



Space-time Trefftz-DG approach for elasto-acoustic wave propagation

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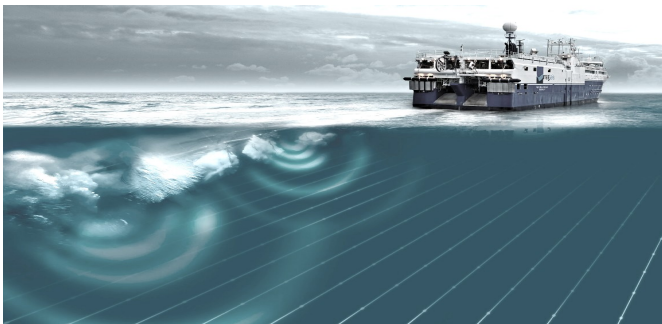
SPACE-TIME TREFFTZ-DG APPROACH FOR ELASTO-ACOUSTIC WAVE PROPAGATION

WCCM 2018 | 13th World Congress on Computational Mechanics

ABSTRACT

SEISMIC SURVEY

FIGURE 1: Ocean Bottom Seismic (OBS) data acquisition



BASIC NUMERICAL METHODS

TABLE 1: Generic properties of the most widely used numerical methods*

Numerical method	Complex geometries	High-order accuracy and hp -adaptivity	Explicit semi-discrete form	Conservation laws	Elliptic problems
FDM	●	●	●	●	●
FVM	●	●	●	●	⊙
FEM	●	●	●	⊙	●
DG-FEM	●	●	●	●	⊙

* J.S.Hesthaven T.Warburton. *Nodal DG methods. Algorithms, analysis, and applications*. 2007

DISCONTINUOUS GALERKIN METHODS

PROS AND CONS



- Adapted to the complex geometries
- High-order accuracy and hp-adaptivity
- Explicit semi-discrete form
- Conservation laws



- Higher number of degrees of freedom,
compared to the methods with continuous approximation

TREFFTZ METHOD

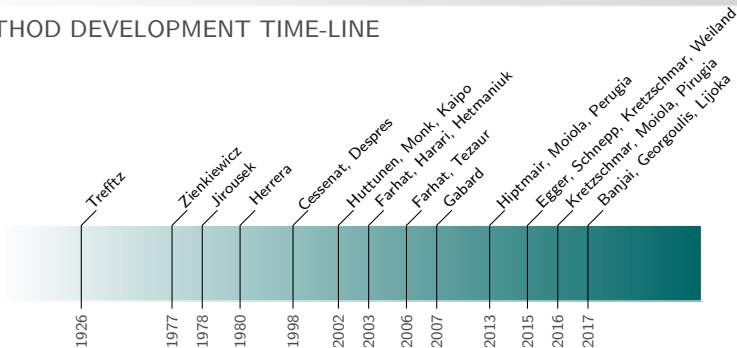
DEFINITION *

Given a region of an Euclidean space of some partitions of that region,
a **Trefftz method** is any procedure for solving BVP of PDE,
using solutions of that PDE or its adjoint, defined in its subregions.

