Domain decomposition methods for linearized Boussinesq type equations

Joao Caldas Steinstraesser, Gaspard Kemlin, Antoine Rousseau

16ème Journées de l'Hydrodynamique
November 2018

Domain decomposition methods for linearized Boussinesq type equations

J.G Caldas Steinstraesser [1,3] G. Kemlin [2] A. Rousseau [1,3]

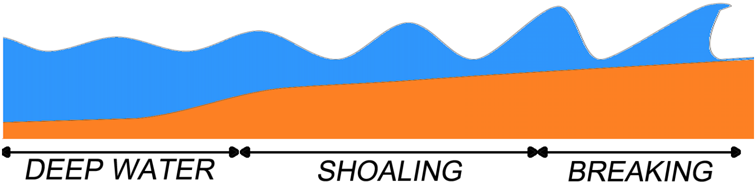
[1] Inria, Team LEMON, Montpellier, France
[2] Inria Chile, Avenida Apoquindo 2827, Las Condes, Santiago, Chile
[3] MERIC, Avenida Apoquindo 2827, Las Condes, Santiago, Chile

1

Introduction

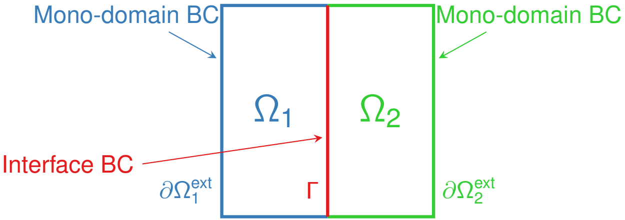
Introduction

- Wave propagation in nearshore area: Boussinesq-type equations;



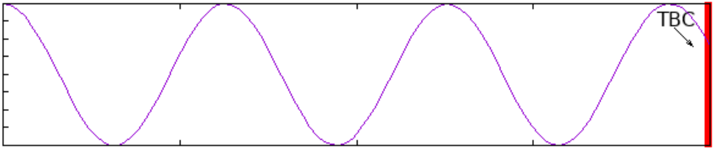
Introduction

- Wave propagation in nearshore area: Boussinesq-type equations;
- Domain Decomposition Methods (DDMs)
  - "Divide and conquer";
  - Main issue: interface boundary conditions (IBCs);
  - Future objective: couple different wave propagation models;



Introduction

- Wave propagation in nearshore area: Boussinesq-type equations;
- Domain Decomposition Methods (DDMs)
  - "Divide and conquer";
  - Main issue: interface boundary conditions (IBCs);
  - Future objective: couple different wave propagation models;
- Transparent Boundary Conditions (TBCs)
  - Numerical resolution in finite computational domain;
  - Optimal interface conditions for DDMs;
  - Cannot be computed exactly.



Objectives

- Derive discrete TBCs for the linearized equations;

Objectives

- Derive discrete TBCs for the linearized equations;
- Test their efficiency as TBCs;

Objectives

- Derive discrete TBCs for the linearized equations;
- Test their efficiency as TBCs;
- Test their efficiency as IBCs in a DDM;

Objectives

- Derive discrete TBCs for the linearized equations;
- Test their efficiency as TBCs;
- Test their efficiency as IBCs in a DDM;
- Test their efficiency for the full nonlinear problem.

The governing equations

- Full equations [Roeber and Cheung, 2012]:

$$\begin{cases} \eta_t + \nabla \cdot [(h+\eta)\, \boldsymbol{u}] + \nabla \cdot \left[\left(\frac{z^2}{2} - \frac{h^2}{6} \right) h \nabla \left(\nabla \cdot \boldsymbol{u} \right) + \left(z + \frac{h}{2} \right) h \nabla \left(\nabla \cdot (h\boldsymbol{u}) \right) \right] = 0, \\ \boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + g\nabla\eta + z \left[\frac{z}{2} \nabla \left(\nabla \cdot \boldsymbol{u}_t \right) + \nabla \left(\nabla \cdot (h\boldsymbol{u}_t) \right) \right] + \boldsymbol{\tau} = 0. \end{cases}$$

