Domain decomposition methods for linearized Boussinesq type equations

Joao Guilherme Caldas Steinstraesser, Gaspard Kemlin, Antoine Rousseau

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Introduction

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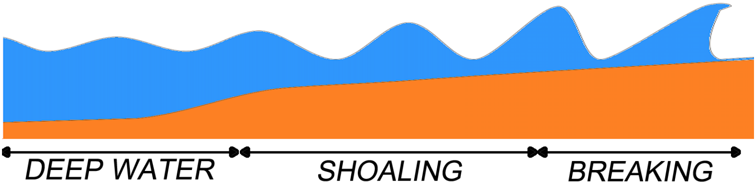
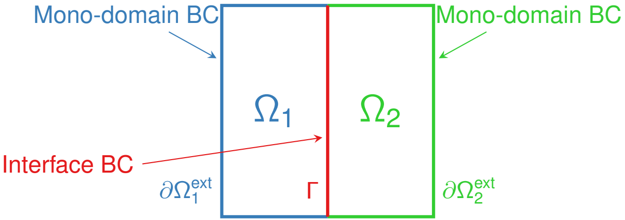
- Wave propagation in nearshore area: Boussinesq-type equations;



Introduction

- Wave propagation in nearshore area: Boussinesq-type equations;
- Domain Decomposition Methods (DDMs)
  - "Divide and conquer";
  - Main issue: interface boundary conditions (IBCs);
  - Future objective: couple different wave propagation models;



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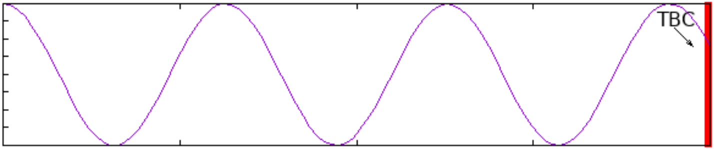
- Domain Decomposition Methods (DDMs)
  - "Divide and conquer";
  - Main issue: interface boundary conditions (IBCs);
  - Future objective: couple different wave propagation models;
- Transparent Boundary Conditions (TBCs)
  - Numerical resolution in finite computational domain;
  - Optimal interface conditions for DDMs;
  - Cannot be computed exactly.



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- Test their efficiency as TBCs;
- Test their efficiency as IBCs in a DDM;
- Test their efficiency for the full nonlinear problem.

The governing equations

- Full equations [Roeber and Cheung, 2012]:

$$\begin{cases} \eta_t + \nabla \cdot [(h+\eta)\,\boldsymbol{u}] + \nabla \cdot \left[\left(\frac{z^2}{2} - \frac{h^2}{6} \right) h \nabla \left(\nabla \cdot \boldsymbol{u} \right) + \left(z + \frac{h}{2} \right) h \nabla \left(\nabla \cdot (h\boldsymbol{u}) \right) \right] = 0, \\ \boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + g\nabla\eta + z \left[\frac{z}{2} \nabla \left(\nabla \cdot \boldsymbol{u}_t \right) + \nabla \left(\nabla \cdot (h\boldsymbol{u}_t) \right) \right] + \boldsymbol{\tau} = 0. \end{cases}$$

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- Simplified framework:
  - 1D;
  - Flat bottom;
  - Linearization around $(\overline{\eta}, \overline{u}) = (0, 0)$;
  - Bottom shear stress $\tau = 0$.

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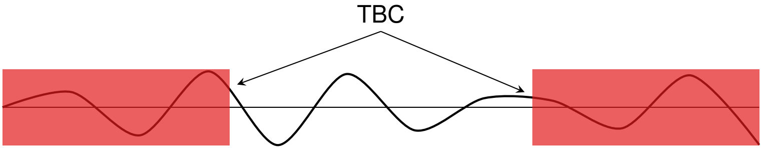
\begin{cases}
\eta_t + h_0 u_x + \tilde{h} u_{xxx} = 0, \\
u_t + g\eta_x + \bar{h} u_{xxt} = 0,
\end{cases}
\implies u_{tt} + \bar{h} u_{xxtt} - g h_0 u_{xx} - g \tilde{h} u_{xxxx} = 0
$$

2

Transparent Boundary Conditions

Transparent Boundary Conditions (TBCs)



- Problems are well posed in an infinite domain, but the numerical resolution must be done in a finite computational domain;