* I.Herrera. *Trefftz method: a general theory*. 2000

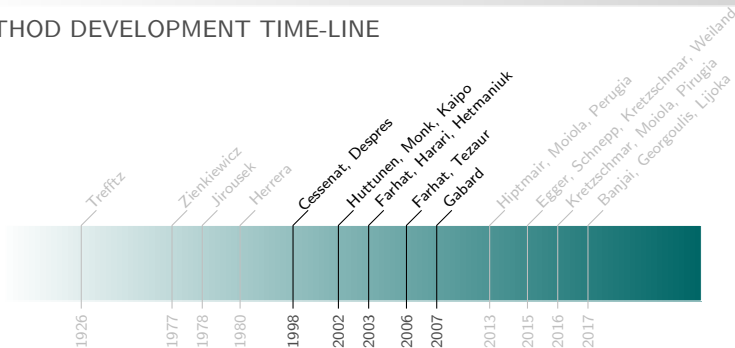
TREFFTZ METHOD

METHOD DEVELOPMENT TIME-LINE



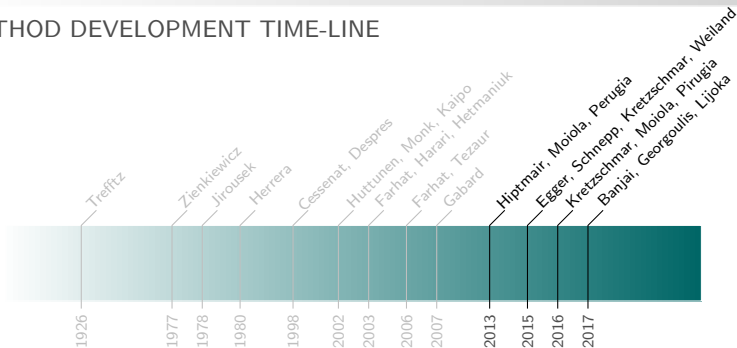
TREFFTZ METHOD

METHOD DEVELOPMENT TIME-LINE



TREFFTZ METHOD

METHOD DEVELOPMENT TIME-LINE



TREFFTZ METHOD

EXPECTED ADVANTAGES AND DRAWBACKS



Higher order of convergence

Flexibility in the choice of basis functions

Low dispersion

Incorporation of propagation directions in the discrete space

Adaptivity and local space-time mesh refinement

TREFFTZ METHOD

EXPECTED ADVANTAGES AND DRAWBACKS



Higher order of convergence

Flexibility in the choice of basis functions

Low dispersion

Incorporation of propagation directions in the discrete space

Adaptivity and local space-time mesh refinement



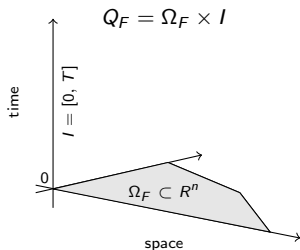
Sparse form of the global space-time matrix

MATHEMATICAL FORMULATION

ACOUSTIC SYSTEM

ACOUSTIC SYSTEM

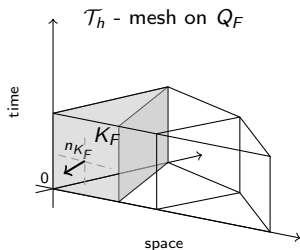
PROBLEM EQUATIONS



$$\left\{ \begin{array}{ll} \frac{1}{c_F^2 \rho_F} \frac{\partial p}{\partial t} + \operatorname{div} \mathbf{v}_F = f & \text{in } Q_F \\ \rho_F \frac{\partial \mathbf{v}_F}{\partial t} + \nabla p = 0 & \text{in } Q_F \\ \mathbf{v}_F = \mathbf{v}_{F0}, \quad p = p_0 & \text{in } \Omega_F \times \{0\} \\ \mathbf{v}_F = \mathbf{g}_F^D & \text{in } \partial\Omega_F \times I \end{array} \right.$$

ACOUSTIC SYSTEM

MESHING



$\mathcal{F}_h = \cup_{K_F \in \mathcal{T}_h} \partial K_F$ - mesh skeleton

$K_F \in \mathcal{Q}_F$ ($c_F, \rho_F \equiv \text{const}$ in K_F)

$v_F, p \in H^1(K_F)$

$\omega_F, q \in H^1(K_F)$

ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION IN THE ELEMENT K_F

AS

(v_F, p)

ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION IN THE ELEMENT K_F

$$\begin{array}{c} \text{AS} \\ (v_F, p) \end{array} \times \begin{array}{c} \text{test} \\ \text{functions} \\ (\omega_F, q) \end{array}$$

ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION IN THE ELEMENT K_F

$$\underbrace{\begin{array}{c} \text{AS} \\ (v_F, p) \end{array}} \times \begin{array}{c} \text{test} \\ \text{functions} \\ (\omega_F, q) \end{array} =$$

SPACE-TIME INTEGRATION ON K_F

ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION IN THE ELEMENT K_F

$$\underbrace{\begin{array}{c} \text{AS} \\ (v_F, p) \end{array}}_{\text{SPACE-TIME INTEGRATION ON } K_F} \times \begin{array}{c} \text{test} \\ \text{functions} \\ (\omega_F, q) \end{array} = \begin{array}{c} \text{VOLUME INTEGRATION TERM} \\ \text{AS} \times (v_F, p) \\ \text{(test functions)} \end{array}$$

ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION IN THE ELEMENT K_F

$$\underbrace{\boxed{\begin{array}{c} AS \\ (v_F, p) \end{array}} \times \boxed{\begin{array}{c} \text{test functions} \\ (\omega_F, q) \end{array}}}_{\text{SPACE-TIME INTEGRATION ON } K_F} = \underbrace{\boxed{\begin{array}{c} AS \\ (\text{test functions}) \end{array}} \times (v_F, p)}_{\text{VOLUME INTEGRATION TERM}} + \underbrace{\boxed{\begin{array}{c} (\text{test functions}) \\ (v_F, p) \end{array}}}_{\text{SURFACE INTEGRATION TERM}}$$

ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION IN THE ELEMENT K_F

$$\underbrace{\boxed{\begin{array}{c} \text{AS} \\ (v_F, p) \end{array}} \times \boxed{\begin{array}{c} \text{test functions} \\ (\omega_F, q) \end{array}}}_{\text{SPACE-TIME INTEGRATION ON } K_F} = \underbrace{\boxed{\begin{array}{c} \text{AS} \\ (\text{test functions}) \end{array}} \times (v_F, p)}_{\text{VOLUME INTEGRATION TERM}} + \underbrace{\boxed{\begin{array}{c} (\text{test functions}) \\ (v_F, p) \end{array}}}_{\text{SURFACE INTEGRATION TERM}}$$

ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION IN THE ELEMENT K_F

$$\underbrace{\boxed{\begin{array}{c} \text{AS} \\ (v_F, p) \end{array}}} \times \boxed{\begin{array}{c} \text{test} \\ \text{functions} \\ (\omega_F, q) \end{array}} = \underbrace{\boxed{\begin{array}{c} \text{test} \\ \text{functions} \end{array}} \text{ in } \mathbf{T}}_{\text{VOLUME INTEGRATION TERM}} + \underbrace{\boxed{\begin{array}{c} (\text{test functions}) \\ (v_F, p) \end{array}}}_{\text{SURFACE INTEGRATION TERM}}$$

SPACE-TIME INTEGRATION ON K_F

ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION IN THE ELEMENT K_F

$$\underbrace{\boxed{\begin{array}{c} \text{AS} \\ (v_F, p) \end{array}} \times \boxed{\begin{array}{c} \text{test functions} \\ (\omega_F, q) \end{array}}}_{\text{SPACE-TIME INTEGRATION ON } K_F} = \boxed{\begin{array}{c} \text{VOLUME INTEGRATION TERM} \\ = 0 \end{array}} + \boxed{\begin{array}{c} \text{SURFACE INTEGRATION TERM} \\ (\text{test functions}) \\ (v_F, p) \end{array}}$$

ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION IN THE ELEMENT K_F

$$\underbrace{\boxed{\begin{array}{c} \text{AS} \\ (v_F, p) \end{array}}} \times \boxed{\begin{array}{c} \text{test} \\ \text{functions} \\ (\omega_F, q) \end{array}} = \boxed{\begin{array}{c} \text{(test functions)} \\ (v_F, p) \end{array}}$$

SPACE-TIME INTEGRATION ON K_F SURFACE INTEGRATION TERM

ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION

$$\begin{array}{c}
 \text{SUM OVER ALL } K_F \\
 \hline
 \boxed{\begin{array}{c} \text{AS} \\ (v_F, p) \end{array}} \times \boxed{\begin{array}{c} \text{test} \\ \text{functions} \\ (\omega_F, q) \end{array}} \\
 \hline
 \text{SPACE-TIME INTEGRATION ON } K_F
 \end{array}
 =
 \begin{array}{c}
 \text{SUM OVER ALL } \partial K_F \\
 \hline
 \text{SURFACE INTEGRATION TERM} \\
 \boxed{\begin{array}{c} (\text{test functions}) \\ (v_F, p) \end{array}}
 \end{array}$$

ACOUSTIC SYSTEM

SPACE-TIME DISCRETIZATION

$$\begin{aligned}
 & - \sum_{K_F} \int_{K_F} \left[p_h \left(\frac{1}{c_F^2 \rho_F} \frac{\partial q}{\partial t} + \operatorname{div} \omega_F \right) + v_{Fh} \cdot \left(\rho_F \frac{\partial \omega_F}{\partial t} + \nabla q \right) \right] dv \\
 & + \sum_{K_F} \int_{\partial K_F} \left[\left(\frac{1}{c_F^2 \rho_F} \hat{p}_h q + \rho_F v_{Fh} \cdot \omega_F \right) n_{K_F}^t + (\hat{p}_h \omega_F + v_{Fh} q) \cdot n_{K_F}^x \right] ds = \sum_{K_F} \int_{K_F} f q dv
 \end{aligned}$$

ACOUSTIC SYSTEM

SPACE-TIME DISCRETIZATION

$$\begin{aligned}
 & - \sum_{K_F} \int_{K_F} \left[p_h \left(\frac{1}{c_F^2 \rho_F} \frac{\partial q}{\partial t} + \operatorname{div} \omega_F \right) + v_{Fh} \cdot \left(\rho_F \frac{\partial \omega_F}{\partial t} + \nabla q \right) \right] dv \\
 & + \sum_{K_F} \int_{\partial K_F} \left[\left(\frac{1}{c_F^2 \rho_F} \hat{p}_h q + \rho_F v_{Fh} \cdot \omega_F \right) n_{K_F}^t + (\hat{p}_h \omega_F + v_{Fh} q) \cdot n_{K_F}^x \right] ds = \sum_{K_F} \int_{K_F} f q dv
 \end{aligned}$$