The governing equations

- Full equations [Roeber and Cheung, 2012]:

$$\begin{cases} \eta_t + \nabla \cdot [(h+\eta)\,\boldsymbol{u}] + \nabla \cdot \left[\left(\frac{z^2}{2} - \frac{h^2}{6} \right) h \nabla \left(\nabla \cdot \boldsymbol{u} \right) + \left(z + \frac{h}{2} \right) h \nabla \left(\nabla \cdot (h\boldsymbol{u}) \right) \right] = 0, \\ \boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + g\nabla\eta + z \left[\frac{z}{2} \nabla \left(\nabla \cdot \boldsymbol{u}_t \right) + \nabla \left(\nabla \cdot (h\boldsymbol{u}_t) \right) \right] + \boldsymbol{\tau} = 0. \end{cases}$$

- Simplified framework:
  - 1D;
  - Flat bottom;
  - Linearization around $(\overline{\eta}, \overline{u}) = (0, 0)$;
  - Bottom shear stress $\tau = 0$.

The governing equations

- Full equations [Roeber and Cheung, 2012]:

$$\begin{cases} \eta_t + \nabla \cdot [(h+\eta)\,\boldsymbol{u}] + \nabla \cdot \left[\left(\frac{z^2}{2} - \frac{h^2}{6} \right) h \nabla (\nabla \cdot \boldsymbol{u}) + \left(z + \frac{h}{2} \right) h \nabla (\nabla \cdot (h\boldsymbol{u})) \right] = 0, \\ \boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + g\nabla\eta + z \left[\frac{z}{2} \nabla (\nabla \cdot \boldsymbol{u}_t) + \nabla (\nabla \cdot (h\boldsymbol{u}_t)) \right] + \boldsymbol{\tau} = 0. \end{cases}$$

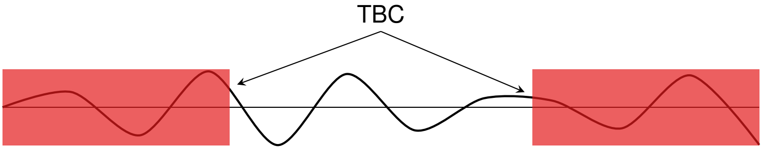
- Simplified framework:
  - 1D;
  - Flat bottom;
  - Linearization around $(\overline{\eta}, \overline{u}) = (0,0)$;
  - Bottom shear stress $\tau = 0$.
- Simplified equations:

$$\begin{cases} \eta_t + h_0 u_x + \tilde{h} u_{xxx} = 0, \\ u_t + g\eta_x + \bar{h} u_{xxt} = 0, \end{cases} \implies u_{tt} + \bar{h} u_{xxtt} - g h_0 u_{xx} - g\tilde{h} u_{xxxx} = 0$$

2

Transparent Boundary Conditions

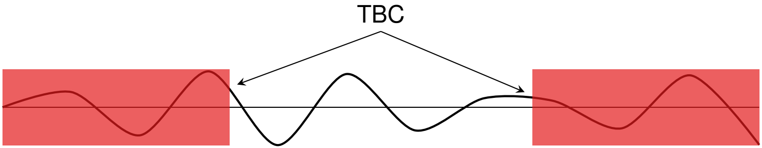
Transparent Boundary Conditions (TBCs)



- Problems are well posed in an infinite domain, but the numerical resolution must be done in a finite computational domain;

$$(A) \begin{cases} \mathcal{L}(u) = 0, & \text{in } \mathbb{R} \\ u \to 0, & x \to \pm\infty \end{cases} \qquad (B) \begin{cases} \mathcal{L}(u) = 0, & \text{in }]0, L[\\ \text{TBCs}, & \text{in } x = 0, L \end{cases}$$

Transparent Boundary Conditions (TBCs)



- Problems are well posed in an infinite domain, but the numerical resolution must be done in a finite computational domain;

$$(A) \begin{cases} \mathcal{L}(u) = 0, & \text{in } \mathbb{R} \\ u \to 0, & x \to \pm\infty \end{cases} \qquad (B) \begin{cases} \mathcal{L}(u) = 0, & \text{in }]0, L[\\ \text{TBCs}, & \text{in } x = 0, L \end{cases}$$

- TBCs assure that the solution of (A), restricted to $[0, L]$, coincides with the solution of (B).