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\text{(A)} \begin{cases} \mathcal{L}(u) = 0, & \text{in } \mathbb{R} \\ u \to 0, & x \to \pm\infty \end{cases} \qquad \text{(B)} \begin{cases} \mathcal{L}(u) = 0, & \text{in }]0, L[\\ \text{TBCs}, & \text{in } x = 0, L \end{cases}
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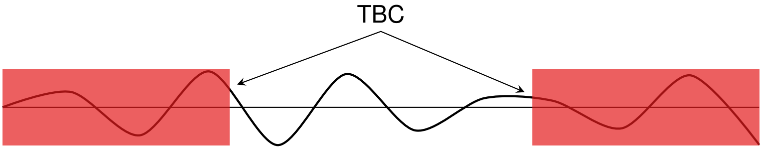
Transparent Boundary Conditions (TBCs)



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$$(A) \begin{cases} \mathcal{L}(u) = 0, & \text{in } \mathbb{R} \\ u \to 0, & x \to \pm\infty \end{cases} \qquad (B) \begin{cases} \mathcal{L}(u) = 0, & \text{in }]0, L[\\ \text{TBCs}, & \text{in } x = 0, L \end{cases}$$

- TBCs assure that the solution of (A), restricted to $[0, L]$, coincides with the solution of (B).

Difficulties

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  - Are non local in time and space;
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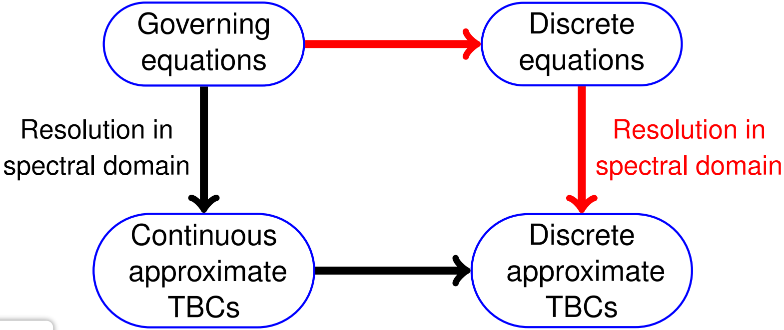
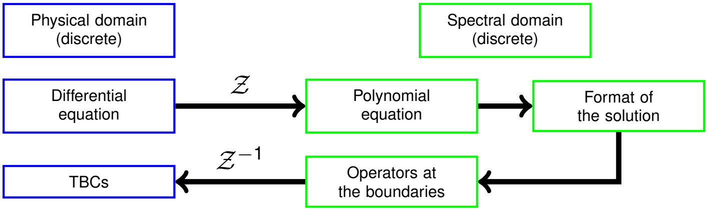
- Two usual approaches:



Derivation of TBCs for the discrete equations

Methodology [Arnold and Ehrhardt, 2001, Besse et al., 2015, Kazakova and Noble, 2017]:

- Discretize the governing equations;



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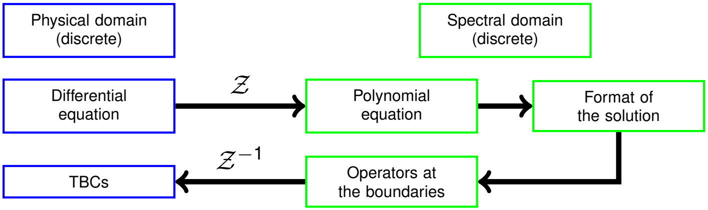
- Discretize the governing equations;
- Use the $\mathcal{Z}$-transform in the complementary set $\mathbb{R}\backslash[0, L]$;



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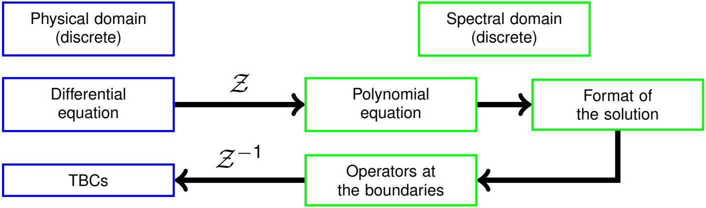