ACOUSTIC SYSTEM

SPACE-TIME DISCRETIZATION

$$\begin{aligned}
 & - \sum_{K_F} \int_{K_F} \left[p_h \left(\frac{1}{c_F^2 \rho_F} \frac{\partial q}{\partial t} + \operatorname{div} \omega_F \right) + v_{Fh} \cdot \left(\rho_F \frac{\partial \omega_F}{\partial t} + \nabla q \right) \right] dv \\
 & + \sum_{K_F} \int_{\partial K_F} \left[\left(\frac{1}{c_F^2 \rho_F} \hat{p}_h q + \rho_F v_{Fh} \cdot \omega_F \right) n_{K_F}^t + (\hat{p}_h \omega_F + v_{Fh} q) \cdot n_{K_F}^x \right] ds = \sum_{K_F} \int_{K_F} f q dv
 \end{aligned}$$

TREFFTZ SPACE

$$\mathbf{T}_F(\mathcal{T}_h) \equiv \left\{ (\omega_F, q) \in H^1(\mathcal{T}_h)^2 : \forall K_F \in \mathcal{T}_h, \rho_F \frac{\partial \omega_F}{\partial t} + \nabla q = \frac{1}{c_F^2 \rho_F} \frac{\partial q}{\partial t} + \operatorname{div} \omega_F = 0 \right\}$$

ACOUSTIC SYSTEM

SPACE-TIME DISCRETIZATION

$$\begin{aligned}
 & - \sum_{K_F} \int_{K_F} \left[p_h \left(\frac{1}{c_F^2 \rho_F} \frac{\partial q}{\partial t} + \operatorname{div} \omega_F \right) + v_{Fh} \cdot \left(\rho_F \frac{\partial \omega_F}{\partial t} + \nabla q \right) \right] dv \\
 & + \sum_{K_F} \int_{\partial K_F} \left[\left(\frac{1}{c_F^2 \rho_F} \hat{p}_h q + \rho_F v_{Fh} \cdot \omega_F \right) n_{K_F}^t + (\hat{p}_h \omega_F + v_{Fh} q) \cdot n_{K_F}^x \right] ds = \sum_{K_F} \int_{K_F} f q dv
 \end{aligned}$$

TREFFTZ SPACE

$$\mathbf{T}_F(\mathcal{T}_h) \equiv \left\{ (\omega_F, q) \in H^1(\mathcal{T}_h)^2 : \forall K_F \in \mathcal{T}_h, \rho_F \frac{\partial \omega_F}{\partial t} + \nabla q = \frac{1}{c_F^2 \rho_F} \frac{\partial q}{\partial t} + \operatorname{div} \omega_F = 0 \right\}$$

SPACE-TIME DISCRETIZATION IN $\mathbf{T}_F$

$$\sum_{K_F} \int_{\partial K_F} \left[\left(\frac{1}{c_F^2 \rho_F} \hat{p}_h q + \rho_F v_{Fh} \cdot \omega_F \right) n_{K_F}^t + (\hat{p}_h \omega_F + v_{Fh} q) \cdot n_{K_F}^x \right] ds = \sum_{K_F} \int_{K_F} f q dv$$

ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION

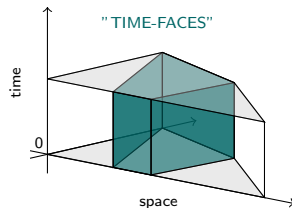
SUM OVER
ALL ∂K_F

SURFACE
INTEGRATION
TERM

ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION

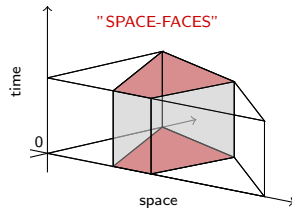
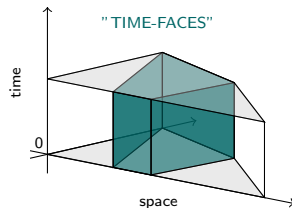
$$\underbrace{\text{SUM THROUGH ALL } \partial K_F}_{\text{SURFACE INTEGRATION TERM}} = \underbrace{\text{SUM THROUGH ALL "TIME-FACES"}}_{\text{SURFACE INTEGRATION TERM}}$$