Difficulties

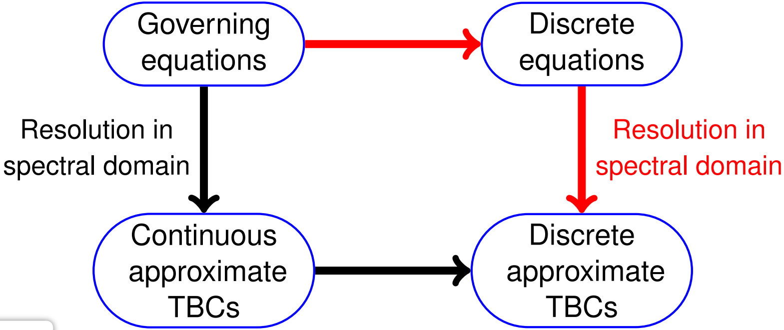
- In general, TBCs
  - Are non local in time and space;
  - Cannot be computed exactly, both analytically and numerically;

Difficulties

- In general, TBCs
  - Are non local in time and space;
  - Cannot be computed exactly, both analytically and numerically;
- Therefore, one must find approximate TBCs;

Difficulties

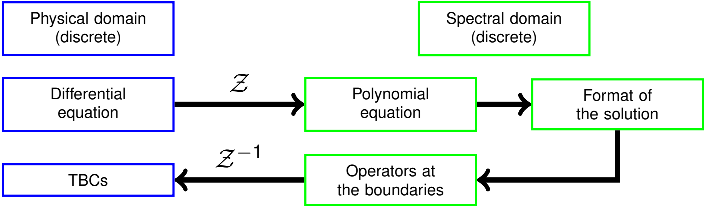
- In general, TBCs
  - Are non local in time and space;
  - Cannot be computed exactly, both analytically and numerically;
- Therefore, one must find approximate TBCs;
- Two usual approaches:



Derivation of TBCs for the discrete equations

Methodology [Arnold and Ehrhardt, 2001, Besse et al., 2015, Kazakova and Noble, 2017]:

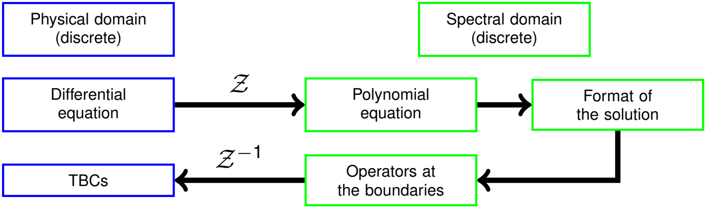
- Discretize the governing equations;



Derivation of TBCs for the discrete equations

Methodology [Arnold and Ehrhardt, 2001, Besse et al., 2015, Kazakova and Noble, 2017]:

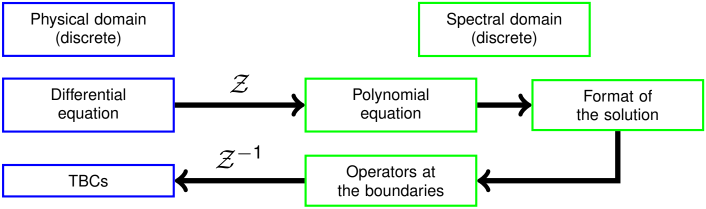
- Discretize the governing equations;
- Use the $\mathcal{Z}$-transform in the complementary set $\mathbb{R}\backslash[0, L]$;



Derivation of TBCs for the discrete equations

Methodology [Arnold and Ehrhardt, 2001, Besse et al., 2015, Kazakova and Noble, 2017]:

