

Domain decomposition methods for linearized Boussinesq type equations

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November 2018

Domain decomposition methods for linearized Boussinesq type equations

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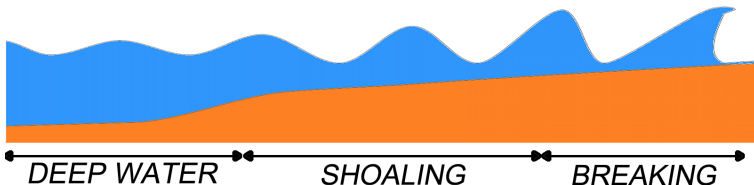
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Introduction

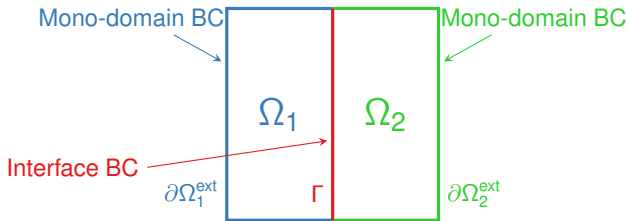
Introduction

- Wave propagation in nearshore area: Boussinesq-type equations;



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- Domain Decomposition Methods (DDMs)
 - “Divide and conquer”;
 - Main issue: interface boundary conditions (IBCs);
 - Future objective: couple different wave propagation models;



Introduction

- Wave propagation in nearshore area: Boussinesq-type equations;
- Domain Decomposition Methods (DDMs)
 - “Divide and conquer”;
 - Main issue: interface boundary conditions (IBCs);
 - Future objective: couple different wave propagation models;
- Transparent Boundary Conditions (TBCs)
 - Numerical resolution in finite computational domain;
 - Optimal interface conditions for DDMs;
 - Cannot be computed exactly.

Objectives

- Derive discrete TBCs for the linearized equations;

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Objectives

- Derive discrete TBCs for the linearized equations;
- Test their efficiency as TBCs;
- Test their efficiency as IBCs in a DDM;
- Test their efficiency for the full nonlinear problem.

The governing equations

- Full equations [Roeber and Cheung, 2012]:

$$\begin{cases} \eta_t + \nabla \cdot [(h + \eta) \mathbf{u}] + \nabla \cdot \left[\left(\frac{z^2}{2} - \frac{h^2}{6} \right) h \nabla (\nabla \cdot \mathbf{u}) + \left(z + \frac{h}{2} \right) h \nabla (\nabla \cdot (h\mathbf{u})) \right] = 0, \\ \mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} + g \nabla \eta + z \left[\frac{z}{2} \nabla (\nabla \cdot \mathbf{u}_t) + \nabla (\nabla \cdot (h\mathbf{u}_t)) \right] + \boldsymbol{\tau} = 0. \end{cases}$$

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- Simplified framework:
 - 1D;
 - Flat bottom;
 - Linearization around $(\bar{\eta}, \bar{\mathbf{u}}) = (0, 0)$;
 - Bottom shear stress $\boldsymbol{\tau} = 0$.

The governing equations

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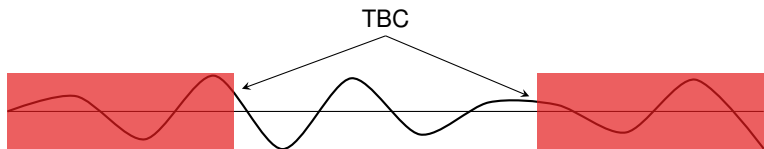
- Simplified framework:
 - 1D;
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 - Linearization around $(\bar{\eta}, \bar{\mathbf{u}}) = (0, 0)$;
 - Bottom shear stress $\tau = 0$.
- Simplified equations:

$$\begin{cases} \eta_t + h_0 u_x + \tilde{h} u_{xxx} = 0, \\ u_t + g \eta_x + \bar{h} u_{xxt} = 0, \end{cases} \quad \implies \quad u_{tt} + \bar{h} u_{xxtt} - g h_0 u_{xx} - g \tilde{h} u_{xxxx} = 0$$

2

Transparent Boundary Conditions

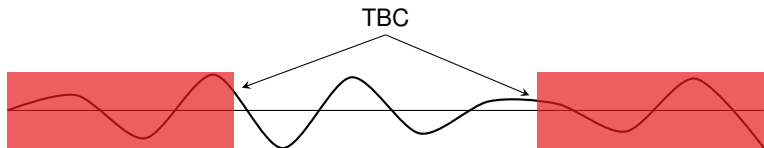
Transparent Boundary Conditions (TBCs)



- Problems are well posed in an infinite domain, but the numerical resolution must be done in a finite computational domain;

$$(A) \begin{cases} \mathcal{L}(u) = 0, & \text{in } \mathbb{R} \\ u \rightarrow 0, & x \rightarrow \pm\infty \end{cases} \quad (B) \begin{cases} \mathcal{L}(u) = 0, & \text{in }]0, L[\\ \text{TBCs}, & \text{in } x = 0, L \end{cases}$$

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- TBCs assure that the solution of (A), restricted to $[0, L]$, coincides with the solution of (B).