ACOUSTIC SYSTEM

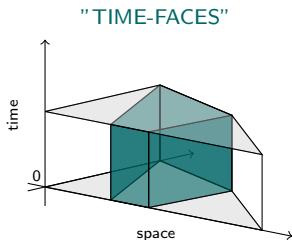
TREFFTZ-DG FORMULATION

$$\underbrace{\text{SUM THROUGH ALL } \partial K_F}_{\text{SURFACE INTEGRATION TERM}} = \underbrace{\text{SUM THROUGH ALL "TIME-FACES"}}_{\text{SURFACE INTEGRATION TERM}} + \underbrace{\text{SUM THROUGH ALL "SPACE-FACES"}}_{\text{SURFACE INTEGRATION TERM}}$$

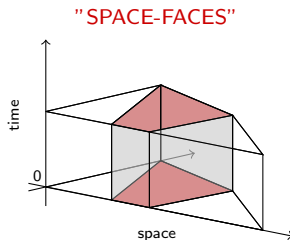


ACOUSTIC SYSTEM

MESH NOTATIONS



internal $\mathcal{F}_h^{I_F} (x - \text{fixed})$
 boundary $\mathcal{F}_h^{D_F} (\partial\Omega_F \times [0, T])$



internal $\mathcal{F}_h^{\Omega_F} (t - \text{fixed})$
 initial $\mathcal{F}_h^{0_F} (\Omega_F \times \{0\})$
 final $\mathcal{F}_h^{T_F} (\Omega_F \times \{T\})$

ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION

$$\sum_{\partial K_F} \int_{\partial K_F} \left[\left(\frac{1}{c_F^2 \rho_F} \hat{p}_h q + \rho_F \mathbf{v}_{Fh} \cdot \omega_F \right) n_{K_F}^t + \left(\hat{p}_h \omega_F + \mathbf{v}_{Fh} q \right) \cdot \mathbf{n}_{K_F}^x \right] ds = \sum_{K_F} \int_{K_F} f q dv$$

NUMERICAL FLUXES THROUGH THE ELEMENT FACES

$$\begin{aligned} \mathcal{F}_h^{I_F} \begin{pmatrix} \mathbf{v}_{Fh} \\ \hat{p}_h \end{pmatrix} &\equiv \begin{pmatrix} \{\mathbf{v}_{Fh}\} + \beta [p_h]_x \\ \{p_h\} + \alpha [\mathbf{v}_{Fh}]_x \end{pmatrix} \\ \mathcal{F}_h^{D_F} \begin{pmatrix} \mathbf{v}_{Fh} \cdot \mathbf{n}_{K_F}^x \\ \hat{p}_h n_{K_F}^x \end{pmatrix} &\equiv \begin{pmatrix} g_{D_F} \cdot \mathbf{n}_{K_F}^x \\ p_h n_{K_F}^x + \alpha (\mathbf{v}_{Fh} - g_{D_F}) \cdot \mathbf{n}_{K_F}^x \end{pmatrix} \end{aligned}$$

ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION

$$\sum_{\partial K_F} \int_{\partial K_F} \left[\left(\frac{1}{c_F^2 \rho_F} \hat{\rho}_h \mathbf{q} + \rho_F \mathbf{v}_F \cdot \omega_F \right) n_{K_F}^t + (\hat{\rho}_h \omega_F + \mathbf{v}_F \cdot \mathbf{q}) \cdot n_{K_F}^x \right] ds = \sum_{K_F} \int_{K_F} f q dv$$

NUMERICAL FLUXES THROUGH THE ELEMENT FACES

$$\begin{aligned} \mathcal{F}_h^{I_F} \begin{pmatrix} v_F \\ \hat{p}_h \end{pmatrix} &\equiv \begin{pmatrix} \{v_F\} + \beta[p_h]_x \\ \{p_h\} + \alpha[v_F]_x \end{pmatrix} \\ \mathcal{F}_h^{D_F} \begin{pmatrix} v_F \cdot n_{K_F}^x \\ \hat{p}_h n_{K_F}^x \end{pmatrix} &\equiv \begin{pmatrix} g_{D_F} \cdot n_{K_F}^x \\ p_h n_{K_F}^x + \alpha(v_F - g_{D_F}) \cdot n_{K_F}^x \end{pmatrix} \\ \mathcal{F}_h^{\Omega_F} \begin{pmatrix} v_F \\ \hat{p}_h \end{pmatrix} &\equiv \begin{pmatrix} v_F^- \\ p_h^- \end{pmatrix} \\ \mathcal{F}_h^{T_F} \begin{pmatrix} v_F \\ \hat{p}_h \end{pmatrix} &\equiv \begin{pmatrix} v_F \\ p_h \end{pmatrix} \\ \mathcal{F}_h^{0_F} \begin{pmatrix} v_F \\ \hat{p}_h \end{pmatrix} &\equiv \begin{pmatrix} v_F^0 \\ p_0 \end{pmatrix} \end{aligned}$$

ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION

Seek $(v_{Fh}, p_h) \in \mathbf{V}(\mathcal{T}_h) \subset \mathbf{T}_F(\mathcal{T}_h)$ s.t. for all $(\omega_F, q) \in \mathbf{V}(\mathcal{T}_h)$ it holds:

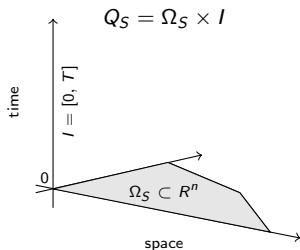
$$\mathcal{A}_{TDG_F}((v_{Fh}, p_h); (\omega_F, q)) = \ell_{TDG_F}(\omega_F, q)$$

MATHEMATICAL FORMULATION

ELASTODYNAMIC SYSTEM

ELASTODYNAMIC SYSTEM

PROBLEM EQUATIONS



$$\left\{ \begin{array}{ll} \mathbf{A} \frac{\partial \boldsymbol{\sigma}}{\partial t} - \epsilon(\mathbf{v}_S) = 0 & \text{in } Q_S \\ \rho_S \frac{\partial \mathbf{v}_S}{\partial t} - \operatorname{div} \boldsymbol{\sigma} = 0 & \text{in } Q_S \\ \mathbf{v}_S = \mathbf{v}_{S0}, \boldsymbol{\sigma} = \boldsymbol{\sigma}_0 & \text{in } \Omega_S \times \{0\} \\ \boldsymbol{\sigma} = \mathbf{g}_{D_S} & \text{in } \partial\Omega_S \times I \end{array} \right.$$

ELASTODYNAMIC SYSTEM

TREFFTZ-DG FORMULATION

$$\sum_{K_S} \int_{\partial K_S} \left[(\mathbf{A} \hat{\boldsymbol{\sigma}}_h : \boldsymbol{\xi} + \rho_S \mathbf{v} \hat{\mathbf{S}}_h \cdot \boldsymbol{\omega}_S) n_{K_S}^t - (\mathbf{v} \hat{\mathbf{S}}_h \boldsymbol{\xi} + \hat{\boldsymbol{\sigma}}_h \boldsymbol{\omega}_S) \cdot \mathbf{n}_{K_S}^x \right] ds = 0$$

NUMERICAL FLUXES THROUGH THE ELEMENT FACES

$$\begin{aligned}
 \mathcal{F}_h^{I_S} \left(\begin{pmatrix} \mathbf{v} \hat{\mathbf{S}}_h \\ \hat{\boldsymbol{\sigma}}_h \end{pmatrix} \right) &\equiv \left(\begin{pmatrix} \{\mathbf{v}_{S_h}\} - \delta[\boldsymbol{\sigma}_h]_x \\ \{\boldsymbol{\sigma}_h\} - \gamma[\mathbf{v}_{S_h}]_x \end{pmatrix} \right) \\
 \mathcal{F}_h^{D_S} \left(\begin{pmatrix} \mathbf{v} \hat{\mathbf{S}}_h \cdot \mathbf{n}_{K_S}^x \\ \hat{\boldsymbol{\sigma}}_h \mathbf{n}_{K_S}^x \end{pmatrix} \right) &\equiv \left(\begin{pmatrix} \mathbf{v}_{S_h} \cdot \mathbf{n}_{K_S}^x - \delta(\boldsymbol{\sigma}_h - \mathbf{g}_{D_S}) \mathbf{n}_{K_S}^x \\ \mathbf{g}_{D_S} \mathbf{n}_{K_S}^x \end{pmatrix} \right) \\
 \mathcal{F}_h^{\Omega_S} \left(\begin{pmatrix} \mathbf{v} \hat{\mathbf{S}}_h \\ \hat{\boldsymbol{\sigma}}_h \end{pmatrix} \right) &\equiv \left(\begin{pmatrix} \mathbf{v}_{S_h}^- \\ \boldsymbol{\sigma}_h^- \end{pmatrix} \right) \\
 \mathcal{F}_h^{T_S} \left(\begin{pmatrix} \mathbf{v} \hat{\mathbf{S}}_h \\ \hat{\boldsymbol{\sigma}}_h \end{pmatrix} \right) &\equiv \left(\begin{pmatrix} \mathbf{v}_{S_h} \\ \boldsymbol{\sigma}_h \end{pmatrix} \right) \\
 \mathcal{F}_h^{0_S} \left(\begin{pmatrix} \mathbf{v} \hat{\mathbf{S}}_h \\ \hat{\boldsymbol{\sigma}}_h \end{pmatrix} \right) &\equiv \left(\begin{pmatrix} \mathbf{v}_{S_0} \\ \boldsymbol{\sigma}_0 \end{pmatrix} \right)
 \end{aligned}$$

