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Abstract: In this article, we propose a new adaptive estimator for compact supported density functions, in the framework of multivariate mixing processes. Several procedures have been proposed in the literature to tackle the boundary bias issue encountered using classical kernel estimators on the unit d-dimensional hypercube. We extend such results to more general bounded domains in $\mathbb{R}^d$. We introduce a specific family of kernel-type estimators adapted to the estimation of compact supported density functions. We then propose a data-driven Goldenshluger and Lepski type procedure to jointly select a kernel and a bandwidth. We prove the optimality of our procedure in the adaptive framework, stating an oracle-type inequality. We illustrate the good behavior of our new class of estimators on simulated data. Finally, we apply our procedure to a real dataset.

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In this paper, we study the classical problem of the estimation of a density function $f : \mathcal{D} \subset \mathbb{R}^d \to \mathbb{R}$, with $\mathcal{D}$ any bounded domain in $\mathbb{R}^d$. In the usual setting, the density is estimated from $X_1, \dots, X_n$, n independent and identically distributed random variables with density f. It is well known that classical kernel-type estimators present a severe bias when the underlying density function does not vanish near the boundary of their bounded support. Several procedures have been proposed in the literature to tackle this issue in the setting where $\mathcal{D} = [0,1]^d$. Schuster (1985), Silverman (1986) and Cline and Hart (1991) studied the reflection of the data near the boundary as well as Marron and Ruppert (1994) who proposed a previous transformation of the data. Müller (1991), Lejeune and Sarda (1992), Jones (1993), Müller and Stadtmüller (1999) and Botev, Grotowski, and Kroese (2010) proposed to construct kernels which take into account the shape of the support of the density. In the same spirit, Chen (1999) studied a new class of kernels constructed using a reparametrization of the family of Beta distributions. For these methods, practical choices of bandwidth or cross-validation selection procedures have generally been proposed. Nevertheless, few papers study the theoretical properties of bandwidth selection procedures in this context. Among others, we point out Bouezmarni and Rombouts (2010) who study the behavior of Beta kernels with a cross-validation selection procedure in a multivariate setting in the specific case of a twice differentiable density. Bertin and Klutchnikoff (2014) study a selection rule based on the Lepski's method (Lepski, 1991) in conjunction with Beta kernels in a univariate setting and prove that the associated estimator is adaptive over Hölder classes of smoothness smaller than or equal to two. More recently, Bertin, El Kolei, and Klutchnikoff (2018) introduced a new family of kernel density estimators that do not suffer from the so-called boundary bias problem and they proposed a data-driven procedure based on the Goldenshluger and Lepski (see Goldenshluger and Lepski, 2014) approach that jointly selects a kernel and a bandwidth. They prove oracle-type inequalities, and adaptivity over a scale of regularity classes.

In the present paper, the support of f is any bounded domain in $\mathbb{R}^d$, not necessarily the unit d-dimensional hypercube $[0,1]^d$. Moreover, we observe n identically distributed random variables $X_1, \dots, X_n$, not necessarily independent. More precisely, we are interested in the framework where the underlying process $(X_i)_{i \geq 0}$ is a stationary β-mixing process. This setting appears in practical frameworks. One motivating example is the following: one aims at estimating the home-range and the core-area of an animal based on tracking data (see the recent paper by Cholaquidis, Fraiman, Mordecki, and Papalardo, 2016). For this case the data is the continuous time trajectory of the animal (obtained for instance from a GPS), therefore it is not reasonable to assume independence of the observations. Roughly speaking, the authors consider that the process under observation behaves in the interior of a compact set $\mathcal{D} \subset \mathbb{R}^d$ like an ordinary Brownian motion with drift, and reflects (normally) at the boundary $\partial\mathcal{D}$. Under regularity assumptions on the drift and geometric constraints on the support $\mathcal{D}$, Cholaquidis et al. (2016) prove the existence of a unique stationary distribution, absolutely continuous with respect to the Lebesgue measure on $\mathcal{D}$. They also prove that the process is geometrically ergodic. Then the trajectories of the animals allow estimating the density of the invariant probability measure.

In the present paper, we extend the results in Bertin et al. (2018), introducing a new family of kernel density estimators that do not suffer from the so-called boundary bias problem, and we adapt the data-driven Goldenshluger and Lepski (see Goldenshluger and Lepski, 2014) procedure to jointly select a kernel and a bandwidth. We prove oracle-type inequalities, and adaptivity over a scale of Hölder classes without bound on the smoothness parameter. Our procedure works for bounded domains, which are not restricted to the unit hypercube $[0,1]^d$ and the data used for the estimation are not assumed to be independent anymore. We first introduce the statistical framework in Section 1. Section 2 is devoted to the description of the new estimation procedure we propose. An oracle-type inequality is stated in Section 3 which leads to the optimality of our procedure in the adaptive framework. Simulation studies are shown in Section 4, as far as the application of our procedure on real data, namely data from the Movebank database, collected from GPS collars placed on elephants in Hwange National Park in Zimbabwe. The proofs are postponed to Section 5.

1. Statistical framework

In what follows, we consider a strongly stationary and β-mixing process $\mathbb{X} = (X_i : i \in \mathbb{Z})$ that lies into a bounded domain $\mathcal{D} \subset \mathbb{R}^d$. We assume that the common marginal distribution of the random variables X_i is absolutely continuous with respect to the Lebesgue measure restricted to $\mathcal{D}$ and we denote by $f\colon \mathcal{D} \to \mathbb{R}$ the density of this distribution. We aim at finding an *accurate* estimation procedure for f based on the observations $X_1, \dots, X_n$, where $n \in \mathbb{N}$. To do so we face two main problems. First, since the observations lie into a bounded set, usual estimators suffer from a boundary bias, see, e.g. Silverman (1986). Next, since the observations are dependent, usual statistical procedures, dedicated to the independent framework, must be adapted.

In the rest of this section, we present the main assumptions we make on the law of the process $\mathbb{X}$ and on the geometry of the domain $\mathcal{D}$. We also present the adaptive minimax framework used to measure the statistical performances of the estimators.

1.1. Assumptions on the law of the process

The assumptions on the process $\mathbb{X}$ are divided into two parts: the assumptions on the marginal density f on the one hand and the assumptions on the dependence structure of the process on the other hand.

To state the assumptions on the marginal density, we recall the definition of a Hölder ball on the domain $\mathcal{D}$. Let γ and L be two positive numbers. A function $f\colon \mathcal{D} \to \mathbb{R}$ belongs to the Hölder class $\mathcal{H}_{\mathcal{D}}(\gamma, L)$ if the following conditions are fulfilled:

i) The partial derivatives $D^\alpha f = \partial^{|\alpha|} f / (\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d})$ exist for any $\alpha \in (\mathbb{N} \cup \{0\})^d$ such that $|\alpha| = \alpha_1 + \cdots + \alpha_d \leq \lfloor \gamma \rfloor$ where $\lfloor \gamma \rfloor = \max\{\ell \in \mathbb{N} \cup \{0\} \colon \ell < \gamma\}$.

ii) For any x, y in $\mathcal{D}$, $\sum_{|\alpha|=\lfloor\gamma\rfloor} |D^\alpha f(y) - D^\alpha f(x)| \le L|y-x|_1^{\gamma-\lfloor\gamma\rfloor}$ where $|u|_p = (\sum_{i=1}^d |u_i|^p)^{1/p}$ if $1 \le p < +\infty$, $|u|_\infty = \max\{|u_i| \colon i = 1, \dots, d\}$.

Assumption 1 *Set $f_\infty > 0$. The sup-norm of f, defined by $\|f\|_\infty = \sup_{x\in\mathcal{D}} |f(x)|$ is less than or equal to f_∞.*

The absolute regularity (or β-mixing) condition of a process was introduced by Volkonskii and Rozanov (1959) and attributed there to Kolmogorov. For the convenience of the reader we recall the definition of the β-mixing coefficients of the strictly stationary process $\mathbb{X}$. For each $k \ge 1 \in \mathbb{N}$, we define:

$$\beta(k) = \sup \frac{1}{2} \sum_{i\in I} \sum_{j\in J} |\mathbf{P}(U_i \cap V_j) - \mathbf{P}(U_i)\mathbf{P}(V_j)|\,,$$

where the supremum is taken over all pairs of finite partitions $\{U_i : i \in I\}$ and $\{V_j : j \in J\}$ of the probability space Ω which are respectively measurable with respect to $\mathcal{F}_{-\infty}^0 = \sigma(X_s : s \le 0)$ and $\mathcal{F}_k^{+\infty} = \sigma(X_s : s \ge k)$.

Assumption 2 *Let c be a positive number and set $0 < \rho < 1$. We assume that the process $\mathbb{X}$ is strictly stationary and β-mixing at a geometric rate. More precisely, for any $k \ge 1$ we have $\beta(k) \le c\rho^k$.*

Assumption 3 *Set $f_\infty > 0$. For any $k \ge 1$ the distribution of the random pair (X_1, X_{k+1}) admits a density $f_k \colon \mathcal{D}^2 \to \mathbb{R}$ (with respect to the Lebesgue measure restricted to $\mathcal{D}^2$) such that $\|f_k\|_\infty \le f_\infty$.*

Without loss of generality we assume that the bounds f_∞ that appear in Assumptions 1 and 3 are the same if the two assumptions hold simultaneously.

1.2. Geometric assumptions on the domain

In this section, we first state technical assumptions on the domain $\mathcal{D}$ and we offer some examples that satisfy these conditions.

Assumption 4 *Set $R > 0$. The domain $\mathcal{D} \neq \emptyset$ is a bounded open connected set such that, for any $x \in \mathcal{D}$, $|x|_\infty \le R$.*

Remark that, since our goal is to consider the estimation on bounded domain, the existence of $R > 0$ is not a restrictive condition. Assuming that $\mathcal{D}$ is connected is also not restrictive since the same estimation procedure could be applied on each connected component. Finally $\mathcal{D}$ is assumed to be open to ensure that the ambiant dimension d is the correct one.

Assumption 5 *There exist $0 < r < 1$ and a finite family $\mathcal{A} = \{\mathbf{A}_1, \dots, \mathbf{A}_\kappa\}$ of distinct elements in $GL_d(\mathbb{R})$ such that:*

i) *For any $j = 1, \dots, \kappa$, $|\det \mathbf{A}_j| = 1$.*
ii) *For any $x \in \mathcal{D}$ there exists $A_x \in \mathcal{A}$ such that $x + A_x^{-1}([0, r]^d) \subset \mathcal{D}$.*

This assumption is mainly about a "neighborhood" of the boundary $\partial\mathcal{D} = \overline{\mathcal{D}} \setminus \mathcal{D}$ of the domain $\mathcal{D}$ where $\overline{\mathcal{D}}$ denotes the adherence of $\mathcal{D}$. Indeed, define $\mathcal{D}_r^\circ = \{x \in \mathcal{D} \colon |x-y|_\infty \geq r, \quad \forall y \in \partial\mathcal{D}\}$ which consists of all points far from the boundary of $\mathcal{D}$, then one can choose A_x as the identity matrix for any $x \in \mathcal{D}_r^\circ$.

Below we present two simple examples that satisfy this technical assumption. The first one was used in Bertin et al. (2018) in a similar context, but more restrictive than ours.

Example 1 (Hypercubes) *We consider the case where $\mathcal{D} = (0,1)^d$ and we define for $u \in \mathcal{D}$ and $x \in \mathcal{D}$: $A_x(u) = (\sigma(x_1)u_1, \dots, \sigma(x_d)u_d)$ where $\sigma(x) = 1 - 2\mathbf{I}_{(1/2,1)}(x)$ for $x \in (0,1)$. In this case we have $r = 1/2$ and $\kappa = 2^d$.*

To state the second example, we denote by $\mathbb{D}_r = \{x \in \mathbb{R}^2 : |x|_2 \leq r\}$ the Euclidean ball with radius $r > 0$. Note that this example is of particular interest since it is used throughout the simulation studies.

Example 2 (Disk) *For simplicity we only consider the case $d = 2$. We assume that $\mathcal{D} = \mathbb{D}_1$. For $k = 1, \dots, 6$ we define:*

$$\mathbf{A}_k = \begin{pmatrix} \cos(\theta_k) & -\sin(\theta_k) \\ \sin(\theta_k) & \cos(\theta_k) \end{pmatrix}$$

where $\theta_k = -3\pi/4 - k\pi/3$. Now, for any $x \in \mathcal{D} \setminus \{0\}$ we identify $x/|x|_2 \in \mathbb{S}^1$ with a real number in $[0, 2\pi)$. Now, since

$$[0, 2\pi) = \bigcup_{k=1}^{6} I_k \quad \textit{where} \quad I_k = \Big[-\frac{\pi}{6} + k\frac{\pi}{3}, \frac{\pi}{6} + k\frac{\pi}{3}\Big),$$

we define $A_x = \mathbf{A}_k$ if k is such that $x/|x|_2 \in I_k$ and $A_0 = \mathbf{A}_1$. Assumption 5 is satisfied with $\kappa = 6$ and $r = 1/4$.

However, generic classes of open subsets of $\mathbb{R}^2$ can be proven to satisfy Assumption 5 as in the two following examples (See proof in Supplemental Material to for more details).

Example 3 (Rolling conditions) *Set $r_0 > 0$. The domain $\mathcal{D}$ is called r_0-regular if, for any $0 < r \leq r_0$, the ball $\mathbb{D}_r$ rolls freely in both $\mathcal{D}$ and $\mathcal{D}^c$. That is, for each $a \in \partial\mathcal{D}$, there exists x_a^r and y_a^r in $\mathbb{R}^d$ such that :*

$$a \in \big(x_a^r + \mathbb{D}_r\big) \subseteq \overline{\mathcal{D}} \quad \textit{and} \quad a \in \big(y_a^r + \mathbb{D}_r\big) \subseteq \mathcal{D}^c$$

Such regularity condition on $\mathcal{D}$ is well-known and widely used in statistics (see Arias-Castro and Rodríguez-Casal, 2017, and references therein).

Finally, since in practical situations the boundary of a domain can be approximated by a simple polygonal path, the following example (see Supplemental Material to for more details) seems to be of prime interest.

Example 4 (Simple polygons) *The interior of any simple polygon satisfies Assumption 5.*

1.3. Adaptive minimax point of view

Under Assumptions 1 and 4, the marginal density f belongs to the space of squared integrable functions that map $\mathcal{D}$ into $\mathbb{R}$. This set, denoted by $\mathbb{L}_2(\mathcal{D})$ is endowed with its natural Hilbertian norm:

$$\|g\|_2 = \left(\int_{\mathcal{D}} g^2(u) du \right)^{1/2}.$$

In this context, an estimator is any measurable map $\tilde{f} \colon \mathcal{D}^n \to \mathbb{L}_2(\mathcal{D})$. To measure the performance of such an estimator, we consider its risk defined by:

$$R_n(\tilde{f}, f) = \left(\mathbf{E} \|\tilde{f} - f\|_2^2 \right)^{1/2}.$$

Let $\mathbb{F}$ be a subset of $\mathbb{L}_2(\mathcal{D})$. The maximal risk of $\tilde{f}$ over $\mathbb{F}$ is defined by:

$$R_n(\tilde{f}, \mathbb{F}) = \sup_{f \in \mathbb{F}} R_n(\tilde{f}, f),$$

whereas the minimax risk over $\mathbb{F}$ (see Tsybakov, 2009) is:

$$\phi_n(\mathbb{F}) = \inf_{\tilde{f}} R_n(\tilde{f}, \mathbb{F})$$

where the infimum is taken over all the estimators. An estimator whose maximal risk is asymptotically bounded, up to a multiplicative factor, by $\phi_n(\mathbb{F})$ is called minimax over $\mathbb{F}$. Such an estimator is well-adapted to the estimation over $\mathbb{F}$ but it can perform poorly over another functional space. The problem of adaptive estimation consists in finding a single estimation procedure that is simultaneously minimax over a scale of functional classes. More precisely, given a family $\{\mathbb{F}_\lambda : \lambda \in \Lambda\}$ of subsets of $\mathbb{L}_2(\mathcal{D})$, the goal is to construct f^* such that $R_n(f^*, \mathcal{F}_\lambda)$ is asymptotically bounded, up to a multiplicative constant, by $\phi_n(\mathbb{F}_\lambda)$ for any $\lambda \in \Lambda$. One of the main tools to prove that an estimation procedure is adaptive over a scale of functional classes is to prove an oracle-type inequality that guarantees that this procedure performs almost as well as the best estimator in a rich family of estimators. Ideally, we would like to have an inequality of the following form:

$$R_n(f^*, f) \leq \inf_{\eta \in H} R_n(\hat{f}_\eta, f), \tag{1}$$

where $\{\hat{f}_\eta \colon \eta \in H\}$ is a family of estimators *well-adapted* to our problem: for any $\lambda \in \Lambda$, there exists $\eta(\lambda)$ such that $\hat{f}_{\eta(\lambda)}$ is minimax over $\mathbb{F}_\lambda$. However, in many situations, (1) is relaxed and we prove a weaker inequality of the type:

$$R_n(f^*, f) \leq \Upsilon_1 \inf_{\eta \in H} R_n^*(f, \eta) + \Upsilon_2 n^{-1/2}, \tag{2}$$

where Υ_1 and Υ_2 are two positive constants and $R_n^*(f, \eta)$ is an appropriate quantity to be determined that can be viewed as a tight upper bound on $R_n(\hat{f}_\eta, f)$. Inequalities of the form (2) are called *oracle-type* inequalities.