Difficulties

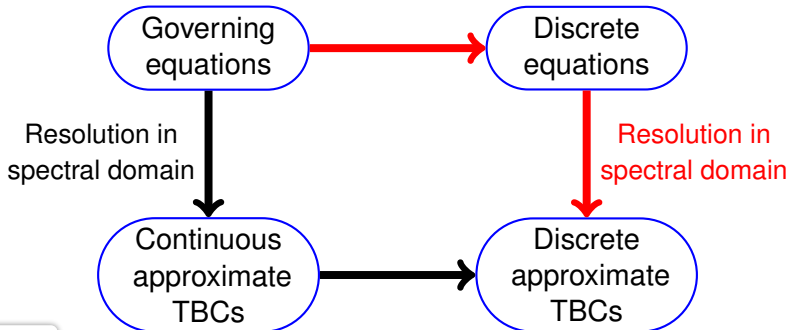
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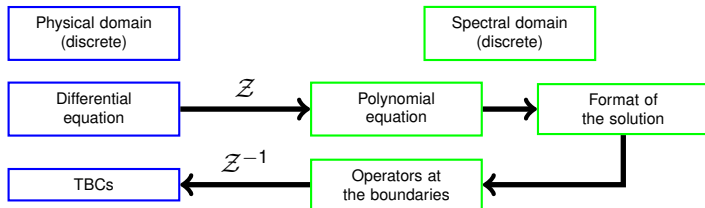
- In general, TBCs
 - Are non local in time and space;
 - Cannot be computed exactly, both analytically and numerically;
- Therefore, one must find approximate TBCs;
- Two usual approaches:



Derivation of TBCs for the discrete equations

Methodology [Arnold and Ehrhardt, 2001, Besse et al., 2015, Kazakova and Noble, 2017]:

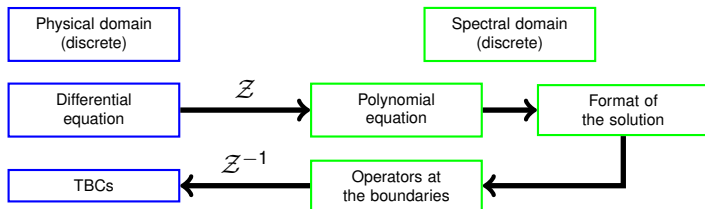
- Discretize the governing equations;



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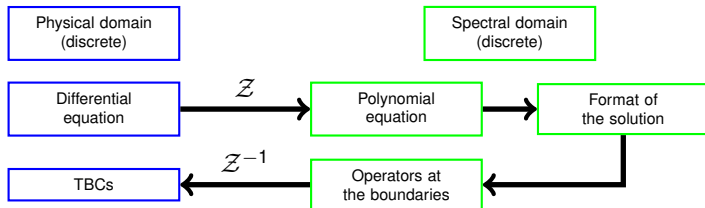
- Discretize the governing equations;
- Use the \mathcal{Z} -transform in the complementary set $\mathbb{R} \setminus [0, L]$;



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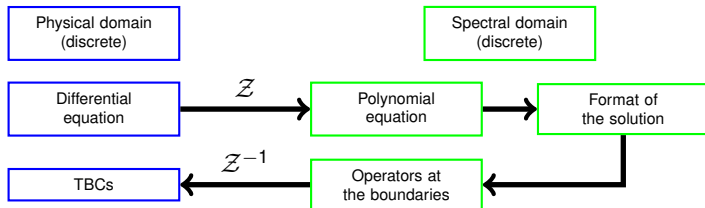
- Discretize the governing equations;
- Use the \mathcal{Z} -transform in the complementary set $\mathbb{R} \setminus [0, L]$;
- Solve the problem in the \mathcal{Z} -space and find the conditions fulfilled by the solutions at the two boundaries;



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$$\text{Left: } \begin{cases} \Gamma_1^l(\{u^n\}) := u_0^n - (Y_5 * u_1)^n + (Y_7 * u_2)^n = 0, \\ \Gamma_2^l(\{u^n\}) := u_0^n - (Y_6 * u_2)^n + 2(Y_9 * u_3)^n - (Y_8 * u_4)^n = 0, \end{cases}$$

where $(Y_i^n)_{n \in \mathbb{N}}$ depend on the equation and $(Y_i * u_j)^n = \sum_{m=0}^n Y_i^m u_j^{n-m}$.