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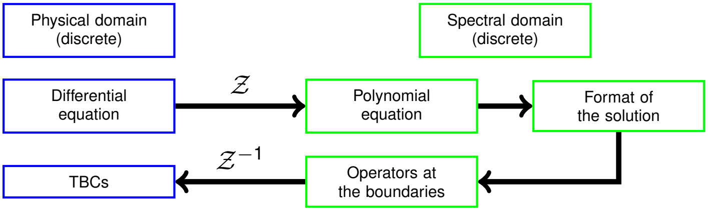

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$$\textbf{Left:} \begin{cases} \Gamma_1^l\left(\{u^n\}\right) := u_0^n - (Y_5 * u_1)^n + (Y_7 * u_2)^n = 0, \\ \Gamma_2^l\left(\{u^n\}\right) := u_0^n - (Y_6 * u_2)^n + 2(Y_9 * u_3)^n - (Y_8 * u_4)^n = 0, \end{cases}$$

where $(Y_i^n)_{n\in\mathbb{N}}$ depend on the equation and $(Y_i * u_j)^n = \sum_{m=0}^n Y_i^m u_j^{n-m}$.

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$$\textbf{Right:} \quad \begin{cases} \Gamma_1^r\left(\{u^n\}\right) := u_J^n - (Y_1 * u_{J-1})^n + (Y_3 * u_{J-2})^n = 0, \\ \\ \Gamma_2^r\left(\{u^n\}\right) := u_J^n - 2(Y_1 * u_{J-1})^n + (Y_2 * u_{J-2})^n - (Y_4 * u_{J-4})^n = 0, \end{cases}$$

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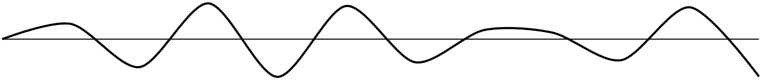
- These operators are non local in time.

Numerical tests

- Reference solution: computed in a bigger domain.



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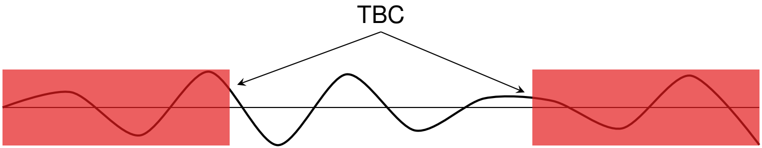

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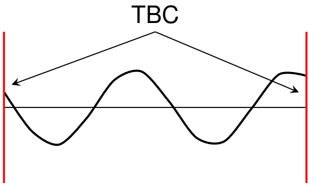

Numerical tests

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t	e^n
0.25	$1.5 \cdot 10^{-9}$
0.5	$9.5 \cdot 10^{-7}$
0.75	$5.2 \cdot 10^{-6}$
1.0	$8.4 \cdot 10^{-6}$

$$e_T = 4.5 \cdot 10^{-6}$$

$$e^n = \frac{\|u_{\text{ref}}(\cdot, t^n) - u_{\text{num}}(\cdot, t^n)\|_{l^2}}{\|u_{\text{ref}}(\cdot, t^n)\|_{l^2}}, \qquad e_T = \sqrt{\delta t \times \sum_{n=1}^{n_{\max}} (e^n)^2}.$$

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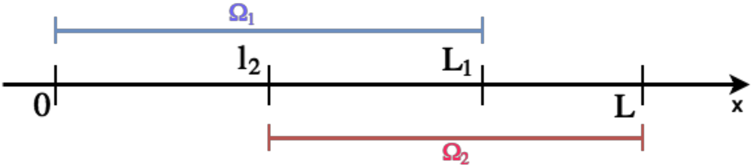
Domain Decomposition Method

Domain Decomposition Methods (DDMs)

- Idea: divide the domain in two or more subdomains (possibly overlapping) and solve the problem in each one of them.

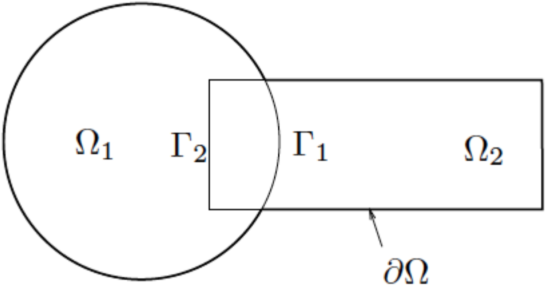


$$\Omega = \Omega_1 \cup \Omega_2, \qquad \partial\Omega_1^{\text{int}} = \partial\Omega_1 \cap \Omega_2, \qquad \partial\Omega_2^{\text{int}} = \partial\Omega_2 \cap \Omega_1$$

- Main difficulty: definition of the interface boundary conditions (IBCs) on $\partial\Omega_1^{\text{int}}$ and $\partial\Omega_2^{\text{int}}$ for the communication between the subdomains.