1.4. Objectives

In this paper we are interested in the adaptive estimation of the marginal density of the strictly stationary β-mixing process $\mathbb{X}$ over the scale: $\mathbb{F}_\lambda = \mathcal{H}_\mathcal{D}(\gamma, L)$, $\lambda = (\gamma, L) \in \Lambda = (0, +\infty)^2$. To do so, we will construct a new family of kernel estimators $\{\hat{f}_\eta \colon \eta \in H\}$ that are well-adapted to our problem and we will propose a procedure that selects $\hat{\eta} \in H$ using the so-called Goldenshluger-Lepski method. We will prove that the resulting plug-in estimator $\hat{f}_{\hat{\eta}}$ satisfies an oracle-type inequality which will imply its optimality in the adaptive framework.

2. Statistical procedure

We propose to construct a specific family of kernel-type estimators which can tackle with the boundary bias problem encountered using classical kernel estimators (see Bertin et al., 2018, and references therein for more details). The construction of this family is linked with geometrical assumptions on the domain $\mathcal{D}$. Before presenting the ideas behind the construction of the family of estimators, we introduce some notations used throughout the paper. A function $K : \mathbb{R} \to \mathbb{R}$ is called a univariate kernel if the support of K is included into $[0, 1]$, $\|K\|_\infty < +\infty$ and $\int_0^1 K(u)du = 1$.

Moreover, we say that K is a kernel of order $m \in \mathbb{N} \cup \{0\}$ if

$$\int_0^1 K(u)u^p du = \delta_{0,p}, \quad \text{for any } 0 \le p \le m.$$

2.1. A first family of kernel estimators

For any bandwidth $h > 0$, the normalized multivariate kernel $\mathbf{K}_h$ is defined by:

$$\mathbf{K}_h(s) = h^{-d} \prod_{i=1}^{d} K(h^{-1}s_i), \qquad s \in \mathbb{R}^d.$$

Equipped with these notations we note that, under Assumption 5 and for h small enough, for any $x \in \mathcal{D}$, $\mathbf{K}_h \circ A_x(u - x) = 0$ as soon as $u \notin \mathcal{D}$. This property implies that there is no loss of mass near the boundary (as in the classical convolution kernel). This leads us to consider the following estimators:

$$\hat{f}_{K,h}(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{K}_h \circ A_x(X_i - x), \qquad x \in \mathcal{D}. \tag{3}$$

The family of estimators $\{\hat{f}_{K,h}\}_{K,h}$ indexed by all bandwidths $h > 0$ and univariate kernels K is well-adapted to our problem. Indeed Proposition 1 below guarantees that there exists an estimator of this form that reaches the minimax rate of convergence over $\mathcal{H}_\mathcal{D}(\gamma, L)$. To state this proposition, we define:

$$\Gamma(K, h) = h^{d/2} \left(\int_\mathcal{D} \mathbf{E} \left(\mathbf{K}_h \circ A_x \right)^2 (X_1 - x) dx \right)^{1/2}. \tag{4}$$

Proposition 1 *Assume that Assumptions 1, 2, 4 and 5 are fulfilled. Set $\gamma > 0$ and $L > 0$. Let K be a kernel of order greater than or equal to $\lfloor \gamma \rfloor$. Then, there exist two absolute constants $C(R, \gamma)$ and $C(\rho, c, f_\infty)$ such that for any $h > 0$ we have:*

$$\|\mathbf{E}\hat{f}_{K,h} - f\|_2 \leq C(R, \gamma) L \|K\|_\infty^d h^\gamma \tag{5}$$

and

$$\mathbf{E}\|\hat{f}_{K,h} - \mathbf{E}\hat{f}_{K,h}\|_2^2 \leq \frac{4\kappa \|K\|_2^{2d}}{nh^d} \frac{c}{1-\rho}. \tag{6}$$

With the additional Assumption 3, we get

$$\mathbf{E}\|\hat{f}_{K,h} - \mathbf{E}\hat{f}_{K,h}\|_2^2 \leq \frac{\Gamma^2(K,h)}{nh^d} + \frac{\kappa\|K\|_2^{2d}}{nh^d} C(c, \rho, f_\infty) h^{d/2} \tag{7}$$

$$\leq \frac{\kappa\|K\|_2^{2d}}{nh^d} \left(1 + C(c, \rho, f_\infty) h^{d/2}\right) \tag{8}$$

Moreover, in both cases, taking $h = n^{-1/(2\gamma+d)}$ the estimator $\hat{f}_{K,h}$ reaches the minimax rate $\phi_n(\gamma, L) \asymp n^{-\gamma/(2\gamma+d)}$.

2.2. A second family of estimators

Even if the previous family of estimators is well-adapted to our framework, it is a very large family since we do not impose any restriction on the kernel nor on the bandwidth. This implies that selecting in a data driven way an element in this family is difficult from both theoretical and practical points of view. In this section we construct a one-parameter subfamily which consists of predefined well-chosen pairs of kernels and bandwidths. To do so, we fix a family of kernels $\mathcal{K} = \{K_m : m \in \mathbb{N} \cup \{0\}\}$ and we define, for any $\ell \in \mathbb{N}$:

$$h(\ell) = e^{-\ell} \quad \text{and} \quad m_n(\ell) = \left[\frac{\log(n)}{2\ell} + \frac{1}{2}\right]$$

where $[\cdot]$ denotes the integer part. Using the notation introduced in (3) we define, for any $\ell \in \mathbb{N}$ the estimator: $\hat{f}_\ell = \hat{f}_{K_{m_n(\ell)}, h(\ell)}$. To obtain a finite collection of estimators that have a good behavior, we impose an additional restriction on $h(\ell)$ by considering only $\ell \in \mathcal{L}_n$ where

$$\mathcal{L}_n = \{\ell \in \mathbb{N} \colon \underline{h}_n < h(\ell) < \overline{h}_n\}.$$

Here, the bandwidths $\underline{h}_n$ and $\overline{h}_n$ are defined, for given $c_1 > 0$ and $c_2 > 0$, by

$$\overline{h}_n = (\log n)^{-2(1+c_1)/d} \quad \text{and} \quad n\underline{h}_n^d = \left(\log(n)\right)^{c_2}.$$

Finally, the one-parameter family $\left\{\hat{f}_\ell : \ell \in \mathcal{L}_n\right\}$ depends only on the choice of the set $\mathcal{K}$. Let us state two assumptions such a family of kernels can satisfy.

Assumption 6 *The family $\mathcal{K} = \{K_m : m \in \mathbb{N} \cup \{0\}\}$ is such that, for any $m \in \mathbb{N}$, the kernel K_m is of order m.*

Assumption 7 *The family $\mathcal{K} = \{K_m : m \in \mathbb{N} \cup \{0\}\}$ satisfies: there exist $A > 0$ and $B \geq 1$ such that $\|K_m\|_\infty \leq A(m+1)^B$, for any $m \in \mathbb{N}$.*

Assumption 6 is used to obtain the following proposition that guarantees that our family of estimators is well-adapted to our problem. Assumption 7 is more technical.

Proposition 2 *Assume that Assumptions 1, 2, 4, 5 and 6 are fulfilled. Set $\gamma > 0$ and define $\ell_\gamma = [(2\gamma + d)^{-1} \log n]$. The estimator $\hat{f}_{\ell_\gamma}$ reaches the minimax rate of convergence over $\mathcal{H}_{\mathcal{D}}(\gamma, L)$ for any $L > 0$.*

Proposition 2 is a direct consequence of Proposition 1 and Assumption 6 ensures that the order of the kernel of $\hat{f}_{\ell_\gamma}$ is larger than γ.

To end this section, remark that a family of kernels satisfying Assumptions 6 and 7 is given, for example, by:

$$K_m(u) = \sum_{r=0}^{m} \varphi_r(0)\varphi_r(u),\; u \in [0,1],\; m \in \mathbb{N} \cup \{0\},\; \varphi_k(u) = \sqrt{2k+1}Q_k(2u-1)$$

and Q_k is the Legendre Polynomial of degree k on $[-1,1]$. The kernel K_m is of order m and satisfies Assumption 7 with $A = 1$ and $B = 2$. More precisely it can be proven that: $\|K_m\|_2 = m+1$ and $\|K_m\|_\infty = (m+1)^2$. See Lemma 2 in Bertin et al. (2018) for more details.

2.3. Selection rule

Let $\tau > 0$. For any $\ell, \ell' \in \mathcal{L}_n$ we define the following *majorants*:

$$\widehat{M}(\ell) = \sqrt{2}\,\frac{\hat{\Gamma}\big(K_{m_n(\ell)}, h(\ell)\big) + \tau\|K_{m_n(\ell)}\|_2^d}{\sqrt{nh^d(\ell)}} \text{ and } \widehat{M}(\ell,\ell') = \widehat{M}(\ell') + \widehat{M}(\ell' \wedge \ell),$$

where $\ell \wedge \ell'$ denotes the minimum between ℓ and ℓ' and $\hat{\Gamma}(K,h)$ is defined by

$$\hat{\Gamma}(K,h) = h^{d/2}\left(\int_{\mathcal{D}} \frac{1}{n}\sum_{i=1}^{n} \left(\mathbf{K}_h \circ A_x(X_i - x)\right)^2 dx\right)^{1/2}.$$

Now, we define: $\widehat{B}(\ell) = \max_{\ell' \in \mathcal{L}_n} \left\{\|\hat{f}_{\ell\wedge\ell'} - \hat{f}_{\ell'}\|_2 - \widehat{M}(\ell,\ell')\right\}_+$ with $x_+ = \max(x,0)$ denotes the positive part of x. The final estimator, $\widehat{f}$ is then defined by:

$$\widehat{\ell} = \arg\min_{\ell \in \mathcal{L}_n}\left(\widehat{B}(\ell) + \widehat{M}(\ell)\right) \qquad \text{and} \qquad \widehat{f} = \hat{f}_{\widehat{\ell}}. \tag{9}$$

The selection rule is an adaptation of the so-called Goldenshluger-Lepski (GL) method which consists in selecting, in a data-driven way, an estimator that realizes the trade-off (9) beetwen $\widehat{B}$ and $\widehat{M}$, estimators of respectively the bias term and the stochastic term. Finding tight *majorants* is the key-point of this procedure. Let us briefly comment on the form of the *majorant* $\widehat{M}(\ell)$. The ideal *majorant* would be

$$\overline{M}(\ell) = \frac{\Gamma\big(K_{m_n(\ell)}, h(\ell)\big)}{\sqrt{nh^d(\ell)}}.$$

However the term $\Gamma\big(K_{m_n(\ell)}, h(\ell)\big)$ depends on the unknown density f and is bounded, see (10), by the deterministic constant $\sqrt{\kappa}\|K\|_2^d$ which is rough in some situations, like the one in Example 1. Recall that this is due to the specific form of our boundary kernels. To circumvent this drawback we replace this quantity by a simple estimator $\hat{\Gamma}\big(K_{m_n(\ell)}, h(\ell)\big)$. For technical reasons that appear in the proof of Lemma 7 we add the small corrective term $\tau\|K\|_2^d$. Finally, the extra $\sqrt{2}$ factor allows us to take into account the dependence structure of the observations using classical Berbee coupling techniques (see Berbee, 1979; Comte, Prieur, and Samson, 2017) in the proofs of Theorem 3.

Remark 1 *The final procedure depends on the parameters τ, c_1 and c_2 that can be chosen, at least from a theoretical point of view, as small as desired. We refer the reader to the simulation study for practical details.*

3. Results

Theorem 3 (Oracle-type inequality) *Assume Assumptions 1, 2, 3, 4, 5, 6 and 7 are fulfilled. We have*

$$R_n(\hat{f}, f) \leq \Upsilon_1 \inf_{\ell \in \mathcal{L}_n} \left(\max_{\ell' \geq \ell} \|\mathbf{E}\hat{f}_{\ell'} - f\|_2 + \frac{\|K_{m_n(\ell)}\|_2^d}{\sqrt{nh^d(\ell)}} \right) + \Upsilon_2 n^{-1/2}$$

with Υ_1, Υ_2 two positive constants depending on τ, c, ρ, f_∞, κ, A, B.

Note that this oracle-type inequality is of the form (2) with:

$$R_n^*(f, \ell) = \max_{\ell' \geq \ell} \|\mathbf{E}\hat{f}_{\ell'} - f\|_2 + \frac{\|K_{m_n(\ell)}\|_2^d}{\sqrt{nh^d(\ell)}}$$

which is, up to a multiplicative constant, an upper bound of the risk of $\hat{f}_\ell$ since, using (6):

$$\begin{aligned} R_n(\hat{f}_\ell, f) &= \left(\|\mathbf{E}\hat{f}_\ell - f\|_2^2 + \mathbf{E}\|\hat{f}_\ell - \mathbf{E}\hat{f}_\ell\|_2^2 \right)^{1/2} \\ &\leq \|\mathbf{E}\hat{f}_\ell - f\|_2 + \left(\frac{4c\kappa}{1-\rho} \right)^{1/2} \frac{\|K_{m_n(\ell)}\|_2^d}{\sqrt{nh^d(\ell)}}. \end{aligned}$$

This allows us to prove that the procedure is adaptive over a large scale of Hölder spaces.

Theorem 4 (Adaptive estimation) *Assume Assumptions 1, 2, 3, 4, 5, 6 and 7 are fulfilled. For any $\gamma > 0$ and $L > 0$, we have:*

$$\limsup_{n \to +\infty} n^{\gamma/(2\gamma+d)} \sup_{f \in \mathcal{H}_{\mathcal{D}}(\gamma, L)} R_n(\widehat{f}, f) < +\infty.$$

The proof of this theorem relies on Proposition 2 as well as the fact that, if $f \in \mathcal{H}_{\mathcal{D}}(\gamma, L)$ for some $\gamma > 0$, then $\max_{\ell' \geq \ell_\gamma} \|\mathbf{E}\hat{f}_{\ell'} - f\|_2$ is a tight upper bound of $\|\mathbf{E}\hat{f}_{\ell_\gamma} - f\|_2$ (see the proof for more details). Note that this result is obtained without any restriction on the smoothness parameter $\gamma > 0$. This follows from the simultaneous choice of a bandwidth and a kernel and differs from the usual bandwidth selection procedure where the kernel remains fixed.

4. Simulation study

In this section, we study the performance of our procedure using simulated data in Sections 4.1 and 4.2 and we apply it to a real data set of elephant trajectories in Section 4.3. More precisely, in Section 4.1, we aim at estimating several densities defined on the disk that exhibit various behaviors near the boundary of the support and we consider independent data. In Section 4.2, we study the stationary density of a two dimensional reflected Langevin diffusion on the disk. In each situation, we study the accuracy of our procedure as well as usual kernel estimators, calculating empirical risks using $M = 500$ Monte-Carlo replications. In the following, we detail our simulation scheme and comment on the obtained results.