ELASTODYNAMIC SYSTEM

TREFFTZ-DG FORMULATION

Seek $(v_{Sh}, \sigma_h) \in \mathbf{V}(\mathcal{T}_h) \subset \mathbf{T}_S(\mathcal{T}_h)$, s.t. for all $K_S \in \mathcal{T}_h$, $(\omega_S, \xi) \in \mathbf{V}(\mathcal{T}_h)$ it holds:

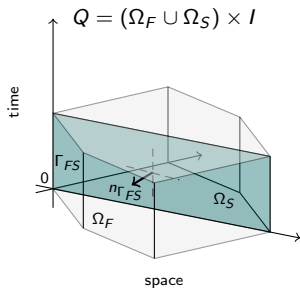
$$\mathcal{A}_{TDG_S}((v_{Sh}, \sigma_h); (\omega_S, \xi)) = \ell_{TDG_S}(\omega_S, \xi)$$

MATHEMATICAL FORMULATION

ELASTO-ACOUSTIC SYSTEM

ELASTO-ACOUSTIC SYSTEM

TRANSMISSION CONDITIONS



$$\begin{cases} \mathbf{v}_F \cdot \mathbf{n}_{\Gamma_{FS}} = \mathbf{v}_S \cdot \mathbf{n}_{\Gamma_{FS}} & \text{on } \Gamma_{FS} \\ \boldsymbol{\sigma} \mathbf{n}_{\Gamma_{FS}} = -\mathbf{p} \mathbf{n}_{\Gamma_{FS}} & \text{on } \Gamma_{FS} \end{cases}$$

ELASTO-ACOUSTIC SYSTEM

TREFFTZ SPACE

$$\mathbf{T}(\mathcal{T}_h) \equiv \left\{ (\omega_F, q, \omega_S, \xi) \in H^1(\mathcal{T}_h)^4 : \rho_F \frac{\partial \omega_F}{\partial t} + \nabla q = \frac{1}{c_F^2 \rho_F} \frac{\partial q}{\partial t} + \operatorname{div} \omega_F = 0, \right. \\ \left. \rho_S \frac{\partial \omega_S}{\partial t} - \operatorname{div} \xi = \frac{\partial \mathbf{A} \xi}{\partial t} - \varepsilon(\omega_S) = 0, \forall K_F, K_S \in \mathcal{T}_h \right\}$$

ELASTO-ACOUSTIC SYSTEM

TREFFTZ SPACE

$$\mathbf{T}(\mathcal{T}_h) \equiv \left\{ (\omega_F, q, \omega_S, \xi) \in H^1(\mathcal{T}_h)^4 : \rho_F \frac{\partial \omega_F}{\partial t} + \nabla q = \frac{1}{c_F^2 \rho_F} \frac{\partial q}{\partial t} + \operatorname{div} \omega_F = 0, \right. \\ \left. \rho_S \frac{\partial \omega_S}{\partial t} - \operatorname{div} \xi = \frac{\partial \mathbf{A} \xi}{\partial t} - \varepsilon(\omega_S) = 0, \forall K_F, K_S \in \mathcal{T}_h \right\}$$

SPACE-TIME DISCRETIZATION IN T

$$\sum_{K_F} \int_{\partial K_F} \left[\left(\frac{1}{c_F^2 \rho_F} \hat{\rho}_h q + \rho_F \mathbf{v}_{Fh} \cdot \omega_F \right) n_{K_F}^t + (\hat{\rho}_h \omega_F + \mathbf{v}_{Fh} q) \cdot \mathbf{n}_{K_F}^x \right] ds \\ + \sum_{K_S} \int_{\partial K_S} \left[(\mathbf{A} \hat{\sigma}_h : \xi + \rho_S \mathbf{v}_{Sh} \cdot \omega_S) n_{K_S}^t - (\mathbf{v}_{Sh} \xi + \hat{\sigma}_h \omega_S) \cdot \mathbf{n}_{K_S}^x \right] ds = \sum_{K_F} \int_{K_F} f q dv$$