The Schwarz method

- The first and one of the most important types of DDM [Gander, 2008];



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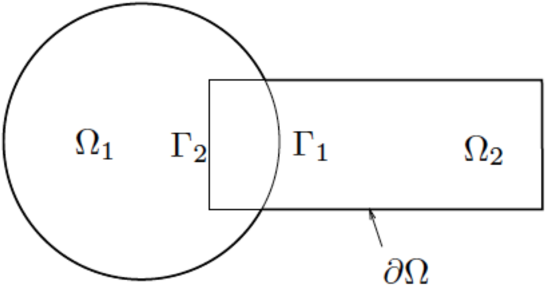
- Iterative method;



The Schwarz method (1870)

- Equations solved in each subdomain (in parallel or additive form), for updating the solution to time t_{n+1}:

$$\begin{cases} \mathcal{L}(u_1^{n+1,k+1}) = 0, & \text{in } \Omega_1 \\ u_1^{n+1,k+1} = g, & \text{on } \partial\Omega_1^{\text{ext}} \\ \mathcal{B}_1(u_1^{n+1,k+1}) = \mathcal{B}_1(u_2^{n+1,k}), & \text{on } \partial\Omega_1^{\text{int}} \end{cases},$$

$$\begin{cases} \mathcal{L}(u_2^{n+1,k+1}) = 0, & \text{in } \Omega_2 \\ u_2^{n+1,k+1} = g, & \text{on } \partial\Omega_2^{\text{ext}} \\ \mathcal{B}_2(u_2^{n+1,k+1}) = \mathcal{B}_2(u_1^{n+1,k}), & \text{on } \partial\Omega_2^{\text{int}} \end{cases},$$

The Schwarz method

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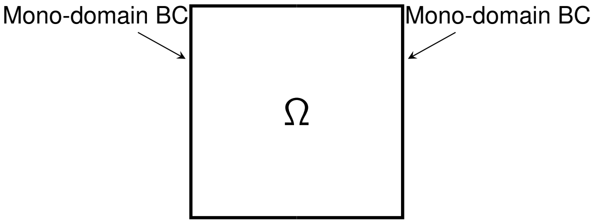
The Schwarz method

- Optimal IBCs? TBCs [Japhet and Nataf, 2001]
  - Convergence to the solution of the monodomain;
  - Fastest convergence
- We implement the discrete TBCs as IBCs.

Numerical results

- Reference solution: numerical solution of the problem solved in the monodomain;



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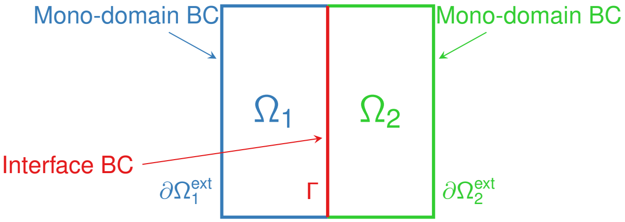

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  - DDM using Dirichlet IBCs (original Schwarz method):

$$\begin{cases} u_1^{n+1,k+1} = u_2^{n+1,k}, & \text{on } \partial\Omega_1^{\text{int}} \\ u_2^{n+1,k+1} = u_1^{n+1,k}, & \text{on } \partial\Omega_2^{\text{int}} \end{cases}$$

Numerical results

- Reference solution: numerical solution of the problem solved in the monodomain;

- Objective: compare the number of iterations for convergence to the reference solution in two cases:
  - DDM using the discrete TBCs as IBCs:

$$\begin{cases} \Gamma_1^r\left(\left\{u_1^{n+1}\right\}^{k+1}\right) = \Gamma_1^r\left(\left\{u_2^{n+1}\right\}^{k}\right) \\ \Gamma_2^r\left(\left\{u_1^{n+1}\right\}^{k+1}\right) = \Gamma_2^r\left(\left\{u_2^{n+1}\right\}^{k}\right) \end{cases} \quad \text{on } \partial\Omega_1^{\text{int}}$$

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(these IBCs are local in time).