4.1. Densities on the disk

Simulation scheme We consider a family of densities $\{\mathrm{f}_{a,b,c}\}_{(a,b,c)\in(0,+\infty)^3}$ such that $\begin{cases} \mathrm{f}_{a,b,c} : \mathbb{D}_1 \to [0,+\infty) \\ \mathrm{f}_{a,b,c}(x,y) = \mathrm{g}_{a,b}(x^2+y^2)\mathrm{g}_{c,c}\left(\frac{\mathrm{atan2}(y,x)}{2\pi} \mod 1\right) \end{cases}$ with $\mathrm{g}_{a,b}$ the usual density of the beta distribution with parameters a and b. Equivalently, (X,Y) is distributed as $(\sqrt{R}\cos\Theta, \sqrt{R}\sin\Theta)$ with R and Θ independent with respective distributions $\mathrm{Beta}(a,b)$ and $\mathrm{Beta}(c,c)$. Four densities, plotted in Figure 1, are studied:

Case 1 $a = 1, b = 1, c = 1$. The density $\mathrm{f}_1 := \mathrm{f}_{1,1,1}$ is in fact the uniform density on the disk.

Case 2 $a = 1.5, b = 1, c = 1$. The density $\mathrm{f}_2 := \mathrm{f}_{1.5,1,1}$ takes small values in the centre of the disk and its values increase slowly as one gets close to the boundary.

Case 3 $a = 2, b = 1, c = 1$. The density $\mathrm{f}_3 := \mathrm{f}_{2,1,1}$ takes very small values in most of the disk and only on a small strip of the boundary takes larger values.

Case 4 $a = 1.5, b = 1, c = 3$. The density $\mathrm{f}_4 := \mathrm{f}_{1.5,1,3}$, contrary to the others, is not invariant by rotations. The mass is more important near the point $(-1, 0)$.

Quality criteria For each density function $f \in \{\mathrm{f}_1, \mathrm{f}_2, \mathrm{f}_3, \mathrm{f}_4\}$, we simulate $M = 500$ sequences of observations $(X_1, \ldots, X_n)$ with $n \in \{500, 1000, 2000, 5000\}$.

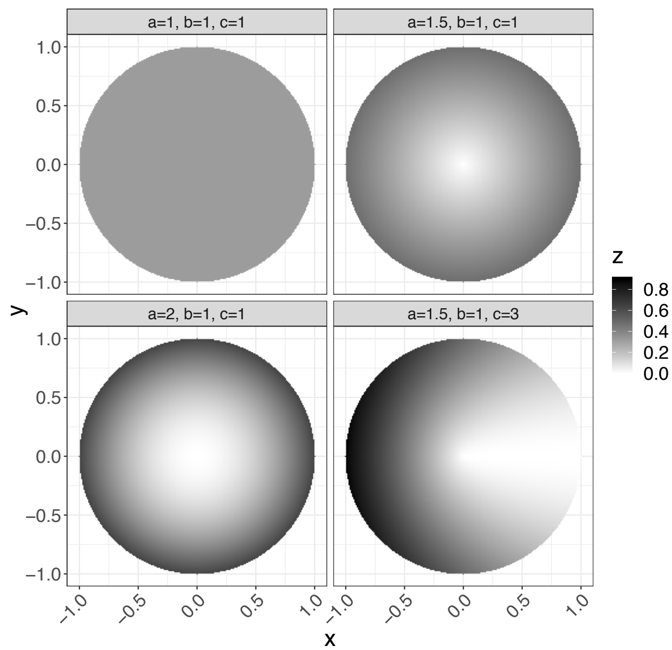

FIGURE 1. *Representation of the four densities*

Given an estimation procedure $\hat{f}$, we calculate M estimators $\hat{f}^{(1)}, \dots, \hat{f}^{(M)}$. We consider the integrated square error (ISE):

$$ISE(\hat{f}^{(j)}) = \int_{\mathbb{D}_1} \left(f(x) - \hat{f}^{(j)}(x) \right)^2 dx \, .$$

Comparison of estimation procedures We consider a set $\mathcal{H}$ of 39 equally spaced bandwidths between 0.05 and 1 with step 0.025. For each $h \in \mathcal{H}$ we define the *usual* kernel estimator $\widetilde{f}_h$ and $\hat{f}_h$ which is a modified version of our estimator defined in Section 2.2 that we call *boundary* estimator. These estimators are defined as follows:

$$\widetilde{f}_h(x) = \frac{1}{nh^2} \sum_{i=1}^{n} \widetilde{K}(h^{-1}(X_i - x)) \text{ with } \widetilde{K}(u_1, \dots, u_d) = \prod_{j=1}^{d} \mathbf{I}_{\left[-\frac{1}{2}, \frac{1}{2}\right]}(u_j)$$

and $\hat{f}_h(x) = \begin{cases} \widetilde{f}_h(x) & \text{if } x \in \mathbb{D}_{1-h/\sqrt{2}} \\ \hat{f}_{K,h}(x) & \text{otherwise} \end{cases}$. Here $K = \mathbf{I}_{[0,1]}$ and the transformation A_x is the one in Example 2. We consider only uniform kernels to allow an easier comparison of the performances of the different estimators. Boxplots of the ISE are given in Figure 2 for $n \in \{500, 1000\}$. For $n \in \{2000, 5000\}$, the behaviours are quite similar and boxplots can be found in Figures 1 and 2 of Supplemental Material to. More precisely we compare:

- the oracle of the *boundary* estimators,
- the Goldenshluger Lepki procedure based on the *boundary* estimators,
- the oracle of *usual* Kernel estimators,
- the Goldenshluger Lepki procedure based on the *usual* Kernel estimators.

In almost all the cases, the GL procedure based on *boundary* estimators outperforms the GL procedure based on *usual* kernels. Note also that the ratio between the MISE of the GL procedure and the one of the oracle, both based on boundary estimator is around 1.6 (see Table 1 in Supplemental Material to) which means the the GL procedure mimics quite well the oracle.

4.2. *Diffusion*

Let us consider in this section the following two dimensional reflected Langevin diffusion on the disk:

$$\begin{cases} dX_t &= dW_t^1 - \beta \frac{X_t}{(1+X_t^2+Y_t^2)^\beta} dt + n_1(X_t, Y_t) dL_t \\ dY_t &= dW_t^2 - \beta \frac{Y_t}{(1+X_t^2+Y_t^2)^\beta} dt + n_2(X_t, Y_t) dL_t \end{cases}$$

with $\beta > 1$, $(n_1(x,y), n_2(x,y))$, $(x,y) \in \partial\mathcal{D}$ defined the normal vector to the boundary of the domain $\mathcal{D} = \mathbb{D}_r$, W^1 and W^2 are two independent standard Brownian motions and L the local time on $\partial\mathcal{D}$. The process $Z_t = (X_t, Y_t)_{t \geq 0}$ is well known as Brownian motion with drift. This process is ergodic and exponential Φ-mixing. The invariant measure is absolutely continuous with respect to the Lebesgue measure restricted to the disk $\mathbb{D}_r$. The invariant density writes as follows:

$$f(x,y) = \frac{1-\beta}{\pi[(1+r^2)^{1-\beta} - 1]} \frac{1}{(1+x^2+y^2)^\beta}.$$

Note that this density has most of its mass concentrated in the centre of the disk. In the simulations we fix $\beta = 2$ and we run as in Cattiaux, León, and Prieur (2017) the Euler reflected scheme introduced in Bossy, Gobet, and Talay (2004). As we are interested in the stationary regime, we throw away the first runs of the scheme. As in Section 4.1, we simulate $M = 500$ Monte-Carlo replications with sample size $n \in \{500, 1000, 2000, 5000\}$. We obtain that for all sample size, the GL procedure based on *boundary* estimators outperforms the GL procedure based on *usual* kernels. (See Figure 3 in Supplemental Material to).

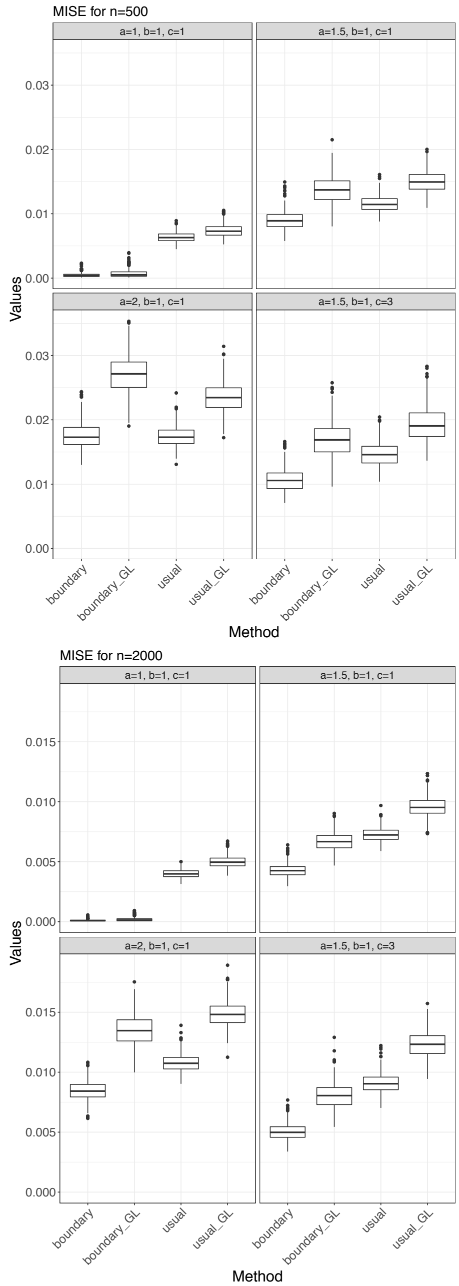

FIGURE 2. *Boxplots of the integrated squared error (ISE) on the disk for the models described by Cases* 1 *to* 4*, and sample sizes equal to* 500 *and* 2000*.*

4.3. Application to a real dataset

In this section our method is applied to a database obtained from the study "African Elephant (Migration) Chamaillé-Jammes Hwange NP" that resides in the repository Movebank (Wikelski and Kays, 2018, accessed on 2018/11/05).

The data was obtained from the GPS tracking of the migratory trajectories of 30 elephants evolving in the Hwange National Park in Zimbabwe (HNP in the following). The daily observations of their displacements were taken by means of GPS devices located in collars installed in individuals of different herds (see Tshipa, Valls-Fox, Fritz, Collins, Sebele, Mundy, and Chamaillé-Jammes, 2017; Valls-Fox, De Garine-Wichatitsky, Fritz, and Chamaillé-Jammes, 2018, and references therein). The date of installation of the collars was as follows: August 2009 (10 elephants), November 2012 (10 elephants), November 2014 (8 elephants) and February 2015 (2 elephants). Each elephant is observed during approximately 2 years.

We are interested in the spatial density of the whole set of elephants into the park. As in Cholaquidis et al. (2016) we assume that the animal movements can be modeled by a reflected diffusion. This allows us to guarantee that the process that modeled the trajectories of the elephants satisfies the main assumptions of our model. Moreover, nine elephants were removed from the initial database as their behavior seems atypic. Figure 3 (which represents each of the $n = 17501$ GPS positions as a point) shows that most observations near a boundary are located on the east side of the park.

The boundary of the park was approximated by a simple polygon which allows to apply the methodology develops in this paper in dimension $d = 2$. In Figure 3 below, we have drawn the density estimation on a spatial grid with step $\delta = 0.01$ of the polygon defining our boundary. Here $K = \mathbf{I}_{[0,1]}$ and we have constructed a finite family $\mathcal{A} = \{A_1, A_2, A_3\}$ of size 3. We used a slope heuristic to determine the constant τ in the penalty. The bandwidth which was selected is equal to $h = 0.07$. The result we obtain seems to confirm that the density of elephants is related to the placement of artificial water pumps. It would be interesting to investigate further that issue with the owners of this database. We give more details on how we proceed in Supplemental Material to.

5. Proofs

5.1. Preliminary notations and technical lemmas

For any $\ell \in \mathcal{L}_n$, define

$$M(\ell) = \frac{\Gamma(\ell) + \tau \|K_{m_n(\ell)}\|_2^d}{\sqrt{nh^d(\ell)}} \text{ where } \Gamma(\ell) = \Gamma(K_{m_n(\ell)}, h(\ell))$$

with $\Gamma(K, h)$ defined in (4). For any kernel K and bandwidth $h > 0$, define:

$$\xi_{K,h}(x) = \frac{\sqrt{nh^d}}{n} \sum_{i=1}^{n} \left(\mathbf{K}_h \circ A_x(X_i - x) - \mathbf{E}\mathbf{K}_h \circ A_x(X_i - x) \right), \, x \in \mathcal{D}.$$

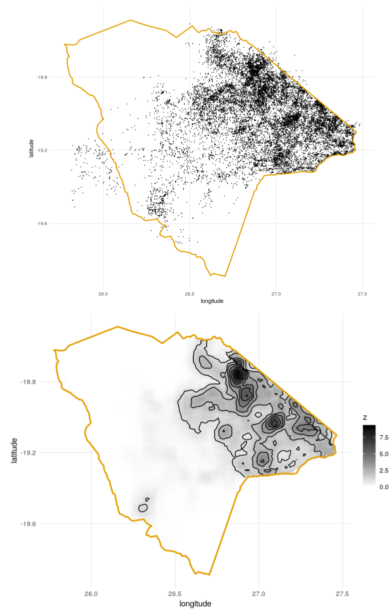

FIGURE 3. *(left) Hwange NP* © **OpenStreetMap contributors**, *(right) estimated density on the park.*

We denote $\mathcal{D}_i = \{x \in \mathcal{D} \colon A_x = A_i\}$. The following lemma, whose proof is postponed in Section 5.5, provides some properties of $\xi_{K,h}$.

Lemma 5 *We have*

$$\Gamma(K, h) \leq \sqrt{\kappa}\|K\|_2^d. \tag{10}$$

Moreover under Assumptions 1, 2, 3, 4 and 5, there exists an absolute constant $C(c, \rho, f_\infty)$ such that we have

$$\mathbf{E}\|\xi_{K,h}\|_2^2 \leq \Gamma^2(K, h) + \kappa\|K\|_2^{2d} C(c, \rho, f_\infty) h^{d/2} \tag{11}$$

$$\leq \kappa\|K\|_2^{2d} \left(1 + C(c, \rho, f_\infty) h^{d/2}\right) \tag{12}$$

and for $h \leq \overline{h}_n$ and n large enough we also have:

$$\mathbf{E}\|\xi_{K,h}\|_2 \leq \Gamma(K, h) + \|K\|_2^d \frac{\mathcal{T}}{4}. \tag{13}$$

The proof uses the Bousquet inequality from Boucheron, Lugosi, and Bousquet (2004) (see also Theorem 12.5 in Boucheron, Lugosi, and Massart (2013)). It also makes use of ingredients in Lemma 4.2 in Viennet (1997). Both Lemmas are recalled in Supplemental Material to.

5.2. Proof of Proposition 1

Proof of bound (5) Set $f \in \mathcal{H}_{\mathcal{D}}(\gamma, L)$ and let K be a kernel of order $m = \lfloor \gamma \rfloor$. Using that $|\det(A_x)| = 1$, we have for $x \in \mathcal{D}$

$$\mathcal{B}_{K,h}(f,x) = \mathbf{E}\left(\hat{f}_{K,h}(x)\right) - f(x) = \int_{\mathcal{D}} \mathbf{K}_h \circ A_x(u-x) f(u)du - f(x)$$
$$= \int_{[0,1]^d} \prod_{i=1}^{d} K(s_i) \left[f(x + hA_x^{-1}(s)) - f(x)\right] ds.$$

Using a Taylor expansion we obtain

$$f(x + hA_x^{-1}(s)) - f(x) = \sum_{|\alpha| \le m} \frac{D^\alpha f(x)}{\alpha!} h^{|\alpha|} (A_x^{-1}(s))^\alpha$$
$$+ m \sum_{|\alpha|=m} \frac{h^m (A_x^{-1}(s))^\alpha}{\alpha!} \int_0^1 (1-t)^{m-1} \left[D^\alpha f(x + thA_x^{-1}(s)) - D^\alpha f(x)\right] dt,$$

where for $x \in \mathbb{R}^d$ and $\alpha \in (\mathbb{N} \cup \{0\})^d$ we define $x^\alpha = \prod_{i=1}^{d} x_i^{\alpha_i}$. Using that K is a kernel of order m and that $(A_x^{-1}(s))^\alpha$ is a polynomial in $(s_1, \dots, s_d)$ of degree α, we have

$$|\mathcal{B}_{K,h}(f,x)| \le Lh^\gamma \sum_{|\alpha|=m} \int_{[0,1]^d} \prod_{i=1}^{d} |K(s_i)| |(A_x^{-1}(s))^\alpha| |A_x^{-1}(s)|^{\gamma - m} ds$$
$$\le Lh^\gamma \|K\|_\infty^d C(R,\gamma),$$

where $C(R,\gamma)$ is a positive constant that depends only on R and γ.