Derivation of TBCs for the discrete equations

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$$\text{Right: } \begin{cases} \Gamma_1^r(\{u^n\}) := u_j^n - (Y_1 * u_{j-1})^n + (Y_3 * u_{j-2})^n = 0, \\ \Gamma_2^r(\{u^n\}) := u_j^n - 2(Y_1 * u_{j-1})^n + (Y_2 * u_{j-2})^n - (Y_4 * u_{j-4})^n = 0, \end{cases}$$

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Derivation of TBCs for the discrete equations

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$$\text{Left: } \begin{cases} \Gamma_1^l(\{u^n\}) = 0 \\ \Gamma_2^l(\{u^n\}) = 0 \end{cases} \quad \text{Right: } \begin{cases} \Gamma_1^r(\{u^n\}) = 0 \\ \Gamma_2^r(\{u^n\}) = 0 \end{cases}$$

- These operators are non local in time.

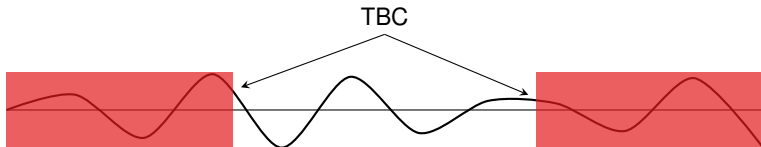
Numerical tests

- Reference solution: computed in a bigger domain.



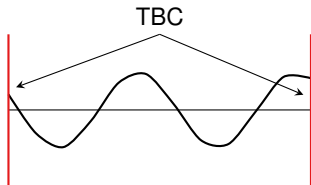
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t	e^n
0.25	$1.5 \cdot 10^{-9}$
0.5	$9.5 \cdot 10^{-7}$
0.75	$5.2 \cdot 10^{-6}$
1.0	$8.4 \cdot 10^{-6}$

$$e_T = 4.5 \cdot 10^{-6}$$

$$e^n = \frac{\|u_{\text{ref}}(\cdot, t^n) - u_{\text{num}}(\cdot, t^n)\|_{L^2}}{\|u_{\text{ref}}(\cdot, t^n)\|_{L^2}},$$

$$e_T = \sqrt{\delta t \times \sum_{n=1}^{n_{\text{max}}} (e^n)^2}.$$

3

Domain Decomposition Method

Domain Decomposition Methods (DDMs)

- Idea: divide the domain in two or more subdomains (possibly overlapping) and solve the problem in each one of them.

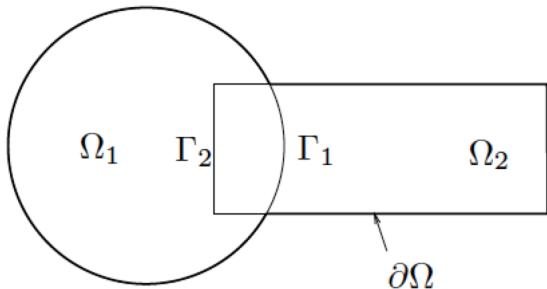


$$\Omega = \Omega_1 \cup \Omega_2, \quad \partial\Omega_1^{\text{int}} = \partial\Omega_1 \cap \Omega_2, \quad \partial\Omega_2^{\text{int}} = \partial\Omega_2 \cap \Omega_1$$

- Main difficulty: definition of the interface boundary conditions (IBCs) on $\partial\Omega_1^{\text{int}}$ and $\partial\Omega_2^{\text{int}}$ for the communication between the subdomains.