Numerical results

t	TBC	Dirichlet
0.25	2	604
0.5	2	612
0.75	2	546
1.0	2	556

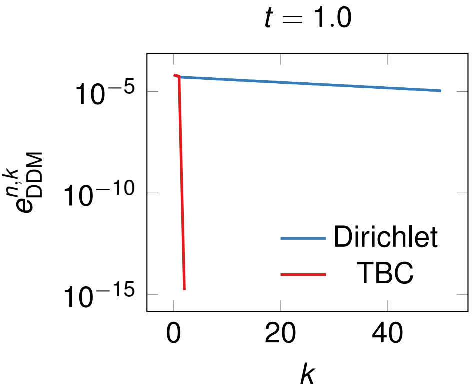

Table: Convergence of the additive Schwarz method, for a precision of $\varepsilon = 10^{-15}$, for an overlap size of 5.

4

Towards TBCs for nonlinear problems

The Serre equations

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- Analogous procedure lead to similar discrete TBCs as obtained for Boussinesq equations, differing only by the coefficients of the TBCs;

Numerical simulations

- Use discrete TBCs:
  - for simulating nonlinear problem;
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Numerical simulations

- Use discrete TBCs:
  - for simulating nonlinear problem;
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- Compare results for nonlinear Serre equations using
  - Discrete TBCs coefficients obtained for linearized Serre equations;
  - Discrete TBCs coefficients obtained for linearized Boussinesq equations.

Numerical simulations

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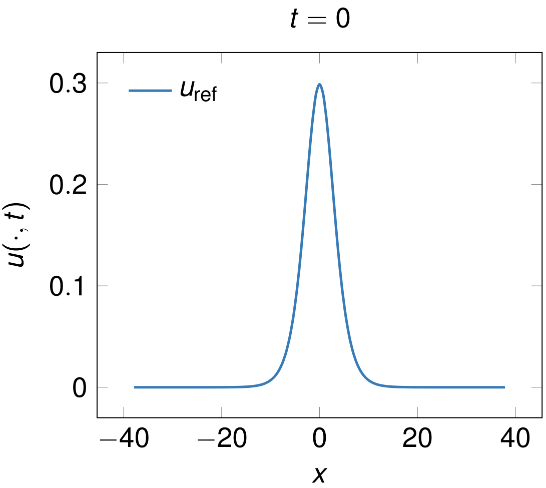
- Compare results for nonlinear Serre equations using
  - Discrete TBCs coefficients obtained for linearized Serre equations;
  - Discrete TBCs coefficients obtained for linearized Boussinesq equations.
- Two testcases:
  - Gaussian wave;
  - Solitary wave;

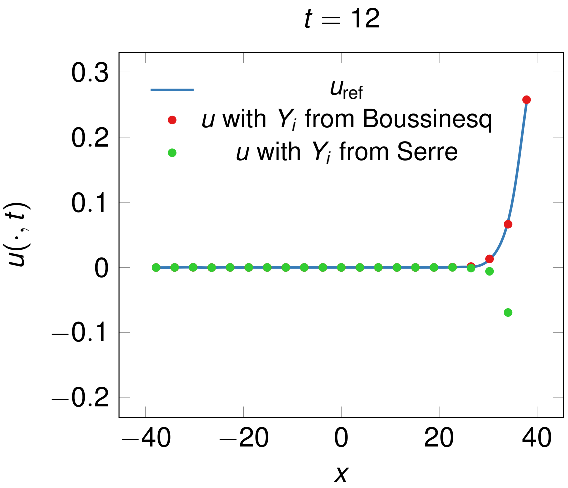
Numerical results - TBCs - Solitary wave

Initial solution:



Numerical results - TBCs - Solitary wave

TBCs with different coefficients:



Numerical results - DDM

Case	Dirichlet	TBC	
		Serre	Boussinesq
Gaussian	$120 \sim 160$	18	9
Solitary	11	8	8

Table: Convergence of the additive Schwarz method, for a precision of $\varepsilon = 10^{-15}$, for an overlap size of 5.

Question: Coefficients from linearized Boussinesq equations seem to be more efficient than the coefficients from linearized Serre equations.

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- The operators provide good and efficient results
  - as TBCs;
  - as IBCs in a DDM;
- Application to the numerical resolution of nonlinear problems shall be investigated in more details.
- More detailed paper under review [Caldas Steinstraesser et al., 2018].

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