Proof of Bound (6) We have

$$\mathbf{E}\|\hat{f}_{K,h} - \mathbf{E}\hat{f}_{K,h}\|_2^2 = \int_{\mathcal{D}} \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{K}_h \circ A_x(X_i - x)\right) dx$$
$$\le \frac{\kappa}{n^2} \sup_{j=1,\dots,\kappa} \int_{\mathcal{D}_j} \mathrm{Var}\left(\sum_{i=1}^{n} \mathbf{K}_h \circ A_j(X_i - x)\right) dx.$$

Then using Lemma 2 in Supplemental Material to, we deduce that there exists a sequence of random variables $(b_k(X_0))_{k \ge 0}$ such that $0 \le b_k(X_0) \le 1$ and $\mathbf{E}(b_k(X_0)) = \beta(k)$ and we have

$$\mathbf{E}\|\hat{f}_{K,h} - \mathbf{E}\hat{f}_{K,h}\|_2^2 \le \frac{4n\kappa}{n^2} \sup_{j=1,\dots,\kappa} \mathbf{E}\left(\sum_{k=0}^{n} b_k(X_0) \int_{\mathcal{D}_j} \mathbf{K}_h^2 \circ A_j(X_0 - x)dx\right)$$
$$\le \frac{4\kappa\|K\|_2^{2d}}{nh^d} \sum_{k=0}^{n} \beta(k) \le \frac{4\kappa\|K\|_2^{2d}}{nh^d} \frac{c}{1-\rho}.$$

Final step Note that (7) and (8) are direct consequences of (11) and (12) of Lemma 5. Finally choosing $h = n^{-1/(2\gamma+d)}$, the estimator $\hat{f}_{K,h}$ reaches the minimax rate $n^{-2\gamma/(2\gamma+d)}$.

5.3. Proof of Theorem 3

To prove Theorem 3, we use Berbee's coupling method as in Viennet (1997) (proof of Proposition 5.1) and proof of Theorem 1 of Comte et al. (2017). We assume $n = 2p_n q_n$ with $q_n = [\log n]^2$. Then there exist random variables X_i^*, $i = 1, ..., n$ satisfying the following properties:

- For $r = 1, ..., p_n$, the random vectors $\vec{U}_{r,1} = (X_{2(r-1)q_n+1}, ..., X_{(2r-1)q_n})^T$ and $\vec{U}^*_{r,1} = (X^*_{2(r-1)q_n+1}, ..., X^*_{(2r-1)q_n})^T$ have the same distribution, and so have the vectors $\vec{U}_{r,2} = (X_{(2r-1)q_n+1}, ..., X_{2rq_n})^T$ and $\vec{U}^*_{r,2} = (X^*_{(2r-1)q_n+1}, ..., X^*_{2rq_n})^T$.
- For $r = 1, ..., p_n$, $\mathbb{P}(\vec{U}_{r,1} \neq \vec{U}^*_{r,1}) \le \beta(q_n)$ and $\mathbb{P}(\vec{U}_{r,2} \neq \vec{U}^*_{r,2}) \le \beta(q_n)$.
- For each $i \in \{1, 2\}$, the random vectors $\vec{U}^*_{1,i}, ..., \vec{U}^*_{p_n,i}$ are independent.

We define $\Omega^* = \{X_i = X_i^*, i = 1, \dots, n\}$ and $\overline{\Omega}^*$ its complementary set in Ω. We have (see Comte et al., 2017)

$$\mathbb{P}(\overline{\Omega}^*) \le 2p_n\beta(q_n) \le n\beta(q_n). \tag{14}$$

Now denoting $\mathbf{K}_\ell(\cdot) = h(\ell)^{-d} \bigotimes_{i=1}^d K_{m_n(\ell)}\left((h(\ell))^{-1}\cdot\right)$, we define $\hat{f}^*_\ell = (\hat{f}^{*(1)}_\ell + \hat{f}^{*(2)}_\ell)/2$ where

$$\hat{f}^{*(1)}_\ell(x) = \frac{2}{n}\sum_{r=1}^{p_n}\sum_{s=1}^{q_n} \mathbf{K}_\ell \circ A_x(X^*_{2(r-1)q_n+s} - x),$$

$$\hat{f}^{*(2)}_\ell(x) = \frac{2}{n}\sum_{r=1}^{p_n}\sum_{s=1}^{q_n} \mathbf{K}_\ell \circ A_x(X^*_{(2r-1)q_n+s} - x).$$

For any kernel K and any bandwith h, let $\xi^{*(1)}_{K,h}(x)$ (*resp.* $\xi^{*(2)}_{K,h}(x)$) defined as

$$2\frac{\sqrt{nh^d}}{n}\sum_{r=1}^{p_n}\sum_{s=1}^{q_n}\left(\mathbf{K}_h \circ A_x(X^*_{2(r-1)q_n+s} - x) - \mathbf{E}\mathbf{K}_h \circ A_x(X^*_{2(r-1)q_n+s} - x)\right),$$

$$2\frac{\sqrt{nh^d}}{n}\sum_{r=1}^{p_n}\sum_{s=1}^{q_n}\left(\mathbf{K}_h \circ A_x(X^*_{(2r-1)q_n+s} - x) - \mathbf{E}\mathbf{K}_h \circ A_x(X^*_{(2r-1)q_n+s} - x)\right),$$

and $\xi^*_{K,h}(x) = \frac{1}{2}\left(\xi^{*(1)}_{K,h}(x) + \xi^{*(2)}_{K,h}(x)\right)$. The two following lemmas give some properties of $\xi^{*(1)}_{K,h}$ and $\xi^{*(2)}_{K,h}$. Lemma 6 follows immediately from Lemma 5.

Lemma 6 *We have* $\mathbf{E}\|\xi^{*(i)}_{K,h}\|_2 \le \left(2\Gamma^2(K,h) + 2\kappa\|K\|_2^{2d}C(c,\rho,f_\infty)h^{d/2}\right)^{1/2}$. *Moreover,* $\exists\, n_0(c,\rho,f_\infty,c_1,\kappa,\tau)$ *such that for* $n \ge n_0(c,\rho,f_\infty,c_1,\kappa,\tau)$ *and* $h \le \overline{h_n}$ *we also have:* $\mathbf{E}\|\xi^{*(i)}_{K,h}\|_2 \le \sqrt{2}\left(\Gamma(K,h) + \|K\|_2^d \frac{\tau}{4}\right)$.

Lemma 7 *For any $\delta > 0$, $x \geq 0$, any $i = 1, 2$ and $n^{-1} \leq h \leq \overline{h}_n$, $\mathbf{P}(\|\xi^{*(i)}_{K,h}\|_2 - \mathbf{E}\|\xi^{*(i)}_{K,h}\|_2 > \delta\|K\|_2^d + x)$ is bounded by*

$$\widetilde{C}_0 \exp\left(-\frac{\widetilde{C}_1 x^2 \overline{h}_n^{-d/2}}{\|K\|_2^{2d} + x\|K\|_2^d}\right) \exp\left(-\widetilde{C}_2 \overline{h}_n^{-d/2}\right)$$

where $\widetilde{C}_0$, $\widetilde{C}_1$, $\widetilde{C}_2$ positive constant that depends only on δ, ρ, c, κ and f_∞.

We are now able to prove the oracle inequality. Set $\ell \in \mathcal{L}_n$. Using the triangular inequality we get:

$$\|f - \widehat{f}\|_2 \leq \|f - \hat{f}_\ell\|_2 + \|\hat{f}_{\widehat{\ell}\wedge\ell} - \hat{f}_\ell\|_2 + \|\hat{f}_{\widehat{\ell}\wedge\ell} - \hat{f}_{\widehat{\ell}}\|_2.$$

Note that if $\ell \geq \widehat{\ell}$, using the definitions of $\widehat{B}(\ell)$ and $\widehat{M}(\ell)$, we easily obtain:

$$\begin{aligned}
\|f - \widehat{f}\|_2 &\leq \|f - \hat{f}_\ell\|_2 + \|\hat{f}_{\widehat{\ell}\wedge\ell} - \hat{f}_\ell\|_2 \leq \|f - \hat{f}_\ell\|_2 + \widehat{B}(\widehat{\ell}) + \widehat{M}(\widehat{\ell}, \ell) \\
&\leq \|f - \hat{f}_\ell\|_2 + \widehat{B}(\widehat{\ell}) + \widehat{M}(\ell) + \widehat{M}(\widehat{\ell}) \\
&\leq \|f - \hat{f}_\ell\|_2 + 2\left(\widehat{B}(\ell) + \widehat{M}(\ell)\right).
\end{aligned}$$

Last inequality comes from the definition of $\widehat{\ell}$. The same bound remains valid if $\ell \leq \widehat{\ell}$. This implies:

$$\left(\mathbf{E}\|\widehat{f} - f\|_2^2\right)^{1/2} \leq \left(\mathbf{E}\|\hat{f}_\ell - f\|_2^2\right)^{1/2} + 2\left(\mathbf{E}\widehat{B}^2(\ell)\right)^{1/2} + 2\left(\mathbf{E}\widehat{M}^2(\ell)\right)^{1/2}. \quad (15)$$

It remains to bound each term of the right hand side of this inequality.

1 We have:

$$\begin{aligned}
\left(\mathbf{E}\|\hat{f}_\ell - f\|_2^2\right)^{1/2} &\leq \|\mathbf{E}\hat{f}_\ell - f\|_2 + \left(\mathbf{E}\|\hat{f}_\ell - \mathbf{E}\hat{f}_\ell\|_2^2\right)^{1/2} \\
&\leq \|\mathbf{E}\hat{f}_\ell - f\|_2 + \sqrt{\kappa}\|K_{m_n(\ell)}\|_2^d \sqrt{\frac{1}{nh^d(\ell)} + \frac{C(\rho, c, f_\infty)}{nh^{d/2}(\ell)}}. \quad (16)
\end{aligned}$$

Last line follows from Bound (8) of Proposition 1.

2 Using triangular inequality and (10), last term can be bounded by:

$$\left(\mathbf{E}\widehat{M}^2(\ell)\right)^{1/2} \leq \sqrt{2}\, M(\ell) \leq \frac{\sqrt{2}(\sqrt{\kappa} + \tau)\|K_{m_n(\ell)}\|_2^d}{\sqrt{nh^d(\ell)}}. \quad (17)$$

3 Remark that, using the triangular inequality, we have:

$$\begin{aligned}
\widehat{B}(\ell) &\leq 2 \max_{\ell' \in \mathcal{L}_n} \left\{\|\hat{f}_{\ell'} - \mathbf{E}\hat{f}_{\ell'}\|_2 - \widehat{M}(\ell')\right\}_+ + \max_{\ell' \in \mathcal{L}_n} \|\mathbf{E}\hat{f}_{\ell'} - \mathbf{E}\hat{f}_{\ell\wedge\ell'}\|_2 \\
&\leq 2 \max_{\ell' \in \mathcal{L}_n} \left\{\|\hat{f}_{\ell'} - \mathbf{E}\hat{f}_{\ell'}\|_2 - \widehat{M}(\ell')\right\}_+ + 2\max_{\ell' \geq \ell} \|\mathbf{E}\hat{f}_{\ell'} - f\|_2 \\
&\leq \max_{\ell' \in \mathcal{L}_n} \left\{\|\hat{f}^{*(1)}_{\ell'} - \mathbf{E}\hat{f}^{*(1)}_{\ell'}\|_2 - \widehat{M}(\ell')\right\}_+ + \max_{\ell' \in \mathcal{L}_n} \left\{\|\hat{f}^{*(2)}_{\ell'} - \mathbf{E}\hat{f}^{*(2)}_{\ell'}\|_2 - \widehat{M}(\ell')\right\}_+ \\
&\quad + 2\max_{\ell' \geq \ell} \|\mathbf{E}\hat{f}_{\ell'} - f\|_2 + 2\max_{\ell' \in \mathcal{L}_n} \|\hat{f}^*_{\ell'} - \hat{f}_{\ell'}\|_2.
\end{aligned}$$

This implies that $\left(\mathbf{E}\widehat{B}^2(\ell)\right)^{1/2}$ is bounded by:

$$2(\mathbf{E}(\max_{\ell'\in\mathcal{L}_n}\|\hat{f}^*_{\ell'}-\hat{f}_{\ell'}\|_2)^2)^{1/2}+2\max_{\ell'\geq\ell}\|\mathbf{E}\hat{f}_{\ell'}-f\|_2+(\sqrt{\Delta_n^{(1)}}+\sqrt{\Delta_n^{(2)}}), \quad (18)$$

where for $i=1,2$ $\Delta_n^{(i)}=\mathbf{E}\left(\max_{\ell'\in\mathcal{L}_n}\left\{\|\hat{f}^{*(i)}_{\ell'}-\mathbf{E}\hat{f}^{*(i)}_{\ell'}\|_2-\widehat{M}(\ell')\right\}_+\right)^2$. It remains to study the terms of the right hand side of (18).

4. Study of $\Delta_n^{(i)}$ First define for $m\in\mathbb{N}$, the kernel $K^*_m=K^2_m/\|K_m\|_2^2$ and consider, for $\ell'\in\mathcal{L}_n$ the event

$$\mathcal{D}_{\ell'}=\left\{\|\xi_{K^*_{m_n(\ell')},h(\ell')}\|_1^{1/2}\leq\frac{\tau}{2}\left(nh^d(\ell')\right)^{1/4}\right\}.$$

Now, remark that, on the event $\mathcal{D}_{\ell'}$ we have:

$$\begin{aligned}|\widehat{M}(\ell')-\sqrt{2}\,M(\ell')|&=\sqrt{2}\left|\frac{\hat{\Gamma}(\ell')-\Gamma(\ell')}{\sqrt{nh^d(\ell')}}\right|\\&=\sqrt{2}\frac{\|K_{m_n(\ell')}\|_2^d}{\sqrt{nh^d(\ell')}}\left|\|\hat{f}_{K^*_{m_n(\ell')},h(\ell')}\|_1^{1/2}-(\mathbf{E}\|\hat{f}_{K^*_{m_n(\ell')},h(\ell')}\|_1)^{1/2}\right|\\&\leq\sqrt{2}\frac{\|K_{m_n(\ell')}\|_2^d}{\sqrt{nh^d(\ell')}}\left((nh^d(\ell'))^{-1/2}\|\xi_{K^*_{m_n(\ell')},h(\ell')}\|_1\right)^{1/2}\\&\leq\sqrt{2}\frac{\|K_{m_n(\ell')}\|_2^d}{(nh^d(\ell'))^{3/4}}\|\xi_{K^*_{m_n(\ell')},h(\ell')}\|_1^{1/2}\leq\sqrt{2}\frac{\tau\|K_{m_n(\ell')}\|_2^d}{2(nh^d(\ell'))^{1/2}}.\end{aligned}$$

This implies that, on $\mathcal{D}_{\ell'}$

$$\widehat{M}(\ell')\geq\sqrt{2}\frac{\Gamma(\ell')+\frac{\tau}{2}\|K_{m_n(\ell')}\|_2^d}{(nh^d(\ell'))^{1/2}}. \quad (19)$$

Using (19), we obtain for $i=1,2$

$$\begin{aligned}&\left\{\|\hat{f}^{*(i)}_{\ell'}-\mathbf{E}\hat{f}^{*(i)}_{\ell'}\|_2-\widehat{M}(\ell')\right\}_+\\&\quad\leq\|\hat{f}^{*(i)}_{\ell'}-\mathbf{E}\hat{f}^{*(i)}_{\ell'}\|_\infty\mathbf{I}_{\bar{\mathcal{D}}_{\ell'}}+\left\{\|\hat{f}^{*(i)}_{\ell'}-\mathbf{E}\hat{f}^{*(i)}_{\ell'}\|_2-\widehat{M}(\ell')\right\}_+\mathbf{I}_{\mathcal{D}_{\ell'}},\\&\quad\leq\frac{2\|K_{m_n(\ell')}\|_\infty^d}{\underline{h}_n^d}\mathbf{I}_{\bar{\mathcal{D}}_{\ell'}}+\left\{\|\hat{f}^{*(i)}_{\ell'}-\mathbf{E}\hat{f}^{*(i)}_{\ell'}\|_2-\sqrt{2}\frac{\Gamma(\ell')+\frac{\tau}{2}\|K_{m_n(\ell')}\|_2^d}{(nh(\ell')^d)^{1/2}}\right\}_+.\end{aligned}$$