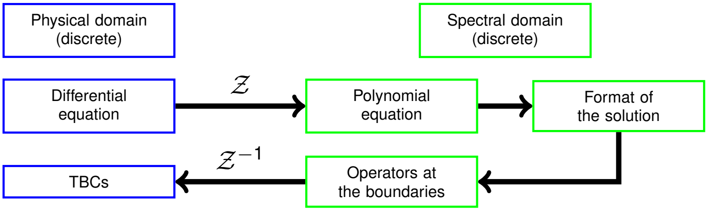
- Discretize the governing equations;
- Use the $\mathcal{Z}$-transform in the complementary set $\mathbb{R}\backslash[0, L]$;
- Solve the problem in the $\mathcal{Z}$-space and find the conditions fulfilled by the solutions at the two boundaries;



Derivation of TBCs for the discrete equations

Methodology [Arnold and Ehrhardt, 2001, Besse et al., 2015, Kazakova and Noble, 2017]:

- Discretize the governing equations;
- Use the $\mathcal{Z}$-transform in the complementary set $\mathbb{R} \backslash [0, L]$;
- Solve the problem in the $\mathcal{Z}$-space and find the conditions fulfilled by the solutions at the two boundaries;
- Use the inverse $\mathcal{Z}$-transform to find the TBCs.



Derivation of TBCs for the discrete equations

Methodology [Arnold and Ehrhardt, 2001, Besse et al., 2015, Kazakova and Noble, 2017]:

- Discretize the governing equations;
- Use the $\mathcal{Z}$-transform in the complementary set $\mathbb{R}\backslash[0, L]$;
- Solve the problem in the $\mathcal{Z}$-space and find the conditions fulfilled by the solutions at the two boundaries;
- Use the inverse $\mathcal{Z}$-transform to find the TBCs.

Left:
$$\begin{cases} \Gamma_1^l\left(\{u^n\}\right) := u_0^n - (Y_5 * u_1)^n + (Y_7 * u_2)^n = 0, \\ \Gamma_2^l\left(\{u^n\}\right) := u_0^n - (Y_6 * u_2)^n + 2(Y_9 * u_3)^n - (Y_8 * u_4)^n = 0, \end{cases}$$

where $(Y_i^n)_{n\in\mathbb{N}}$ depend on the equation and $(Y_i * u_j)^n = \sum_{m=0}^{n} Y_i^m u_j^{n-m}$.

Derivation of TBCs for the discrete equations

Methodology [Arnold and Ehrhardt, 2001, Besse et al., 2015, Kazakova and Noble, 2017]:

- Discretize the governing equations;
- Use the $\mathcal{Z}$-transform in the complementary set $\mathbb{R} \backslash [0, L]$;
- Solve the problem in the $\mathcal{Z}$-space and find the conditions fulfilled by the solutions at the two boundaries;
- Use the inverse $\mathcal{Z}$-transform to find the TBCs.

Right:
$$\begin{cases} \Gamma_1^r\left(\{u^n\}\right) := u_J^n - (Y_1 * u_{J-1})^n + (Y_3 * u_{J-2})^n = 0, \\ \Gamma_2^r\left(\{u^n\}\right) := u_J^n - 2(Y_1 * u_{J-1})^n + (Y_2 * u_{J-2})^n - (Y_4 * u_{J-4})^n = 0, \end{cases}$$

where $(Y_i^n)_{n \in \mathbb{N}}$ depend on the equation and $(Y_i * u_j)^n = \sum_{m=0}^n Y_i^m u_j^{n-m}$.

Derivation of TBCs for the discrete equations

Methodology [Arnold and Ehrhardt, 2001, Besse et al., 2015, Kazakova and Noble, 2017]:

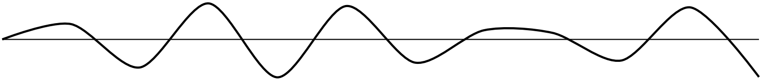
- Discretize the governing equations;
- Use the $\mathcal{Z}$-transform in the complementary set $\mathbb{R}\backslash[0, L]$;
- Solve the problem in the $\mathcal{Z}$-space and find the conditions fulfilled by the solutions at the two boundaries;
- Use the inverse $\mathcal{Z}$-transform to find the TBCs.

Left: $\begin{cases} \Gamma_1^l(\{u^n\}) = 0 \\ \Gamma_2^l(\{u^n\}) = 0 \end{cases}$ **Right:** $\begin{cases} \Gamma_1^r(\{u^n\}) = 0 \\ \Gamma_2^r(\{u^n\}) = 0 \end{cases}$

- These operators are non local in time.

