered in the proposed method.

6. SIMULATIONS

6.1 State estimation results

In this section, an example of the decoupled linearized longitudinal motion dynamics of the F-18 aircraft adapted from Yang and Ye (2006) is used to demonstrate the effectiveness of the proposed method. The state space equation of the consider system is as follows

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} = A_{long}^{m7h14} \begin{bmatrix} \alpha \\ q \end{bmatrix} + B_{long}^{m7h14} \begin{bmatrix} \delta_E \\ \delta_{PTV} \end{bmatrix} + w, \tag{36}$$

where α is the angle of attack, q is the pitch rate, δ_E si the symmetric elevator position, δ_{PTV} is the symmetric pitch thrust velocity nozzle position and w is the external disturbance. A_{long}^{m7h14} and B_{long}^{m7h14} denote the system matrices at Mach 0.7 and $14-kft$ altitude. q is assumed to be measurable. By using the Euler method to discretize the continuous-time model (36) with sample time $T_s = 0.1s$, we obtain a discrete-time system in the form of (2) with

$$A = \begin{bmatrix} 0.8825 & 0.0987 \\ -0.8458 & 0.9122 \end{bmatrix}, \quad B = \begin{bmatrix} -0.0194 & -0.0036 \\ -1.9290 & -0.3803 \end{bmatrix},$$
$$E = 0.01I_2, \quad C = [0\ \ 1], \quad F = 0.01.$$

The disturbance w_k and measurement noise v_k are bounded by

$$|w_k| \le [1\ \ 1]^T, \quad |v_k| \le 1.$$

The system suffers an unknown constant actuator bias, which can be described by $G = B$ and $\theta \in \langle 0,\, 0.03I_3 \rangle$.

The initial sate is $x_0 = [0.2\ \ -0.2]^T$. The uncertainties are set as

$$p_0 = [0\ \ 0]^T, \quad H_0 = 0.5I_2, \quad W = I_2, \quad V = 1$$

and let $\hat{x}_0 = p_0$.

By solving the optimization problem (13), we obtain the observer gain as

$$L = \begin{bmatrix} -0.0051 \\ 1.0117 \end{bmatrix}$$

with the minimal $\gamma = 0.0646$.

In the simulation study, the proposed method is compared with ZKF. The simulation results are shown in Figure $1-2$. They show that the guaranteed state estimations obtained by the proposed method are more accurate than those by ZKF, which is due to that the proposed method has considered the time-invariance of θ.

6.2 Fault detection results

In order to demonstrate the effectiveness of the zonotopic fault detection based on the proposed method, we consider a case where an actuator fault occurs, which can be simulated as follows

$$f_k = \begin{cases} [0,\, 0]^T, & k < 50; \\ [0.15,\, 0]^T, & k \ge 50. \end{cases}$$

The simulation results are shown in Figure 3. It shows that y_k quickly exceeds the interval estimation obtained based on the proposed method after the occurrence of fault, which indicates that a fault has been detected. On the other hand, it also shows that the interval estimations based on ZKF has lower fault detection rate because of the extra conservatism introduced by neglecting the time invariant nature of the uncertainties.

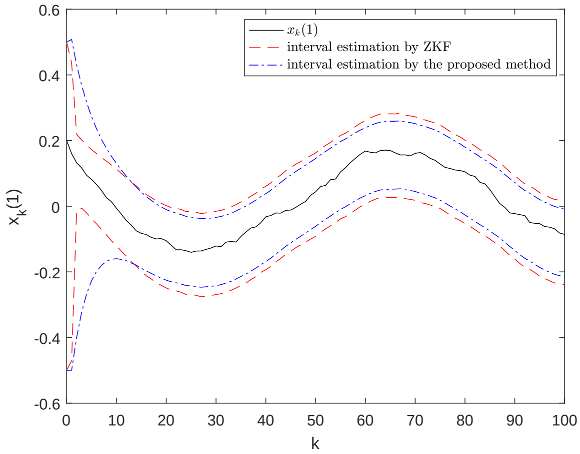

Fig. 1. $x_k(1)$ and its interval estimations by ZKF and the proposed method.

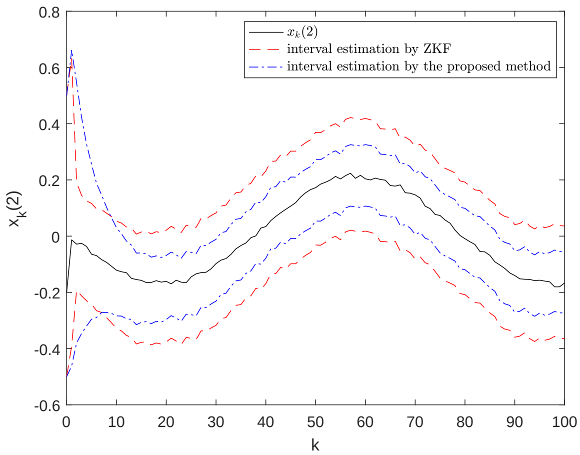

Fig. 2. $x_k(2)$ and its interval estimations by ZKF and the proposed method.