Now, using triangular inequality

$$\left(\Delta_n^{(i)}\right)^{1/2}\leq\sum_{\ell'\in\mathcal{L}_n}\left(\mathbf{E}\left\{\|\hat{f}^{*(i)}_{\ell'}-\mathbf{E}\hat{f}^{*(i)}_{\ell'}\|_2-\widehat{M}(\ell')\right\}_+^2\right)^{1/2}\leq$$

$$\sum_{\ell'\in\mathcal{L}_n} \frac{2\|K_{m_n(\ell')}\|_\infty^d}{\underline{h}_n^d}\sqrt{\mathbf{P}(\bar{\mathcal{D}}_{\ell'})}+\sqrt{\mathbf{E}\left\{\|\hat{f}^{*(i)}_{\ell'}-\mathbf{E}\hat{f}^{*(i)}_{\ell'}\|_2-\sqrt{2}\frac{\Gamma(\ell')+\frac{\tau}{2}\|K_{m_n(\ell')}\|_2^d}{(nh(\ell')^d)^{1/2}}\right\}_+^2} \tag{20}$$

Assume that here and after we have $n\geq n_0(c,\rho,f_\infty,c_1,\kappa,\tau)$. We consider:

$$\begin{aligned}
&\mathbf{E}\left\{\|\hat{f}^{*(i)}_{\ell'}-\mathbf{E}\hat{f}^{*(i)}_{\ell'}\|_2-\sqrt{2}\frac{\Gamma(\ell')+\frac{\tau}{2}\|K_{m_n(\ell')}\|_2^d}{(nh^d(\ell'))^{1/2}}\right\}_+^2\\
&\quad=\frac{1}{nh^d(\ell')}\mathbf{E}\left\{\|\xi^{*(i)}_{K_{m_n(\ell')},h(\ell')}\|_2-\sqrt{2}\Gamma(\ell')-\sqrt{2}\frac{\tau}{2}\|K_{m_n(\ell')}\|_2^d\right\}_+^2\\
&\quad=\frac{1}{nh^d(\ell')}\int_0^{+\infty}\mathbf{P}\left(\|\xi^{*(i)}_{K_{m_n(\ell')},h(\ell')}\|_2-\sqrt{2}\Gamma(\ell')-\sqrt{2}\frac{\tau}{2}\|K_{m_n(\ell')}\|_2^d>x^{1/2}\right)dx
\end{aligned}$$

bounded by

$$\int_0^{+\infty}\frac{2x}{nh^d(\ell')}\mathbf{P}\left(\|\xi^{*(i)}_{K_{m_n(\ell')},h(\ell')}\|_2-\mathbf{E}\|\xi^{*(i)}_{K_{m_n(\ell')},h(\ell')}\|_2>\frac{\tau}{\sqrt{2}}\|K_{m_n(\ell')}\|_2^d+x-\mathcal{H}(\ell')\right)dx\,.$$

where $\mathcal{H}(\ell')=\frac{\tau}{2\sqrt{2}}\|K_{m_n(\ell')}\|_2^d$. Last line follows from Lemma 6. Using Lemma 7 with $\delta=\tau/(2\sqrt{2})$ and $x=0$ we obtain:

$$\begin{aligned}
&\mathbf{E}\left\{\|\hat{f}^{*(i)}_{\ell'}-\mathbf{E}\hat{f}^{*(i)}_{\ell'}\|_2-\sqrt{2}\frac{\Gamma(\ell')+\frac{\tau}{2}\|K_{m_n(\ell')}\|_2^d}{(nh(\ell')^d)^{1/2}}\right\}_+^2\\
&\quad\leq\frac{2\widetilde{C}_0\exp\left(-\widetilde{C}_2\overline{h}_n^{-d/2}\right)}{nh^d(\ell')}\int_0^{+\infty}x\exp\left(-\widetilde{C}_1\frac{x^2\overline{h}_n^{-d/2}}{\|K_{m_n(\ell')}\|_2^{2d}+x\|K_{m_n(\ell')}\|_2^d}\right)dx\\
&\quad\leq\frac{2\widetilde{C}_0\exp\left(-\widetilde{C}_2\overline{h}_n^{-d/2}\right)}{nh^d(\ell')}\int_0^{+\infty}x\exp\left(-\widetilde{C}_1\frac{x^2\overline{h}_n^{-d/2}}{\alpha_n^2+x\alpha_n}\right)dx
\end{aligned}$$

where $\alpha_n=A^d(\log n+3/2)^{Bd}$. This follows from the fact that the kernel $K_{m_n(\ell')}$ satisfies Assumption 7 combined with the expression of $m_n(\ell')$. Now, splitting the integral into two terms, depending on the position of x with respect to α_n, we obtain:

$$\begin{aligned}
&\mathbf{E}\left\{\|\hat{f}^{*(i)}_{\ell'}-\mathbf{E}\hat{f}^{*(i)}_{\ell'}\|_2-\sqrt{2}\frac{\Gamma(\ell')+\frac{\tau}{2}\|K_{m_n(\ell')}\|_2^d}{(nh(\ell')^d)^{1/2}}\right\}_+^2\\
&\qquad\leq\frac{\widetilde{C}_3}{2}\frac{\exp\left(-\widetilde{C}_2\overline{h}_n^{-d/2}\right)}{n\underline{h}_n^d}\alpha_n^2\left(\overline{h}_n^{d/2}+\overline{h}_n^d\right)\leq\widetilde{C}_3\frac{\exp\left(-\widetilde{C}_2\overline{h}_n^{-d/2}\right)}{n\underline{h}_n^d}\alpha_n^2\overline{h}_n^{d/2}
\end{aligned}$$

where $\widetilde{C}_3$ is a positive constant that depends on τ, ρ, c, κ, A, B and f_∞. This implies that there exists $\widetilde{C}_4$ is a positive constant that depends on τ, ρ, c, κ, A,

B and f_∞ such that we have:

$$\sum_{\ell'\in\mathcal{L}_n}\left(\mathbf{E}\left\{\|\hat{f}^{*(i)}_{\ell'}-\mathbf{E}\hat{f}^{*(i)}_{\ell'}\|_2-\sqrt{2}\frac{\Gamma(\ell')+\frac{\tau}{2}\|K_{m_n(\ell')}\|_2^d}{(nh(\ell')^d)^{1/2}}\right\}_+^2\right)^{1/2}\leq \widetilde{C}_4 n^{-1/2}. \tag{21}$$

It remains to bound $\sum_{\ell'\in\mathcal{L}_n}\frac{2\|K_{m_n(\ell')}\|_\infty^d}{\underline{h}_n^d}\sqrt{\mathbf{P}(\bar{\mathcal{D}}_{\ell'})}$. As $\|\cdot\|_1\leq\sqrt{\mathrm{Vol}(\mathcal{D})}\|\cdot\|_2$:

$$\begin{aligned}\mathbf{P}(\bar{\mathcal{D}}_{\ell'}) &\leq \mathbf{P}\left(\|\xi_{K^*_{m_n(\ell')},h(\ell')}\|_2\geq\frac{\tau^2(nh^d(\ell'))^{1/2}}{4\sqrt{\mathrm{Vol}(\mathcal{D})}}\right)\\ &\leq 2\mathbf{P}\left(\|\xi^{*(1)}_{K^*_{m_n(\ell')},h(\ell')}\|_2\geq\frac{2}{3}\mathcal{U}(\ell')\right)+3\mathbf{E}\|\xi_{K^*_{m_n(\ell')},h(\ell')}-\xi^*_{K^*_{m_n(\ell')},h(\ell')}\|_2\frac{1}{\mathcal{U}(\ell')}\end{aligned}$$

with $\mathcal{U}(\ell')=\frac{\tau^2(nh^d(\ell'))^{1/2}}{4\sqrt{\mathrm{Vol}(\mathcal{D})}}$. Then, using Lemma 5 and Lemma 6, we have:

$$\mathbf{E}\|\xi^{*(1)}_{K^*_{m_n(\ell')},h(\ell')}\|_2+\|K^*_{m_n(\ell')}\|_2^d\leq\left(\frac{\tau}{2\sqrt{2}}+\sqrt{2}\sqrt{\kappa}+1\right)\|K^*_{m_n(\ell')}\|_2^d.$$

Now using that $\|K_{m_n(\ell')}\|_2\geq 1$ and Assumption 7, we deduce that

$$\|K^*_{m_n(\ell')}\|_2^d=\frac{\|K_{m_n(\ell')}\|_4^{2d}}{\|K_{m_n(\ell')}\|_2^{2d}}\leq\|K_{m_n(\ell')}\|_\infty^{2d}\leq A^{2d}(m_n(\ell')+1)^{2Bd}\leq A^{2d}\left(\frac{\log n}{2\ell'}+\frac{3}{2}\right)^{2Bd}.$$

Now remark that:

$$\left(\frac{\log n}{2\ell'}+\frac{3}{2}\right)^{2Bd}\leq(nh^d(\ell'))^{1/2}\left(\frac{n^{-1/(4Bd)}\log n}{2\ell' e^{-\ell'/(4B)}}+\frac{3(\log n)^{-c_2/(4Bd)}}{2}\right)^{2Bd}.$$

Moreover it is easily seen that, for n large enough:

$$\begin{aligned}\frac{n^{-1/(4Bd)}\log n}{2\ell' e^{-\ell'/(4B)}} &\leq\max\left(\frac{n^{-1/(4Bd)}\log n}{2e^{-1/(4B)}},\frac{n^{-1/(4Bd)}\log n}{2\ell^* e^{-\ell^*/(4B)}}\right)\\ &\leq\max\left(\frac{n^{-1/(4Bd)}\log n}{2e^{-1/(4B)}},d(\log n)^{-c_2/(4Bd)}\right)\\ &\leq d(\log n)^{-c_2/(4Bd)},\end{aligned}$$

where, $\ell^*=[\frac{1}{d}(\log n-c_2\log\log n)]$ is such that, for n large enough:

$$\frac{\log n}{2d}\leq\ell^*\leq\frac{1}{d}\left(\log n-c_2\log\log n\right).$$

Then we obtain for n large enough:

$$\begin{aligned}&\mathbf{E}\|\xi^{*(1)}_{K^*_{m_n(\ell')},h(\ell')}\|_2+\|K^*_{m_n(\ell')}\|_2^d\\ &\leq\sqrt{nh^d(\ell')}\left(\tfrac{\tau}{2\sqrt{2}}+\sqrt{2}\sqrt{\kappa}+1\right)A^{2d}\left(d+\tfrac{3}{2}\right)^{2Bd}(\log n)^{-c_2/2}\leq\tfrac{\tau^2(nh^d(\ell'))^{1/2}}{6\sqrt{\mathrm{Vol}(\mathcal{D})}}.\end{aligned}$$

This implies, using Lemma 7 with $\delta = 1$ and $x = 0$:

$$\exists \widetilde{C}_5 > 0 \text{ s.t. } \mathbf{P}\left(\|\xi^{*(1)}_{K^*_{m_n(\ell')},h(\ell')}\|_2 \geq \frac{2}{3}\mathcal{U}(\ell')\right) \leq \widetilde{C}_5 \exp(-\widetilde{C}_2 \overline{h}_n^{-d/2}).$$

We also have:

$$\begin{aligned}
&\mathbf{E}\|\xi_{K^*_{m_n(\ell')},h(\ell')} - \xi^*_{K^*_{m_n(\ell')},h(\ell')}\|_2 \\
&\leq \sqrt{nh(\ell')^d}\sqrt{\frac{4}{h(\ell')^d}\|K^*_{m_n(\ell')}\|_2^d\,\mathbf{P}(\overline{\Omega}^*)} \leq 2n\sqrt{n}\beta(q_n)\|K^*_{m_n(\ell')}\|_\infty^{2d}.
\end{aligned} \tag{22}$$

Thus we have:

$$\mathbf{P}(\bar{\mathcal{D}}_{\ell'}) \leq 2\widetilde{C}_5 \exp(-\widetilde{C}_2(\overline{h}_n)^{-d/2}) + 2n\beta(q_n)\sqrt{n}\|K^*_{m_n(\ell')}\|_\infty^{2d}\frac{3\times 4\sqrt{\mathrm{Vol}(\mathcal{D})}}{\tau^2\sqrt{nh^d}}.$$

This implies that we have that $\sum_{\ell'\in\mathcal{L}_n}\frac{2\|K_{m_n(\ell')}\|_\infty^d}{\underline{h}_n^d}\sqrt{\mathbf{P}(\bar{\mathcal{D}}_{\ell'})}$ is bounded by:

$$\begin{aligned}
&\textstyle\sum_{\ell'\in\mathcal{L}_n}\frac{2\|K_{m_n(\ell')}\|_\infty^d}{\underline{h}_n^d}\sqrt{2\widetilde{C}_5\exp(-\widetilde{C}_2\overline{h}_n^{-d/2})} \\
&\qquad + \sum_{\ell'\in\mathcal{L}_n}\frac{2\|K_{m_n(\ell')}\|_\infty^{2d}}{\underline{h}_n^d}\sqrt{2n\sqrt{n}\beta(q_n)}\frac{\sqrt{3}\times 2(\mathrm{Vol}(\mathcal{D}))^{1/4}}{\tau(nh^d)^{1/4}}.
\end{aligned}$$

Thus there exists a positive constant $\widetilde{C}_6$ such that

$$\sum_{\ell'\in\mathcal{L}_n}\frac{2\|K_{m_n(\ell')}\|_\infty^d}{\underline{h}_n^d}\sqrt{\mathbf{P}(\bar{\mathcal{D}}_{\ell'})} \leq \widetilde{C}_6 n^{-1/2} \tag{23}$$

as $q_n = [\log n]^2$. Using (20), (21) and (23), we can conclude for $i = 1, 2$ that

$$\left(\Delta_n^{(i)}\right)^{1/2} \leq (\widetilde{C}_4 + \widetilde{C}_6)n^{-1/2} \tag{24}$$

5. Final step Using (14) and proceeding as in (22) we obtain that

$$\sqrt{\mathbf{E}\left(\max_{\ell'\in\mathcal{L}_n}\|\hat{f}^*_{\ell'} - \hat{f}_{\ell'}\|_2\right)^2} \leq \widetilde{C}_7 n^{-1/2} \tag{25}$$

and taking (18), (24) and (25) together we obtain:

$$\left(\mathbf{E}\hat{B}^2(\ell)\right)^{1/2} \leq \widetilde{C}_8 n^{-1/2} + 2\max_{\ell'\geq\ell}\|\mathbf{E}\hat{f}_{\ell'} - f\|_2 \tag{26}$$

with $\widetilde{C}_7$ and $\widetilde{C}_8$ are positive constants depending on τ, c, ρ, f_∞, κ, A, B.

6. Conclusion Combining (15), (16), (17), (26) we finally obtain

$$\left(\mathbf{E}\|\hat{f}-f\|_2^2\right)^{1/2} \leq 5\max_{\ell'\geq\ell}\|\mathbf{E}\hat{f}_{\ell'}-f\|_2 + \widetilde{C}_9\left(\frac{\|K_{m_n(\ell)}\|_2^d}{\sqrt{nh^d(\ell)}}+n^{-1/2}\right)$$

with $\widetilde{C}_9$ a positive constant depending on τ, c, ρ, f_∞, κ, A, B. We thus get the result with $\widetilde{\Upsilon}_1 = 5+\widetilde{C}_9$ and $\widetilde{\Upsilon}_2 = \widetilde{C}_9$.