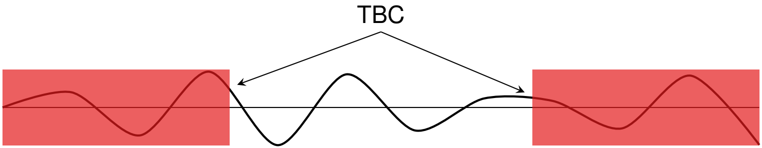
Numerical tests

- Reference solution: computed in a bigger domain.



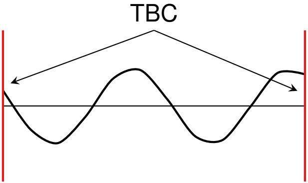
Numerical tests

- Reference solution: computed in a bigger domain.



Numerical tests

- Reference solution: computed in a bigger domain.



Numerical tests

- Reference solution: computed in a bigger domain.

t	e^n
0.25	$1.5 \cdot 10^{-9}$
0.5	$9.5 \cdot 10^{-7}$
0.75	$5.2 \cdot 10^{-6}$
1.0	$8.4 \cdot 10^{-6}$

$$e_T = 4.5 \cdot 10^{-6}$$

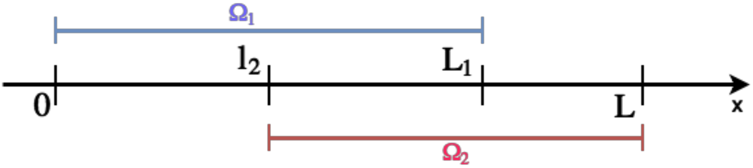
$$e^n = \frac{\|u_{\text{ref}}(\cdot, t^n) - u_{\text{num}}(\cdot, t^n)\|_{l^2}}{\|u_{\text{ref}}(\cdot, t^n)\|_{l^2}}, \qquad e_T = \sqrt{\delta t \times \sum_{n=1}^{n_{\max}} (e^n)^2}.$$

3

Domain Decomposition Method

Domain Decomposition Methods (DDMs)

- Idea: divide the domain in two or more subdomains (possibly overlapping) and solve the problem in each one of them.

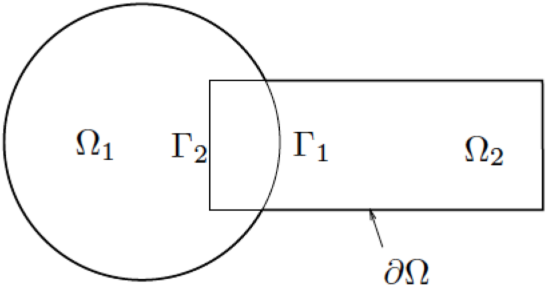


$$\Omega = \Omega_1 \cup \Omega_2, \qquad \partial\Omega_1^{\text{int}} = \partial\Omega_1 \cap \Omega_2, \qquad \partial\Omega_2^{\text{int}} = \partial\Omega_2 \cap \Omega_1$$

- Main difficulty: definition of the interface boundary conditions (IBCs) on $\partial\Omega_1^{\text{int}}$ and $\partial\Omega_2^{\text{int}}$ for the communication between the subdomains.

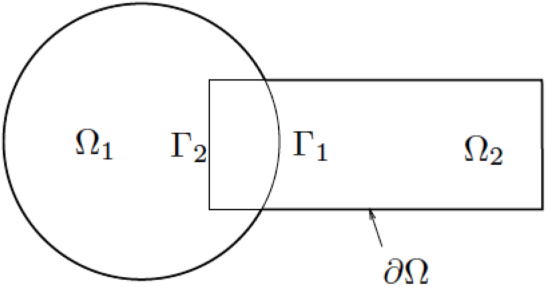
The Schwarz method

- The first and one of the most important types of DDM [Gander, 2008];



The Schwarz method

- The first and one of the most important types of DDM [Gander, 2008];
- Iterative method;



The Schwarz method (1870)