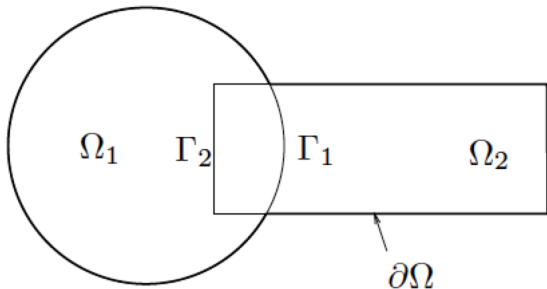
The Schwarz method

- The first and one of the most important types of DDM [Gander, 2008];



The Schwarz method

- The first and one of the most important types of DDM [Gander, 2008];
- Iterative method;



The Schwarz method (1870)

- Equations solved in each subdomain (in parallel or additive form), for updating the solution to time t_{n+1} :

$$\begin{cases} \mathcal{L}(u_1^{n+1,k+1}) = 0, & \text{in } \Omega_1 \\ u_1^{n+1,k+1} = g, & \text{on } \partial\Omega_1^{\text{ext}}, \\ \mathcal{B}_1(u_1^{n+1,k+1}) = \mathcal{B}_1(u_2^{n+1,k}), & \text{on } \partial\Omega_1^{\text{int}} \end{cases}$$

$$\begin{cases} \mathcal{L}(u_2^{n+1,k+1}) = 0, & \text{in } \Omega_2 \\ u_2^{n+1,k+1} = g, & \text{on } \partial\Omega_2^{\text{ext}}, \\ \mathcal{B}_2(u_2^{n+1,k+1}) = \mathcal{B}_2(u_1^{n+1,k}), & \text{on } \partial\Omega_2^{\text{int}} \end{cases}$$

The Schwarz method

- Optimal IBCs?

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 - Convergence to the solution of the monodomain;

The Schwarz method

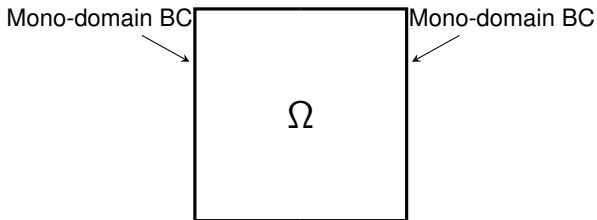
- Optimal IBCs? **TBCs** [Japhet and Nataf, 2001]
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 - Fastest convergence

The Schwarz method

- Optimal IBCs? **TBCs** [Japhet and Nataf, 2001]
 - Convergence to the solution of the monodomain;
 - Fastest convergence
- We implement the discrete TBCs as IBCs.

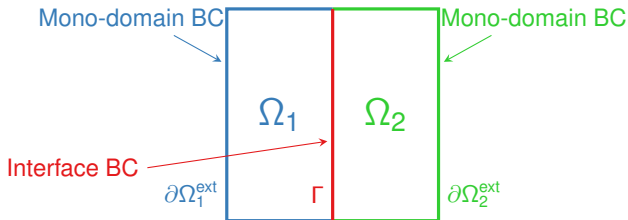
Numerical results

- Reference solution: numerical solution of the problem solved in the monodomain;



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- Objective: compare the number of iterations for convergence to the reference solution in two cases:

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- Reference solution: numerical solution of the problem solved in the monodomain;
- Objective: compare the number of iterations for convergence to the reference solution in two cases:
 - DDM using Dirichlet IBCs (original Schwarz method):

$$\begin{cases} u_1^{n+1,k+1} = u_2^{n+1,k}, & \text{on } \partial\Omega_1^{\text{int}} \\ u_2^{n+1,k+1} = u_1^{n+1,k}, & \text{on } \partial\Omega_2^{\text{int}} \end{cases}$$

Numerical results

- Reference solution: numerical solution of the problem solved in the monodomain;
- Objective: compare the number of iterations for convergence to the reference solution in two cases:
 - DDM using the discrete TBCs as IBCs:

$$\begin{cases} \Gamma_1^r \left(\{u_1^{n+1}\}^{k+1} \right) = \Gamma_1^r \left(\{u_2^{n+1}\}^k \right) \\ \Gamma_2^r \left(\{u_1^{n+1}\}^{k+1} \right) = \Gamma_2^r \left(\{u_2^{n+1}\}^k \right) \end{cases} \quad \text{on } \partial\Omega_1^{\text{int}}$$

$$\begin{cases} \Gamma_1^l \left(\{u_2^{n+1}\}^{k+1} \right) = \Gamma_1^l \left(\{u_1^{n+1}\}^k \right) \\ \Gamma_2^l \left(\{u_2^{n+1}\}^{k+1} \right) = \Gamma_2^l \left(\{u_1^{n+1}\}^k \right) \end{cases} \quad \text{on } \partial\Omega_2^{\text{int}}$$

(these IBCs are **local in time**).