5.4. Proof of Theorem 4

Set $\alpha>0$ and $L>0$ and let $f\in\mathcal{H}_{\mathcal{D}}(\alpha,L)$. Define $\ell_0 = \lceil(2\alpha+d)^{-1}\log n\rceil$ where for $x\in\mathbb{R}$ $\lceil x\rceil$ is the smallest integer greater or equal to x. Since ℓ_0 belongs to $\mathcal{L}_n$ for n large enough, Theorem 3 implies that we only have to bound the two following quantities:

$$\max_{\ell'\geq\ell_0}\|\mathbf{E}\hat{f}_{\ell'}-f\|_2 \quad\text{and}\quad \frac{\|K_{m_n(\ell_0)}\|_2^d}{\sqrt{nh^d(\ell_0)}}.$$

Defining $h_0 = n^{-\frac{1}{2\alpha+d}}$ we have:

$$e^{-1}h_0 < h(\ell_0)\leq h_0 \quad\text{and}\quad m_n(\ell_0)\leq \alpha+d/2+\frac{1}{2} \tag{27}$$

This, implies:

$$\|K_{m_n(\ell_0)}\|_2 \leq A\left(\alpha+d/2+\frac{3}{2}\right)^B. \tag{28}$$

Using (27) and (28), we obtain that $\frac{\|K_{m_n(\ell_0)}\|_2^d}{\sqrt{nh^d(\ell_0)}} \leq Cn^{-\alpha/(2\alpha+d)}$, where C depends on A, B and α. It remains to bound the bias term. Set $\ell'\geq\ell_0$. Remark that: $m_n(\ell')\leq m_n(\ell_0)$, $h(\ell')\leq h(\ell_0)$ and $\|K_{m_n(\ell')}\|_\infty \leq A\left(\alpha+d/2+\frac{3}{2}\right)^B$. We consider two cases.

Case 1 Assume that $m_n(\ell')\geq\lfloor\alpha\rfloor$. Using Proposition 1 we obtain that

$$\|\mathbf{E}\hat{f}_{\ell'}-f\|_2 \leq C(\alpha,R)\|K_{m_n(\ell')}\|_\infty^d L(h(\ell'))^\alpha \leq Cn^{-\alpha/(2\alpha+d)}$$

where C depends on L, α, R, τ, κ, A and B.

Case 2 Assume now that $m_n(\ell')<\lfloor\alpha\rfloor$. Define $\alpha' = m_n(\ell')+1\leq\alpha$ and note that there exists L' that depends on α, L and $\mathcal{D}$ such that $f\in\mathcal{H}_{\mathcal{D}}(\alpha',L')$. Using Proposition 1 we have:

$$\begin{aligned}\|\mathbf{E}\hat{f}_{\ell'}-f\|_2 \leq & C(R,\alpha')\|K_{m_n(\ell')}\|_\infty^d L(h(\ell'))^{\alpha'} \leq C\exp\left(-\ell'(m_n(\ell')+1)\right)\\ \leq & C\exp\left(-\ell'\left(\frac{\log n}{2\ell'}+\frac{1}{2}\right)\right)\leq Cn^{-1/2}\end{aligned}$$

where C depends on L, α, R, τ, κ, A and B. Theorem follows.

5.5. Proof of lemma 5

We have

$$\Gamma^2(K,h) = h^d \sum_{i=1}^{\kappa} \int_{\mathcal{D}_i} \int_{\mathcal{D}} |\mathbf{K}_h \circ A_i(u-x)|^2 f(u) du dx$$

$$\leq \sum_{i=1}^{\kappa} \int_{[0,1]^d} \int_{\mathcal{D}} \prod_{j=1}^{d} K^2(v_j) f(u) du dv \leq \kappa \|K\|_2^{2d}. \tag{29}$$

We have $\mathbf{E}\|\xi_{K,h}\|_2^2 = h^d \left\{ \int_{\mathcal{D}} \operatorname{Var} \mathbf{K}_h \circ A_x(X_1 - x)dx + 2\sum_{k=1}^{n-1} \frac{(n-k)}{n} \mathfrak{c}(k) \right\}$ where $\mathfrak{c}(k) = \int_{\mathcal{D}} \operatorname{Cov}\left(\mathbf{K}_h \circ A_x(X_1 - x), \mathbf{K}_h \circ A_x(X_{k+1} - x)\right) dx$.

The variance term can be easily bounded since

$$\int_{\mathcal{D}} \operatorname{Var} \mathbf{K}_h \circ A_x(X_1 - x)dx \leq \int_{\mathcal{D}} \mathbf{E}|\mathbf{K}_h \circ A_x(X_1 - x)|^2 dx = \frac{\Gamma^2(K,h)}{h^d}. \tag{30}$$

Concerning the covariance terms, on one hand we bound, using Assumption 3 the term $\mathfrak{c}(k)$ by

$$\int_{\mathcal{D}} \mathbf{E}|\mathbf{K}_h \circ A_x(X_1 - x)\mathbf{K}_h \circ A_x(X_{k+1} - x)|dx + \int_{\mathcal{D}} \left(\mathbf{E}|\mathbf{K}_h \circ A_x(X_1 - x)|\right)^2 dx$$

$$\leq \sum_{i=1}^{\kappa} \int_{\mathcal{D}_i} \int_{\mathcal{D}} |\mathbf{K}_h \circ A_x(u-x)||\mathbf{K}_h \circ A_x(v-x)| f_k(u,v) du dv dx$$

$$+ \int_{\mathcal{D}} \left(\mathbf{E}|\mathbf{K}_h \circ A_x(X_1 - x)|\right)^2 dx$$

$$\leq f_\infty \kappa \|K\|_1^{2d} + \sum_{i=1}^{\kappa} \int_{\mathcal{D}_i} \mathbf{E}|\mathbf{K}_h \circ \mathbf{A}_i(X_1 - x)| \int_{\mathcal{D}} |\mathbf{K}_h \circ \mathbf{A}_i(u-x)| f(u) du dx$$

$$\leq 2\kappa f_\infty \|K\|_1^{2d} \leq 2\kappa f_\infty \|K\|_2^{2d}. \tag{31}$$

On the other hand, using Lemma 2 in Supplemental Material to, we get:

$$\mathfrak{c}(k) \leq \frac{2\kappa \|K\|_2^{2d} c\rho^k}{h^d}. \tag{32}$$

Combining (31) with (32), we obtain that for any $a \in (0,1)$

$$\sum_{k=1}^{n-1} \mathfrak{c}(k) \leq \frac{2\kappa \|K\|_2^{2d} f_\infty^{1-a} (c\rho)^a}{h^{da}(1-\rho^a)}. \tag{33}$$

Thus, using (29), (30) and (33) with $a = 1/2$, we get

$$\mathbf{E}\|\xi_{K,h}\|_2^2 \leq \Gamma^2(K,h) + \frac{4\kappa \|K\|_2^{2d} \sqrt{f_\infty}\sqrt{c\rho}}{(1-\sqrt{\rho})} h^{d/2} \leq \kappa \|K\|_2^{2d} \left(1 + \frac{4\sqrt{f_\infty}\sqrt{c\rho}}{1-\sqrt{\rho}} h^{d/2}\right)$$

which concludes the proof of (11) and (12). Finally (13) follows from previous inequality and from the fact that $h \leq \overline{h}_n$.

5.6. Proof of lemma 7

Define $Y^{*1} = \|\xi^{*(1)}_{K,h}\|_2$. Using duality arguments and Banach-Alaoglu theorem, there exists a countable set $\Lambda = (\lambda_k)_{k\in\mathbb{N}}$ of functions such that $\|\lambda_k\|_2 \leq 1$ and:

$$\begin{aligned}
Y^{*1} &= \sup_{k\in\mathbb{N}} \int_{\mathcal{D}} \lambda_k(t)\xi^{*(1)}_{K,h}(t)dt \\
&= \sup_{k\in\mathbb{N}} \frac{1}{\sqrt{n/2}} \sum_{r=1}^{p_n}\sum_{s=1}^{q_n} \left(g_{\lambda_k}(X^*_{2(r-1)q_n+s}) - \mathbf{E}g_{\lambda_k}(X^*_{2(r-1)q_n+s})\right) \\
&= \sup_{k\in\mathbb{N}} \frac{1}{\sqrt{p_n}} \sum_{r=1}^{p_n} \left(g_{\lambda_k,q_n}(\vec{U}^*_{r,1}) - \mathbf{E}g_{\lambda_k,q_n}(\vec{U}^*_{r,1})\right),
\end{aligned}$$

where for any $\lambda \in \Lambda$, we define

$$g_{\lambda,q_n}(x_1,\dots,x_{q_n}) = \sqrt{2}h^{d/2}\frac{1}{\sqrt{q_n}}\int_{\mathcal{D}}\lambda(t)\sum_{s=1}^{q_n}\mathbf{K}_h\circ A_t(x_s - t)dt \text{ and}$$

$$\overline{g}_{\lambda,q_n}(x_1,\dots,x_{q_n}) = \sqrt{2}h^{d/2}\frac{1}{\sqrt{q_n}}\int_{\mathcal{D}}\lambda(t)\sum_{s=1}^{q_n}\left(\mathbf{K}_h\circ A_t(x_s - t) - \mathbf{E}\mathbf{K}_h\circ A_t(X_1 - t)\right)dt.$$

Fix $\lambda \in \Lambda$. We then have $\|\overline{g}_{\lambda,q_n}\|_\infty \leq 2\|g_{\lambda,q_n}\|_\infty$ and

$$\begin{aligned}
|g_{\lambda,q_n}(x_1,\dots,x_{q_n})| &\leq \sqrt{2}h^{d/2}\frac{1}{\sqrt{q_n}}\sum_{j=1}^{\kappa}\left|\int_{\mathcal{D}_j}\lambda(t)\sum_{s=1}^{q_n}\mathbf{K}_h\circ A_j(x_s - t)dt\right| \\
&\leq \sqrt{2}h^{d/2}\kappa\sqrt{q_n}\sup_{j=1,\dots,\kappa}\|\lambda\star(\mathbf{K}_h\circ A_j)\|_\infty \\
&\leq \sqrt{2}h^{d/2}\kappa\sqrt{q_n}h^{-d/2}\|K\|_2^d \quad \leq \sqrt{2}\kappa\sqrt{q_n}\|K\|_2^d.
\end{aligned}$$

This implies that $\|\overline{g}_{\lambda,q_n}\|_\infty \leq \mathfrak{b}$ where $\mathfrak{b} = 2\sqrt{2}\kappa\sqrt{q_n}\|K\|_2^d$. Moreover:

$$\begin{aligned}
\mathbf{E}\left(\overline{g}^2_{\lambda,q_n}(\vec{U}^*_{r,1})\right) &= \frac{2h^d}{q_n}\operatorname{Var}\left(\sum_{s=1}^{q_n}\int_{\mathcal{D}}\lambda(t)(\mathbf{K}_h\circ A_t)(X^*_{2(r-1)q_n+s} - t)dt\right) \\
&\leq \frac{2\kappa^2h^d}{q_n}\sup_{j=1,\dots,\kappa}\operatorname{Var}\left(\sum_{s=1}^{q_n}\int_{\mathcal{D}_j}\lambda(t)(\mathbf{K}_h\circ\mathbf{A}_j)(X_s - t)dt\right).
\end{aligned}$$

Fix $j \in \{1,\dots,\kappa\}$. Denote $\Psi = \lambda\star(\mathbf{K}_h\circ\mathbf{A}_j)$. Using these notations we have:

$$\begin{aligned}
\operatorname{Var}\left(\sum_{s=1}^{q_n}\int_{\mathcal{D}_j}\lambda(t)(\mathbf{K}_h\circ\mathbf{A}_j)(X_s - t)dt\right) &= \operatorname{Var}\left(\sum_{s=1}^{q_n}\Psi(X_s)\right) \\
&\leq 4q_n\left(\mathbf{E}\Psi^4(X_1)\right)^{1/2}\left(2\sum_{k=1}^{+\infty}(k+1)\beta(k)\right)^{1/2}.
\end{aligned}$$

Last line is deduced from the third inequations in Lemma 2 in Supplemental Material to with $p = p' = 2$. Using Young's inequality for convolution products with $r = 4$, $p = 4/3$ and $q = 2$, we obtain:

$$\mathbf{E}\Psi^4(X_1) \leq f_\infty \|\Psi\|_4^4 \leq f_\infty \left(\|\lambda\|_2 \cdot \|\mathbf{K}_h \circ \mathbf{A}_j\|_{4/3}\right)^4 .$$

Since $\|\lambda\|_2 = 1$, this leads to $\mathbf{E}\Psi^4(X_1) \leq f_\infty h^{-d} \|K\|_{4/3}^{4d} \leq f_\infty h^{-d} \|K\|_2^{4d}$.

Finally we obtain the following bound:

$$\mathbf{E}\left(\overline{g}^2_{\lambda,q_n}(\vec{U}^*_{r,1})\right) \leq \mathfrak{a} \quad \text{where} \quad \mathfrak{a} = 8\kappa^2 \left(2\sum_{k=1}^{+\infty}(k+1)\beta(k)\right)^{1/2} f_\infty^{1/2} \|K\|_2^{2d} h^{d/2}.$$

Using Lemma 1 in Supplemental Material to, with $X_{r,\lambda} = \overline{g}_{\lambda,q_n}(\vec{U}^*_{r,1})\frac{\sqrt{p_n}}{\mathfrak{b}}$ we get:

$$\mathbf{P}(Y^{*1} - \mathbf{E}Y^{*1} > \delta\|K\|_2^d + x) \leq \exp\left(-\frac{(x + \delta\|K\|_2^d)^2}{2\mathfrak{a} + \frac{4\mathfrak{b}\mathbf{E}Y^{*1}}{\sqrt{p_n}} + \frac{2\mathfrak{b}}{3\sqrt{p_n}}(x + \delta\|K\|_2^d)}\right).$$

Lemma 6 with Lemma 5 implies that

$$\mathbf{E}Y^{*1} \leq \sqrt{2}\sqrt{\kappa}\|K\|_2^d \sqrt{1 + C(c, \rho, f_\infty)h^{d/2}}.$$

This, combined with basic calculations, implies that there exist positive constants $\widetilde{C}$, $\widetilde{C}_0$, $\widetilde{C}_1$, $\widetilde{C}_2$ that depend only on B, δ, κ, c, ρ and f_∞ such that:

$$\begin{aligned}
&\mathbf{P}(Y^{*1} - \mathbf{E}Y^{*1} > \delta\|K\|_2^d + x) \\
&\quad \leq \exp\left(-\frac{\widetilde{C}x^2}{(\|K\|_2^{2d} + x\|K\|_2^d)(h^{d/2} + q_n n^{-1/2})}\right) \exp\left(-\frac{\widetilde{C}}{h^{d/2} + q_n n^{-1/2}}\right) \\
&\quad \leq \widetilde{C}_0 \exp\left(-\frac{\widetilde{C}_1 x^2 \bar{h}_n^{-d/2}}{\|K\|_2^{2d} + x\|K\|_2^d}\right) \exp\left(-\widetilde{C}_2 \bar{h}_n^{-d/2}\right).
\end{aligned}$$

Last line follows from the fact that, for n large enough we have $q_n n^{-1/2} \leq \bar{h}_n^{d/2}$. Similar arguments can be applied to the study of Y^{*2}.

Acknowledgments

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Supplementary Material

Supplemental Material to: *Adaptive density estimation on bounded domains under mixing conditions*
(). This document contains additional explanation about why examples 3 and 4 satisfy Assumption 5, classical lemmas used in the proofs and a complement to the simulation study.

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Supplemental Materials

In this document, we provide supplemental materials, associated to the article *Adaptive density estimation on bounded domains under mixing conditions* by Bertin *et al.* In Section 1, we prove first that regular domains in the sense of Example 3 in the article satisfy Assumption 5 (see Subsection 1.1). Then we prove in Subsection 1.2 that simple polygons also satisfy Assumption 5. In Section 2, we state Bousquet inequality from Boucheron, Lugosi, and Bousquet (2004) (see Subsection 2.1), as far as ingredients from Lemma 4.2 in Viennet (1997) (see Subsection 2.2) required for the proofs of our results. Finally, we provide in Section 3 a complement to our simulation study.

1. Examples 3 and 4 satisfy Assumption 5.

We first recall Assumption 5 in the article.

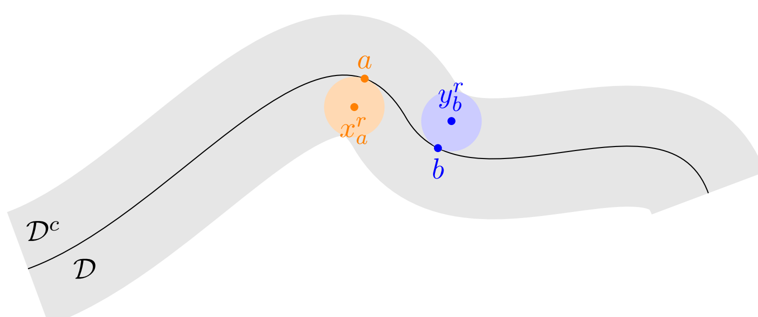
Assumption 5. There exist $0 < r < 1$ and a finite family $\mathcal{A} = \{\mathbf{A}_1, \dots, \mathbf{A}_\kappa\}$ of distinct elements in $GL_d(\mathbb{R})$ such that:

i) For any $j = 1, \dots, \kappa$, $|\det \mathbf{A}_j| = 1$.
ii) For any $x \in \mathcal{D}$ there exists $A_x \in \mathcal{A}$ such that $x + A_x^{-1}([0, r]^d) \subset \mathcal{D}$.