- Equations solved in each subdomain (in parallel or additive form), for updating the solution to time t_{n+1}:

$$\begin{cases} \mathcal{L}(u_1^{n+1,k+1}) = 0, & \text{in } \Omega_1 \\ u_1^{n+1,k+1} = g, & \text{on } \partial\Omega_1^{\text{ext}} \\ \mathcal{B}_1(u_1^{n+1,k+1}) = \mathcal{B}_1(u_2^{n+1,k}), & \text{on } \partial\Omega_1^{\text{int}} \end{cases},$$

$$\begin{cases} \mathcal{L}(u_2^{n+1,k+1}) = 0, & \text{in } \Omega_2 \\ u_2^{n+1,k+1} = g, & \text{on } \partial\Omega_2^{\text{ext}} \\ \mathcal{B}_2(u_2^{n+1,k+1}) = \mathcal{B}_2(u_1^{n+1,k}), & \text{on } \partial\Omega_2^{\text{int}} \end{cases},$$

The Schwarz method

- Optimal IBCs?

The Schwarz method

- Optimal IBCs? TBCs [Japhet and Nataf, 2001]

The Schwarz method

- Optimal IBCs? TBCs [Japhet and Nataf, 2001]
  - Convergence to the solution of the monodomain;

The Schwarz method

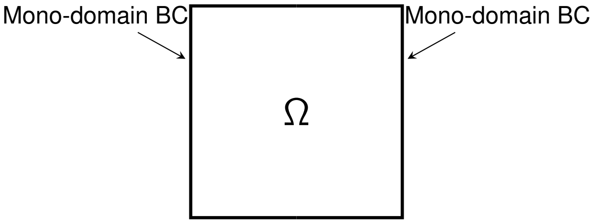
- Optimal IBCs? TBCs [Japhet and Nataf, 2001]
  - Convergence to the solution of the monodomain;
  - Fastest convergence

The Schwarz method

- Optimal IBCs? TBCs [Japhet and Nataf, 2001]
  - Convergence to the solution of the monodomain;
  - Fastest convergence
- We implement the discrete TBCs as IBCs.

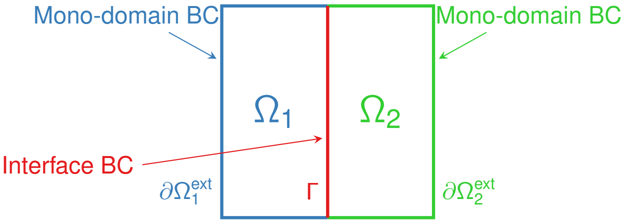
Numerical results

- Reference solution: numerical solution of the problem solved in the monodomain;



Numerical results

- Reference solution: numerical solution of the problem solved in the monodomain;



Numerical results

- Reference solution: numerical solution of the problem solved in the monodomain;
- Objective: compare the number of iterations for convergence to the reference solution in two cases:

Numerical results

- Reference solution: numerical solution of the problem solved in the monodomain;
- Objective: compare the number of iterations for convergence to the reference solution in two cases:
  - DDM using Dirichlet IBCs (original Schwarz method):

$$\begin{cases} u_1^{n+1,k+1} = u_2^{n+1,k}, & \text{on } \partial\Omega_1^{\text{int}} \\ u_2^{n+1,k+1} = u_1^{n+1,k}, & \text{on } \partial\Omega_2^{\text{int}} \end{cases}$$

Numerical results

- Reference solution: numerical solution of the problem solved in the monodomain;

- Objective: compare the number of iterations for convergence to the reference solution in two cases:
  - DDM using the discrete TBCs as IBCs:

$$\begin{cases} \Gamma_1^r\left(\left\{u_1^{n+1}\right\}^{k+1}\right) = \Gamma_1^r\left(\left\{u_2^{n+1}\right\}^{k}\right) \\ \Gamma_2^r\left(\left\{u_1^{n+1}\right\}^{k+1}\right) = \Gamma_2^r\left(\left\{u_2^{n+1}\right\}^{k}\right) \end{cases} \quad \text{on } \partial\Omega_1^{\text{int}}$$

$$\begin{cases} \Gamma_1^l\left(\left\{u_2^{n+1}\right\}^{k+1}\right) = \Gamma_1^l\left(\left\{u_1^{n+1}\right\}^{k}\right) \\ \Gamma_2^l\left(\left\{u_2^{n+1}\right\}^{k+1}\right) = \Gamma_2^l\left(\left\{u_1^{n+1}\right\}^{k}\right) \end{cases} \quad \text{on } \partial\Omega_2^{\text{int}}$$

(these IBCs are local in time).