Numerical results

t	TBC	Dirichlet
0.25	2	604
0.5	2	612
0.75	2	546
1.0	2	556

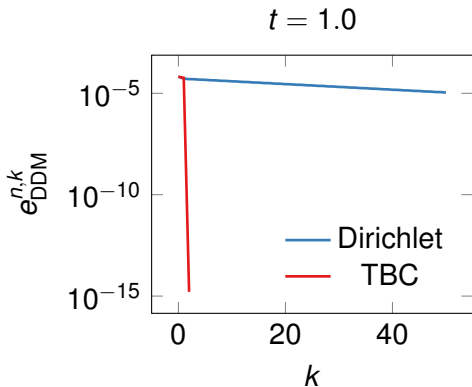


Table: Convergence of the additive Schwarz method, for a precision of $\varepsilon = 10^{-15}$, for an overlap size of 5.

4

Towards TBCs for nonlinear problems

The Serre equations

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- Serre equations [Cienfuegos and Carter, 2011]:

$$\begin{cases} h_t + (hu)_x = 0, \\ u_t + uu_x + gh_x - \frac{1}{3h} \left(h^3 \left(u_{xt} + uu_{xx} - (u_x)^2 \right) \right)_x = 0. \end{cases} \quad (1)$$

The Serre equations

- Objective: use TBCs in the numerical resolution of nonlinear problems;
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- Analogous procedure lead to similar discrete TBCs as obtained for Boussinesq equations, differing only by the coefficients of the TBCs;

Numerical simulations

- Use discrete TBCs:
 - for simulating nonlinear problem;
 - as IBCs in a DDM.

Numerical simulations

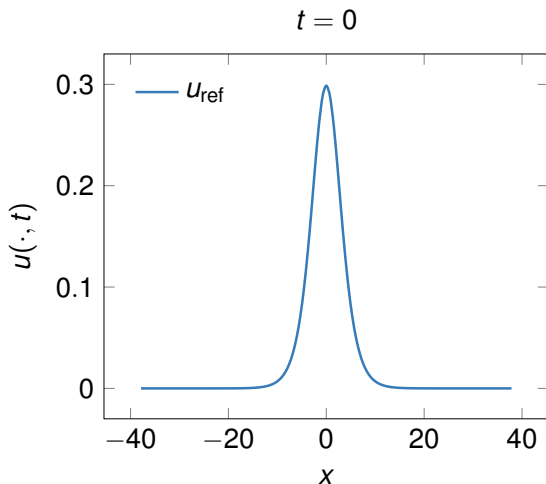
- Use discrete TBCs:
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- Compare results for nonlinear Serre equations using
 - Discrete TBCs coefficients obtained for linearized Serre equations;
 - Discrete TBCs coefficients obtained for linearized Boussinesq equations.

Numerical simulations

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 - as IBCs in a DDM.
- Compare results for nonlinear Serre equations using
 - Discrete TBCs coefficients obtained for linearized Serre equations;
 - Discrete TBCs coefficients obtained for linearized Boussinesq equations.
- Two testcases:
 - Gaussian wave;
 - Solitary wave;

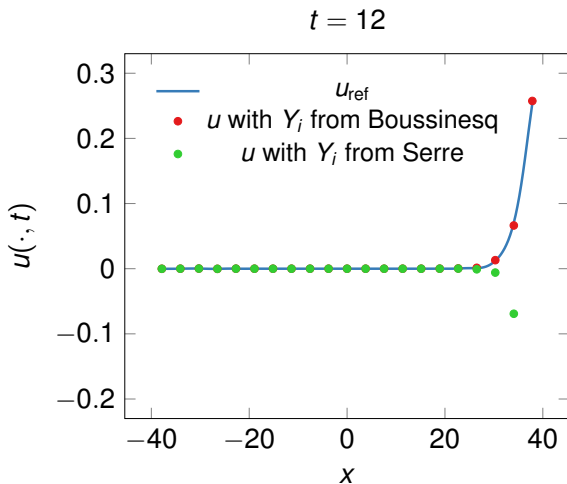
Numerical results - TBCs - Solitary wave

Initial solution:



Numerical results - TBCs - Solitary wave

TBCs with different coefficients:



Numerical results - DDM

Case	Dirichlet	TBC	
		Serre	Boussinesq
Gaussian	120 ~ 160	18	9
Solitary	11	8	8

Table: Convergence of the additive Schwarz method, for a precision of $\varepsilon = 10^{-15}$, for an overlap size of 5.

Question: Coefficients from linearized Boussinesq equations seem to be more efficient than the coefficients from linearized Serre equations.

5

Conclusion

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Conclusion

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- The operators provide good and efficient results
 - as TBCs;
 - as IBCs in a DDM;
- Application to the numerical resolution of nonlinear problems shall be investigated in more details.
- More detailed paper under review [Caldas Steinstraesser et al., 2018].

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