1.1. *Regular domains.*

Example 3. Set $r_0 > 0$. We denote by $\mathbb{D}_r = \{x \in \mathbb{R}^2 : |x|_2 \leq r\}$ the ball of radius $r > 0$. The domain $\mathcal{D}$ is called r_0-regular if, for any $0 < r \leq r_0$, the ball $\mathbb{D}_r$ rolls freely in both $\mathcal{D}$ and $\mathcal{D}^c$. That is, for each $a \in \partial\mathcal{D}$, there exists x_a^r and y_a^r in $\mathbb{R}^d$ such that :

$$a \in (x_a^r + \mathbb{D}_r) \subseteq \overline{\mathcal{D}} \quad \text{and} \quad a \in (y_a^r + \mathbb{D}_r) \subseteq \mathcal{D}^c$$



Step 1. Walther (1999) proved that the domain $\mathcal{D}$ is r_0-regular if, and only if, the following assumption is satisfied: $\partial\mathcal{D}$ is a 1-dimensional C^1 submanifold in $\mathbb{R}^2$ with the outward-pointing unit normal vector $n(a)$ at $a \in \partial\mathcal{D}$ satisfying the Lipschitz condition:

$$|n(a) - n(b)|_2 \leq \frac{1}{r_0}|a - b|_2, \qquad \forall a, b \in \partial\mathcal{D}.$$

Moreover, if x belongs to $\mathcal{D}$ is such that $\inf\{|x - a| : a \in \partial\mathcal{D}\} \leq 2r_0$ then x projects uniquely onto $\partial\mathcal{D}$. We then denote by $a(x)$ this projection which satisfies:

$$a(x) - x = |a(x) - x| \cdot n(a(x)).$$

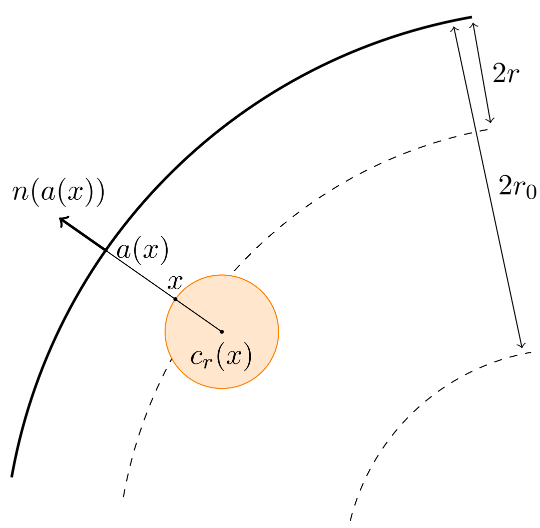
Now we define, for any $0 < r \leq r_0$:

$$\mathcal{C}(r) = \{x \in \mathcal{D} : \inf\{|x - a| : a \in \partial\mathcal{D}\} \leq 2r\} = \bigcup_{a \in \partial\mathcal{D}} (x_a^r + \mathbb{D}_r).$$

Using these notations and the triangle inequality we remark that, for any $0 < r \leq r_0/2$ and any $x \in \mathcal{C}(r)$, we have:

$$x \in (c_r(x) + \mathbb{D}_r) \subseteq \mathcal{C}(r_0) \subset \overline{\mathcal{D}}$$

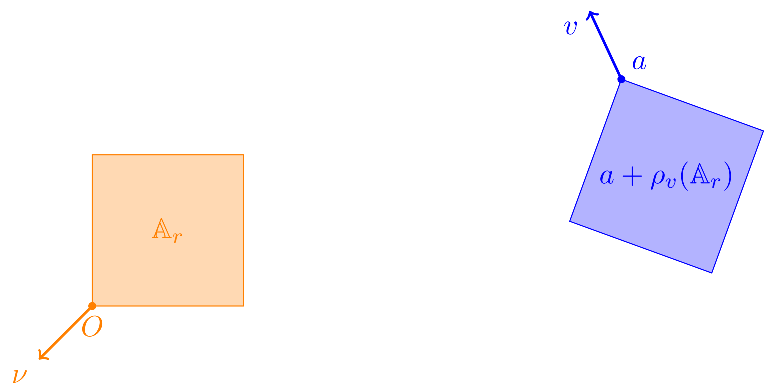
where $c_r(x) = x - rn(a(x))$. The following figure illustrates this property:



Step 2. Define

$$\mathbb{A}_r = \left[0, r\frac{\sqrt{2}}{2}\right]^2$$

and consider the vector $\nu = (-1,-1)/\sqrt{2} \in \mathbb{S}^1$. Note that, for any $v \in \mathbb{S}^1$, there exists a unique rotation $\rho_v(\cdot) \in \mathrm{GL}_2(\mathbb{R})$ such that $\rho_v(\nu) = v$. In the following figure the sets $\mathbb{A}_r$ and $a + \rho_v(\mathbb{A}_r)$ are represented for some point a in $\mathbb{R}^2$ and vector v in $\mathbb{S}^1$.



Now, assume that for any $a \in \partial\mathbb{D}_r = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = r^2\}$, the outward-pointing unit vector is denoted by $v(a)$. Then $a + \rho_{v(a)}(\mathbb{A}_r) \subseteq \mathbb{D}_r$ and moreover $a + \rho_v(\mathbb{A}_r) \subseteq \mathbb{D}_r$ as soon as v is a unit vector such that the angle θ between v and $v(a)$ is less than or equals to $\theta_0 = \arccos(\sqrt{2}/4) - \pi/4$. This is the case if $|v(a) - v| \leq 2\sin(\theta_0/2) = \delta_0$. This results generalizes to any ball centered at $c \in \mathbb{R}^2$ in such a way: if $a \in \partial(c + \mathbb{D}_r)$ then

$$a \in (a + \rho_v(\mathbb{A}_r)) \subseteq (c + \mathbb{D}_r)$$

as soon as v is a unit vector such that $|v - v(a)| \leq \delta_0$. The following figure illustrates this property.

Step 3. Set $0 < r < r_0\delta_0 < r_0/2$. Our goal is to prove the following property: for any $x \in \mathcal{C}(r/6)$, if y satisfies both $y \in \mathcal{C}(r/6)$ and $|x - y| \leq r/3$, then

$$y \in \left(y + \rho_{n(a(x))}(\mathbb{A}_r)\right) \subseteq \mathcal{D}.$$

Note that

$$\begin{aligned} |a(x) - a(y)| &< |a(x) - x| + |x - y| + |y - a(y)| \\ &< 2r/6 + r/3 + 2r/6 \leq r \end{aligned}$$

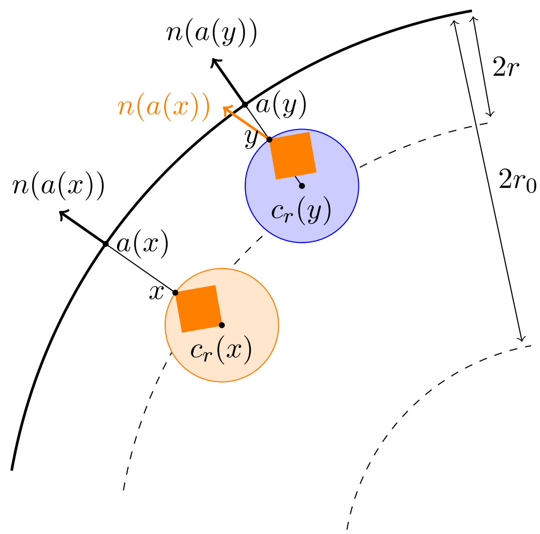
which implies (using the Lipschitz property of the normal $n(\cdot)$) that:

$$|n(a(x)) - n(a(y))| < \frac{r}{r_0} \leq \delta_0.$$

Using the previous steps combined with this result, since $y \in \partial(c_r(y) + \mathbb{D}_r)$, we have

$$y \in \left(y + \rho_{n(a(x))}(\mathbb{A}_r)\right) \subseteq \left(c_r(y) + \mathbb{D}_r\right) \subseteq \mathcal{D}.$$

Last inclusion is straitforward and comes from the triangle inequality. This property is illustrated below:



Step 4. Set $0 < r < r_0\delta_0 < r_0/2$ and define for any $x \in \overline{\mathcal{C}(r/6)}$ the set

$$V(x) = \{y \in \mathcal{C}(r/6) : |x - y| < r/3\},$$

which is an open neighborhood of x. Note also that, using the property proved at the previous step, for any $y \in V(x)$ we have:

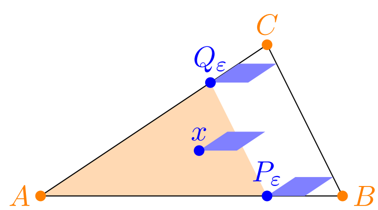
$$y + \rho_{n(a(x))}(\mathbb{A}_r) \subseteq \mathcal{D}$$

Finally $\{V(x) : x \in \overline{\mathcal{C}(r/6)}\}$ is a covering of the compact set $\overline{\mathcal{C}(r/6)}$ by open sets. Thus, there exists a finite number of points $x_1, \ldots, x_N$ such that $\{V(x_n) : n = 1, \ldots, N\}$ is also a covering of $\overline{\mathcal{C}(r/6)}$. The result follows easily.

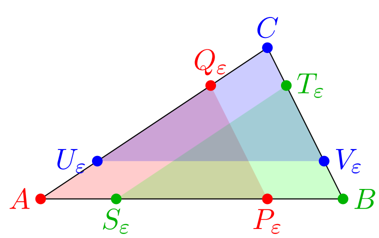
1.2. *Simple polygons.* In practical situations, the boundary of a domain can be approximated by a simple polygonal path. The following example thus seems to be of prime interest.

Example 4. The interior of any simple polygon satisfies Assumption 5 in the article.

Chazelle (1991) proved that any simple polygon can be triangulated (in linear time with respect to the number of vertices). In view of this result, our problem boils down to the case of a triangle. Let ABC be a triangle and let P_ε and Q_ε such that $AP_\varepsilon Q_\varepsilon$ and ABC are similar with homothetic ratio $1 - 2\varepsilon > 0$. Then for any point x in $AP_\varepsilon Q_\varepsilon$, the parallelogram generated by the vectors $\vec{u} = \varepsilon\overrightarrow{AB}$ and $\vec{v} = \varepsilon\overrightarrow{AC}$ with its origin located at x is included into the triangle ABC. This property is illustrated in the following figure.



One can also prove that there exist S_ε, T_ε, U_ε and V_ε such that the triangles $BS_\varepsilon T_\varepsilon$ and $CU_\varepsilon V_\varepsilon$ satisfy similar properties. Now, it is easily seen that, for ε small enough, the union of the triangles $AP_\varepsilon Q_\varepsilon$, $BS_\varepsilon T_\varepsilon$ and $CU_\varepsilon V_\varepsilon$ equals to ABC (this property is illustrated below with $\varepsilon = 0.125$).



Using straightforward arguments, it can be proven that ABC satisfies Assumption 5 with $\kappa = 3$ and $r \le \varepsilon\sqrt{2\,\text{Area}(ABC)}$. This ends the proof.

2. Usefull lemmas from the literature.

2.1. *Bousquet inequality.* The following Lemma is the Bousquet inequality from Boucheron et al. (2004) (see also Theorem 12.5 in Boucheron, Lugosi, and Massart (2013)).

Lemma 1 *Let $X_1, \ldots, X_n$ be independent identically distributed random vectors. Let $\mathcal{S}$ be a countable set of functions. Denote for $s \in \mathcal{S}$ $X_{i,s} = s(X_i)$ and $Z = \sup_{s\in\mathcal{S}} \sum_{i=1}^n s(X_i)$. Assume that for all $i = 1, \ldots, n$ and $s \in \mathcal{S}$ $\mathbf{E}X_{i,s} = 0$, and that $X_{i,s} \leq 1$. Assume also that $v = 2\mathbf{E}Z + \sup_{s\in\mathcal{S}} \sum_{i=1}^n \mathbf{E}(X_{i,s})^2 < \infty$. Then we have for all $t > 0$*

$$\mathbf{P}\left(Z - \mathbf{E}Z \geq t\right) \leq \exp\left\{-\frac{t^2}{2(v + \frac{t}{3})}\right\}.$$

2.2. *Ingredients from Lemma 4.2 in Viennet (1997).*

Lemma 2 *Let $(\chi_i)_{i\in\mathbb{Z}}$ be a stationary $\beta-$mixing process with rate $(\beta(k))_{k\geq 0}$. Let ϕ a measurable function such that $\mathbf{E}\left(\phi^2(\chi_0)\right) < \infty$. There exists a sequence of random variables $(b_k(\chi_0))_{k\geq 0}$ such that $0 \leq b_k(\chi_0) \leq 1$ and $\mathbf{E}(b_k(\chi_0)) = \beta(k)$ that satisfies*

$$\mathrm{Cov}(\phi(\chi_0), \phi(\chi_k)) \leq 2\mathbf{E}\left\{b_k(\chi_0)\phi^2(\chi_0)\right\}$$

and

$$\mathrm{Var}\left(\sum_{i=1}^n \phi(\chi_i)\right) \leq 4n\mathbf{E}\left\{\left(\sum_{k=0}^n b_k(\chi_0)\right)\phi^2(\chi_0)\right\}.$$

For $p \geq 1$, p' such that $\frac{1}{p} + \frac{1}{p'} = 1$, assume $\sum_{k\geq 0}(k+1)^{p-1}\beta(k) < \infty$. Let ϕ a measurable function such that $\mathbf{E}\left(\phi^{2p'}(\chi_0)\right) < \infty$. We have

$$\mathrm{Var}\left(\sum_{i=1}^n \phi(\chi_i)\right) \leq 4n\left(\mathbf{E}\left(\phi^{2p'}(\chi_0)\right)\right)^{1/p'}\left(p\sum_{k\geq 0}(k+1)^{p-1}\beta(k)\right)^{1/p}.$$

3. Complement to the simulation study.

3.1. *Densities on the disk.* In Figures 1 and 2, Boxplots of the ISE for the four procedures (oracle of the *boundary* estimator, GL procedure based on *boundary* estimator, oracle of the *usual* kernel estimator, GL procedure based on *usual* kernel estimator) are given for $n \in \{500, 1000, 2000, 5000\}$ and $M = 500$ replications.