Numerical results

t	TBC	Dirichlet
0.25	2	604
0.5	2	612
0.75	2	546
1.0	2	556

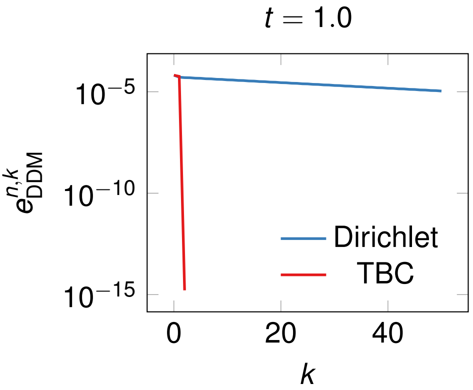


Table: Convergence of the additive Schwarz method, for a precision of $\varepsilon = 10^{-15}$, for an overlap size of 5.

4

Towards TBCs for nonlinear problems

The Serre equations

- Objective: use TBCs in the numerical resolution of nonlinear problems;

The Serre equations

- Objective: use TBCs in the numerical resolution of nonlinear problems;
- Serre equations [Cienfuegos and Carter, 2011]:

$$\begin{cases} h_t + (hu)_x = 0, \\ u_t + uu_x + gh_x - \dfrac{1}{3h}\left(h^3\left(u_{xt} + uu_{xx} - (u_x)^2\right)\right)_x = 0. \end{cases} \tag{1}$$

The Serre equations

- Objective: use TBCs in the numerical resolution of nonlinear problems;
- Serre equations [Cienfuegos and Carter, 2011]:

$$
\begin{cases}
h_t + (hu)_x = 0, \\
u_t + uu_x + gh_x - \dfrac{1}{3h}\left(h^3\left(u_{xt} + uu_{xx} - (u_x)^2\right)\right)_x = 0.
\end{cases}
\tag{1}
$$

- Analogous procedure lead to similar discrete TBCs as obtained for Boussinesq equations, differing only by the coefficients of the TBCs;

Numerical simulations

- Use discrete TBCs:
  - for simulating nonlinear problem;
  - as IBCs in a DDM.

Numerical simulations

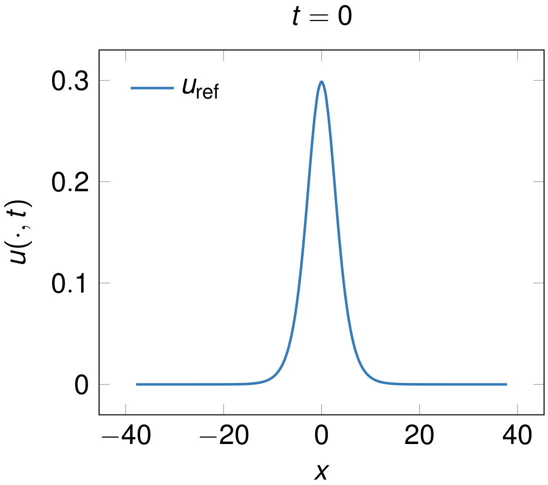
- Use discrete TBCs:
  - for simulating nonlinear problem;
  - as IBCs in a DDM.
- Compare results for nonlinear Serre equations using
  - Discrete TBCs coefficients obtained for linearized Serre equations;
  - Discrete TBCs coefficients obtained for linearized Boussinesq equations.