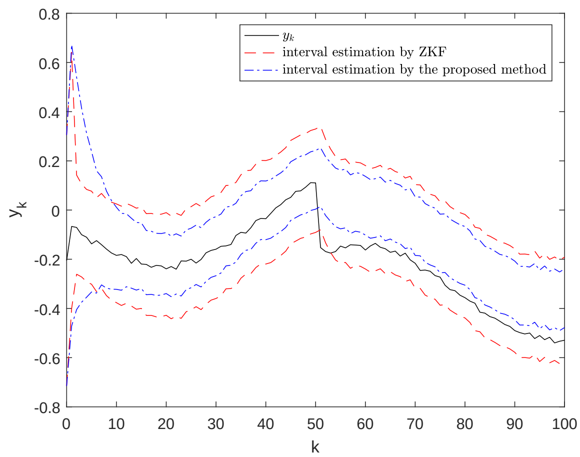

Fig. 3. y_k and its interval estimations by ZKF and the proposed method.

7. CONCLUSION

In this paper, a novel guaranteed state estimation method is proposed for the discrete-time linear systems affected by both disturbances and time-invariant uncertainties. In order to increase the estimation accuracy, the proposed method uses the H_∞ technique to design robust observer. Based on the designed observer, guaranteed state estimation is obtained via zonotopic analysis. The proposed method has considered the time dependency of uncertainties to reduce conservatism. Moreover, the proposed method is applied to actuator fault detection to increase FDR. The simulation results show that the proposed method can obtain more accurate estimation than ZKF, which does not consider explicitly such time-invariance. The fault detection approach based on the proposed method is shown able to obtain higher FDR than that based on ZKF.

REFERENCES

Alamo, T., Bravo, J., and Camacho, E. (2005). Guaranteed state estimation by zonotopes. *Automatica*, 41, 1035–1043.

Althoff, M., Stursberg, O., and Buss, M. (2010). Computing reachable sets of hybrid systems using a combination of zonotopes and polytopes. *Nonlinear Analysis: Hybrid Systems*, 4, 233–249.

Bernard, O. and Gouzé, J.L. (2004). Closed loop observers bundle for uncertain biotechnological models. *Journal of process control*, 14, 765–774.

Chernousko, F. (2005). Ellipsoidal state estimation for dynamical systems. *Nonlinear Analysis: Theory, Methods & Applications*, 63, 872–879.

Combastel, C. (2003). A state bounding observer based on zonotopes. *European Control Conference*, 2589–2594.

Combastel, C. (2011). On computing envelopes for discrete-time linear systems with affine parametric uncertainties and bounded inputs. *IFAC Proceedings Volumes*, 44(1), 4525 – 4533.

Combastel, C. (2015). Zonotopes and Kalman observers: Gain optimality under distinct uncertainty paradigms and robust convergence. *Automatica*, 55, 265–273.

Darouach, M., Boutat-Baddas, L., and Zerrougui, M. (1999). H_∞ observers design for a class of nonlinear singular systems. *Automatica*, 47, 2517–2525.

Efimov, D., Perruquetti, W., Raïssi, T., and Zolghadri, A. (2004). On interval observer design for time-invariant discrete-time systems. *European Control Conference*, 2651–2656.

Kalman, R.E. (1963). New methods in Wiener filtering theory. In *Proceedings of the First Symposium on Engineering Applications of Random Function Theory and Probability*. John Wiley & Sons, New York.

Keller, J. and Darouach, M. (1999). Two-stage Kalman estimator with unknown exogenous inputs. *Automatica*, 35, 339–342.

Kurzhanski, A.B. and Varaiya, P. (2000). Ellipsoidal techniques for reachability analysis: internal approximation. *Systems & control letters*, 41, 201–211.

Le, V.T.H., Stoica, C., Alamo, T., Camacho, E.F., and D, D. (2013). Zonotopic guaranteed state estimation for uncertain systems. *Automatica*, 49, 3418–3424.

Maksarov, D. and Norton, J. (1996). State bounding with ellipsoidal set description of the uncertainty. *International Journal of Control*, 65, 847–866.

Pourasghar, M., Puig, V., Ocampo-Martinez, C., and Zhang, Q. (2017). Reduced-order interval-observer design for dynamic systems with time-invariant uncertainty. *20th World Congress of the International Federation of Automatic Control*, 6445–6450.

Puig, V., Cugueró, P., and Quevedo, J. (2001). Worst-case state estimation and simulation of uncertain discrete-time systems using zonotopes. *European Control Conference*, 1691–1697.

Wang, Z., Huang, B., and Unbehauen, H. (2001). Robust H_∞ observer design of linear time-delay systems with parametric uncertainty. *Systems & Control Letters*, 42, 303–312.

Wang, Z., Shi, P., and Lim, C.C. (2017). H_-/H_∞ fault detection observer in finite frequency domain for linear parameter-varying descriptor systems. *Automatica*, 86, 38–45.

Xu, F., Puig, V., Ocampo-Martinez, C., Olaru, S., and Stoican, F. (2015). Set-theoretic methods in robust detection and isolation of sensor faults. *International Journal of Systems Science*, 46, 2317–2334.

Xu, F., Puig, V., Ocampo-Martinez, C., Stoican, F., and Olaru, S. (2014). Actuator-fault detection and isolation based on set-theoretic approaches. *Journal of Process Control*, 24, 947–956.

Yang, G.H. and Ye, D. (2006). Adaptive fault-tolerant H_∞ control via state feedback for linear systems against actuator faults. In *Proceedings of the 45th IEEE Conference on Decision and Control*, 3530–3535.