In Table 1 are computed for different sample sizes the mean and the standard-deviation for the ratio between the ISE of the GL procedure and

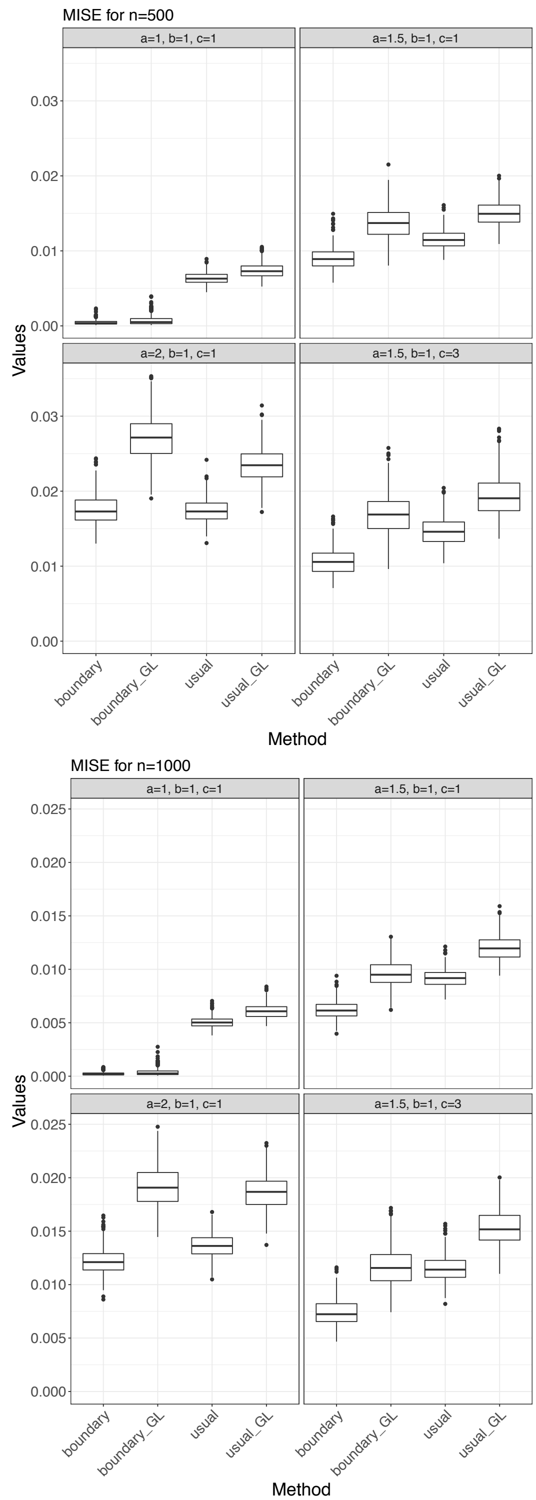


Figure 1. *Boxplots of the integrated squared error (ISE) on the disk for the models described by Cases* 1 *to* 4*, and sample sizes equal to* 500 *and* 1000.

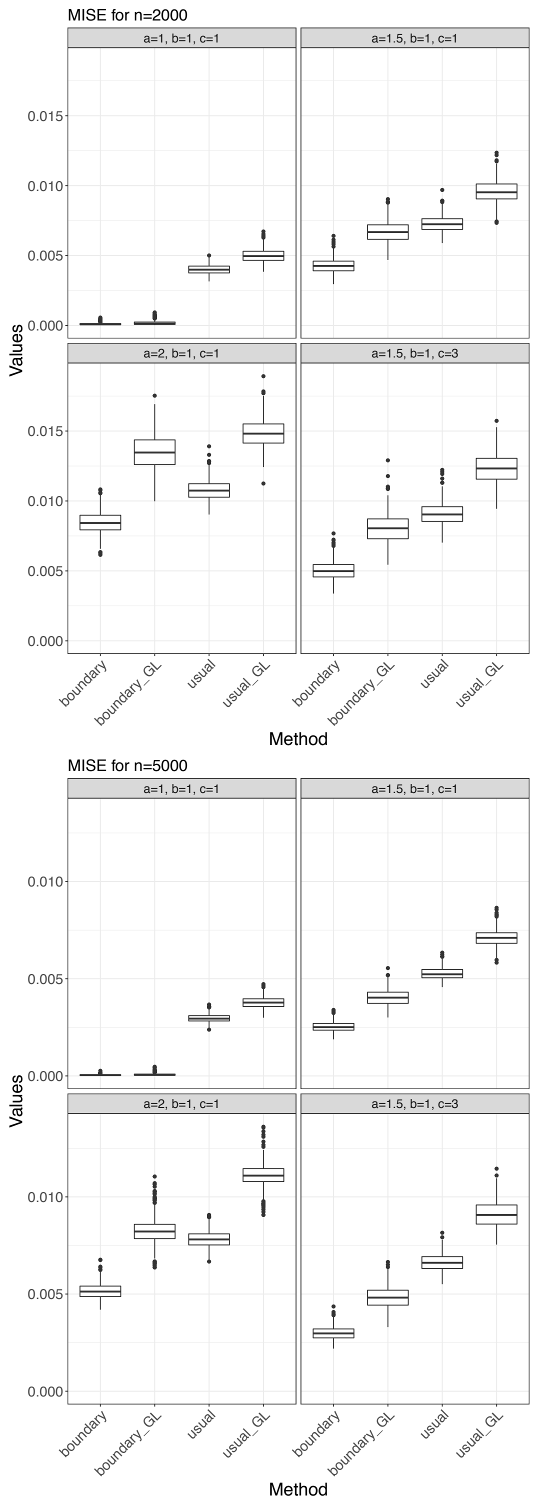


FIGURE 2. *Boxplots of the integrated squared error (ISE) on the disk for the models described by Cases* 1 *to* 4*, and sample sizes equal to* 2000 *and* 5000.

the one of the oracle, both based on the boundary estimator, computed from $M = 500$ replications.

Case	n	mean	sd	n	mean	sd
a=1, b=1, c=1	500	1.63	1.12	2000	1.71	1.23
	1000	1.65	1.18	5000	1.55	1.03
a=1.5, b=1, c=1	500	1.55	0.28	2000	1.57	0.21
	1000	1.57	0.23	5000	1.60	0.19
a=2, b=1, c=1	500	1.56	0.21	2000	1.60	0.16
	1000	1.58	0.17	5000	1.60	0.13
a=1.5, b=1, c=3	500	1.62	0.31	2000	1.62	0.23
	1000	1.59	0.25	5000	1.62	0.19

TABLE 1
Mean and standard-deviation for the ratio between the ISE of the GL procedure and the one of the oracle, both for the boundary estimator, computed from $M = 500$ replications. The results are provided for different sample sizes: $n = 500$, 1000, 2000 and 5000.

3.2. *Diffusion.* In Figure 3 below, we show the boxplots for the ISE of the GL procedure based on the *boundary* and *usual* estimator for sample size $n \in \{500, 1000, 2000, 5000\}$.

4. Application to a real data set. In this section our method is applied to a database obtained from the study "African Elephant (Migration) Chamaillé-Jammes Hwange NP" that resides in the repository Movebank (Wikelski and Kays, 2018, accessed on 2018/11/05).

4.1. *Description of the data.* The data was obtained from the GPS tracking of the migratory trajectories of 30 elephants evolving in the Hwange National Park in Zimbabwe (HNP in the following). The daily observations of their displacements were taken by means of GPS devices located in collars installed in individuals of different herds (see Tshipa, Valls-Fox, Fritz, Collins, Sebele, Mundy, and Chamaillé-Jammes, 2017; Valls-Fox, De Garine-Wichatitsky, Fritz, and Chamaillé-Jammes, 2018, and references therein). The date of installation of the collars was as follows: August 2009 (10 elephants), November 2012 (10 elephants), November 2014 (8 elephants) and February 2015 (2 elephants). Each elephant is observed during approximately 2 years.

We are interested in the spatial density of the whole set of elephants into the park. As in Cholaquidis, Fraiman, Mordecki, and Papalardo (2016) we assume that the animal movements can be modeled by a reflected diffusion. This allows us to guarantee that the process that modeled the trajectories of the elephants satisfies the main assumptions of our model. Moreover, nine

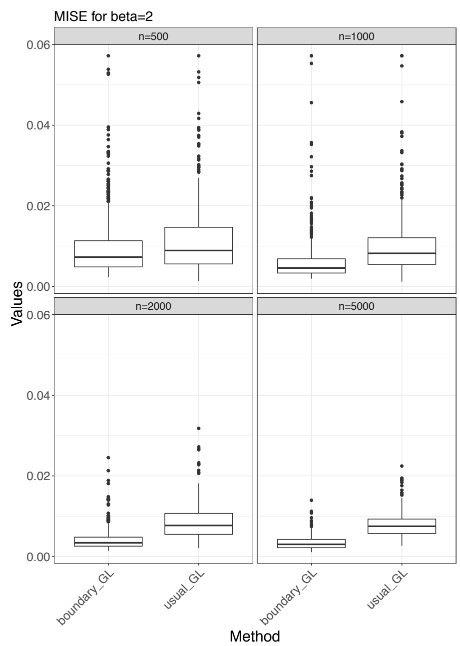
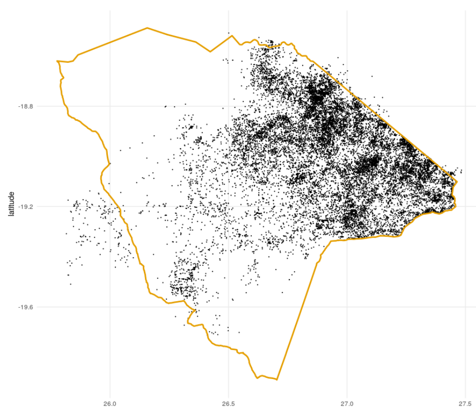


FIGURE 3. *Boxplots of the integrated squared error (ISE) of the GL procedure based on the boundary estimator for* $\beta = 2$ *and different sample sizes,* $n = 500$, 1000, 2000 *and* 5000.

elephants were removed from the initial database as their behavior seems atypic.

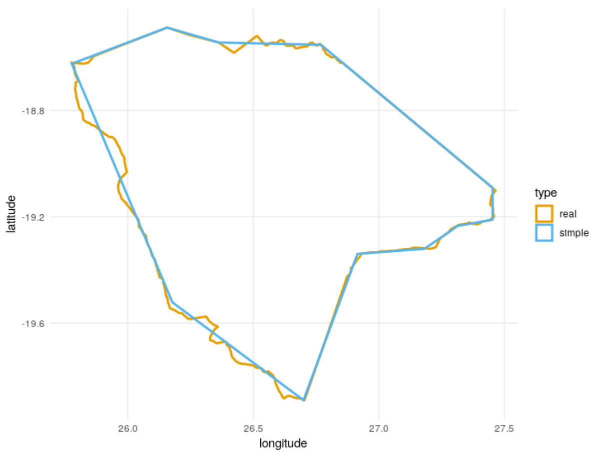
4.2. *Boundary of the park.* The following map, obtained from © *OpenStreetMap contributors*, represents the Hwange National Park.

Note that the boundary is very simple on the east side (it consists mainly in straight lines) and more complex on the west side. However most of the observations lie in the east side. This is especially true for observations that are close to a boundary. The following figure (which represents each of the $n = 17501$ GPS positions as a point) stresses this point.

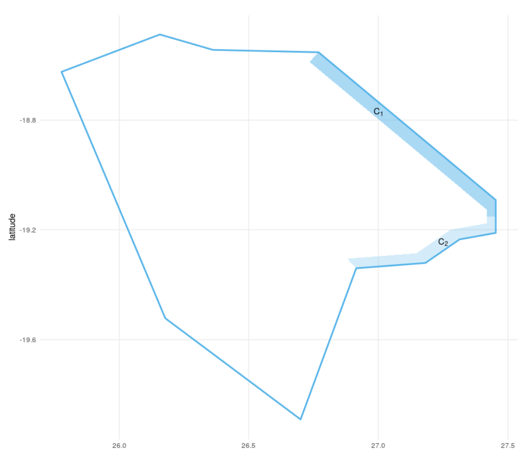


Note that our methodology requires the knowledge of the boundary. Moreover, the more complex the boundary, the more difficult our method is to implement. With this in mind we propose to approximate the boundary of HNP by a simple polygon that adjusts quite well the real boundary in the east part whereas a more rough approximation is used in the west part.

Finally, a simple polygon with only 11 edges was chosen. The following figure represents both the real boundary (in yellow) and the approximating simple polygon (in blue):



4.3. *Family of estimators.* In this section we follow the same strategy as in the simulation study: our boundary estimators are only used in a "neighbohood" of the boundary of the east side of HNP while classical kernel estimators are used otherwhere. More precisely, we define two specific zones: the northeast zone C_1 and the southeast zone C_2 that are represented in the following figure.



Now, our procedure consists in selecting in a data-driven way a bandwidth among a family of 30 bandwidth equally spaced between 0.02 and 0.31. For each bandwidth h and any point x in HNP, the estimator $\hat{f}_h(x)$ is defined in two different ways depending on the position of x. If x belongs to $C_1 \cup C_2$

and if the distance of x to the boundary is less than h, then the boundary estimator is used and we define:

$$\hat{f}_h(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{K}_h \circ A_x(X_i - x),$$

where $A_x = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ if $x \in C_1$ and $A_x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ if $x \in C_2$. Otherwise the usual kernel estimator is used. In this case we define

$$\hat{f}_h(x) = \frac{1}{nh^2}\sum_{i=1}^{n} \widetilde{K}(h^{-1}(X_i - x)) \text{ with } \widetilde{K}(u_1, u_2) = \mathbf{I}_{[-\frac{1}{2},\frac{1}{2}]^2}(u_1, u_2).$$

4.4. *Estimation procedure.* Since our selection procedure depends on a tuning parameter τ that appears in the penalty (also called *majorant* in the terminology introduced by Goldenshluger and Lepski), we propose to use a *slope heuristic* to determine this constant. We refer the reader to Baudry, Maugis, and Michel (2012) for more details. Note that our penalty $\hat{M}(\ell)$ depends on τ via the quantity $\tau\|K\|_2^2/(\sqrt{n}h(\ell))$. As a consequence, for each tuning parameter τ, our procedure selects the quantity $\hat{\ell}(\tau)$ and the slope heuristic consists in defining τ_{min} as the largest slope of the function $\tau \mapsto \hat{M}(\hat{\ell}(\tau))$. As it can be observed in Figure 4, we obtain $\tau_{\min} = 0.8$ so we implement our GL procedure with $\tau = 2\tau_{\min} = 1.6$. The bandwidth which is selected then equals to $h = 0.07$.

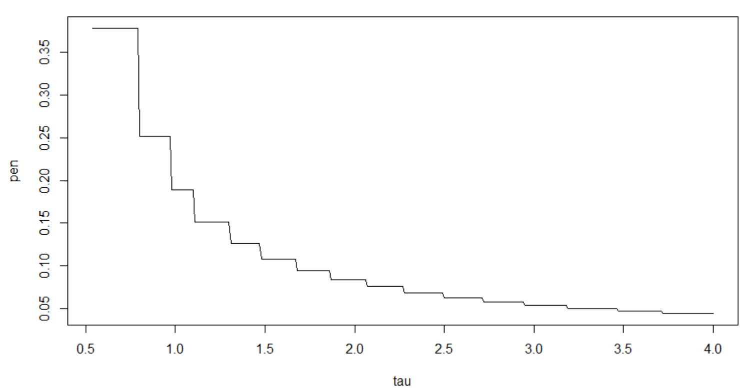
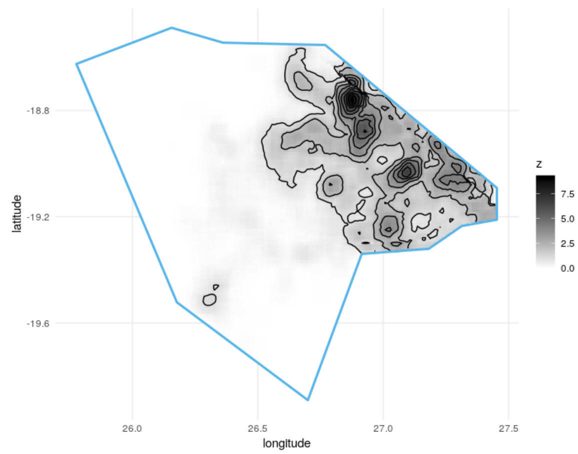
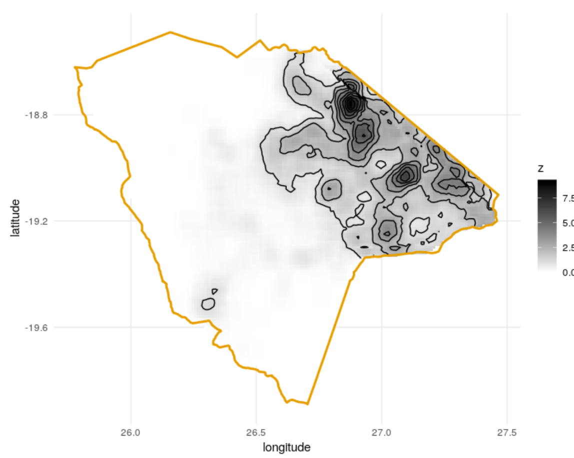


FIGURE 4. *Value of the penalization $\hat{M}(\hat{\ell}(\tau))$ in terms of τ.*

4.5. *Results.* Figure below represents the final density estimation plotted on a spatial regular grid (with step $\delta = 0.01$) of the interior of the simple polygon defining our boundary.



The same density estimation is also plotted within the real boundary for convenience.



The result we obtain seems to confirm that the density of elephants is related to the placement of artificial water pumps. It would be interesting to investigate further that issue with the owners of this database.

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