Numerical simulations

- Use discrete TBCs:
  - for simulating nonlinear problem;
  - as IBCs in a DDM.
- Compare results for nonlinear Serre equations using
  - Discrete TBCs coefficients obtained for linearized Serre equations;
  - Discrete TBCs coefficients obtained for linearized Boussinesq equations.
- Two testcases:
  - Gaussian wave;
  - Solitary wave;

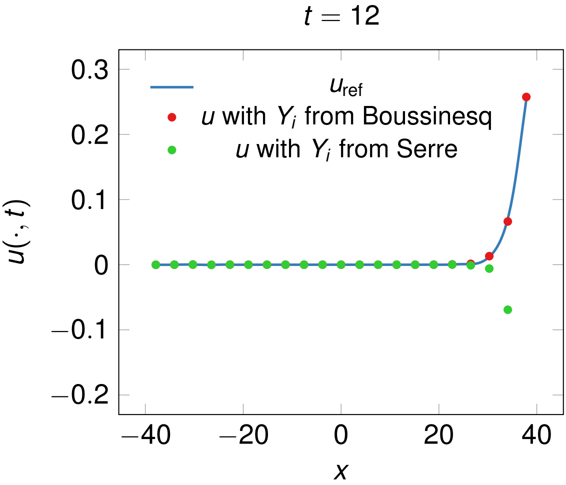
Numerical results - TBCs - Solitary wave

Initial solution:



Numerical results - TBCs - Solitary wave

TBCs with different coefficients:



Numerical results - DDM

Case	Dirichlet	TBC	
		Serre	Boussinesq
Gaussian	$120 \sim 160$	18	9
Solitary	11	8	8

Table: Convergence of the additive Schwarz method, for a precision of $\varepsilon = 10^{-15}$, for an overlap size of 5.

Question: Coefficients from linearized Boussinesq equations seem to be more efficient than the coefficients from linearized Serre equations.

5

Conclusion

Conclusion

- Derivation of discrete TBCs for a class of Boussinesq equations;

Conclusion

- Derivation of discrete TBCs for a class of Boussinesq equations;
- The operators provide good and efficient results
  - as TBCs;
  - as IBCs in a DDM;

Conclusion

- Derivation of discrete TBCs for a class of Boussinesq equations;
- The operators provide good and efficient results
  - as TBCs;
  - as IBCs in a DDM;
- Application to the numerical resolution of nonlinear problems shall be investigated in more details.

Conclusion

- Derivation of discrete TBCs for a class of Boussinesq equations;
- The operators provide good and efficient results
  - as TBCs;
  - as IBCs in a DDM;
- Application to the numerical resolution of nonlinear problems shall be investigated in more details.
- More detailed paper under review [Caldas Steinstraesser et al., 2018].

References I

Arnold, A. and Ehrhardt, M. (2001).
Discrete Transparent Boundary Conditions for the Schrödinger Equation.
Rivista di Matematica della Università di Parma, 6.

Besse, C., Ehrhardt, M., and Lacroix-Violet, I. (2015).
Discrete Artificial Boundary Conditions for the Korteweg-de Vries Equation.
Numerical Methods for Partial Differential Equations.

Caldas Steinstraesser, J. G., Kemlin, G., and Rousseau, A. (2018).
A domain decomposition method for linearized Boussinesq-type equations.
Preprint, available at `https://hal.inria.fr/hal-01797823`.

Cienfuegos, R. and Carter, J. (2011).
The kinematics and stability of solitary and cnoidal wave solutions of the Serre equations.
European Journal of Mechanics B - Fluids, 30(3):259–268.

Gander, M. J. (2008).
Schwarz methods over the course of time.
Electronic Transactions on Numerical Analysis, 31:228–255.

Japhet, C. and Nataf, F. (2001).
The Best Interface Conditions for Domain Decomposition Methods: Absorbing Boundary Conditions.
In *Absorbing Boundaries and Layers, Domain Decomposition Methods: Applications to Large Scale Computations*, pages 348–373, New York. L. Tourrette and L. Halpern (eds.), Nova Science Publisher, Inc.

References II

Kazakova, M. and Noble, P. (2017).
Discrete transparent boundary conditions for the linearized Green-Naghdi system of equations.
https://arxiv.org/pdf/1710.04016.pdf.

Roeber, V. and Cheung, K. F. (2012).
Boussinesq-type model for energetic breaking waves in fringing reef environment.
Coastal Engineering, 70:1–20.