ELASTO-ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION

$$\begin{aligned} & \sum_{K_F} \int_{\partial K_F} \left[\left(\frac{1}{c_F^2 \rho_F} \hat{p}_h q + \rho_F \mathbf{v}_{Fh} \cdot \omega_F \right) n_{K_F}^t + (\hat{p}_h \omega_F + \mathbf{v}_{Fh} q) \cdot \mathbf{n}_{K_F}^x \right] ds \\ & + \sum_{K_S} \int_{\partial K_S} \left[(\mathbf{A} \hat{\boldsymbol{\sigma}}_h : \boldsymbol{\xi} + \rho_S \mathbf{v}_{Sh} \cdot \omega_S) n_{K_S}^t - (\mathbf{v}_{Sh} \boldsymbol{\xi} + \hat{\boldsymbol{\sigma}}_h \omega_S) \cdot \mathbf{n}_{K_S}^x \right] ds = \sum_{K_F} \int_{K_F} f q dv \end{aligned}$$

NUMERICAL FLUXES THROUGH THE FLUID-SOLID INTERFACE

$$\mathcal{F}_h^{FS} \begin{pmatrix} \mathbf{v}_{Fh} \cdot \mathbf{n}_{K_F}^x \\ \hat{p}_h n_{K_F}^x \\ \mathbf{v}_{Sh} \cdot \mathbf{n}_{K_S}^x \\ \hat{\boldsymbol{\sigma}}_h n_{K_S}^x \end{pmatrix} \equiv \begin{pmatrix} \mathbf{v}_{Sh} \cdot \mathbf{n}_{K_F}^x \\ p_h n_{K_F}^x + \alpha (\mathbf{v}_{Fh} \cdot \mathbf{n}_{K_F}^x - \mathbf{v}_{Sh} \cdot \mathbf{n}_{K_F}^x) \\ \mathbf{v}_{Sh} \cdot \mathbf{n}_{K_S}^x - \delta (\boldsymbol{\sigma}_h n_{K_S}^x - p_h n_{K_S}^x) \\ -p_h n_{K_S}^x \end{pmatrix}$$

ELASTO-ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION

Seek $(v_{Fh}, p_h, v_{Sh}, \sigma_h) \in \mathbf{V}(\mathcal{T}_h)$: $\forall K_F, K_S \in \mathcal{T}_h, (\omega_F, q, \omega_S, \xi) \in \mathbf{T}(\mathcal{T}_h)$ it holds:

$$\mathcal{A}_{TDG}((v_{Fh}, p_h, v_{Sh}, \sigma_h); (\omega_F, q, \omega_S, \xi)) = \ell_{TDG}(\omega_F, q, \omega_S, \xi)$$

WELL-POSEDNESS

MESH DEPENDENT L^2 NORMS

ACOUSTIC TDG

ELASTODYNAMIC TDG

ELASTO-ACOUSTIC TDG

WELL-POSEDNESS

MESH DEPENDENT L^2 NORMS

ACOUSTIC TDG
ELASTODYNAMIC TDG
ELASTO-ACOUSTIC TDG

WELL-POSED*

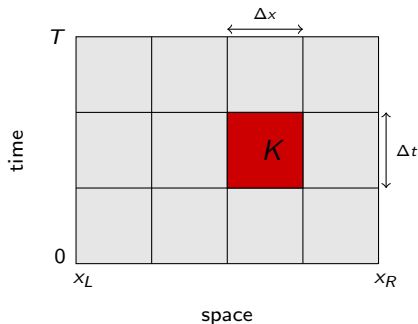
* H.Barucq H.Calandra J.Diaz E.Shishenina. *Space-time Trefftz-Discontinuous Galerkin approximation for elasto-acoustics*. [RR] 2017

IMPLEMENTATION OF THE ALGORITHM

MESH AND GLOBAL MATRIX INVERSION

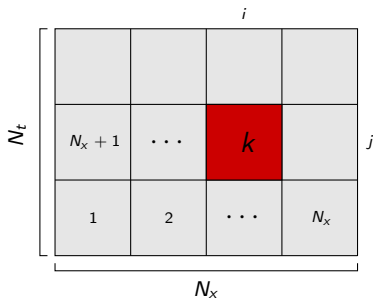
IMPLEMENTATION OF THE ALGORITHM

MESH AND ELEMENT NUMBERING



IMPLEMENTATION OF THE ALGORITHM

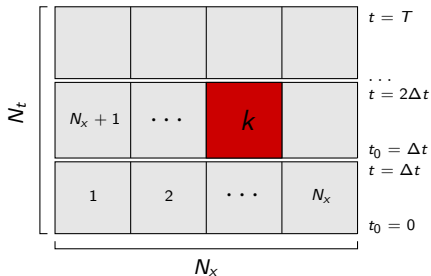
MESH AND ELEMENT NUMBERING



$$k = (j - 1) \times N_x + i$$

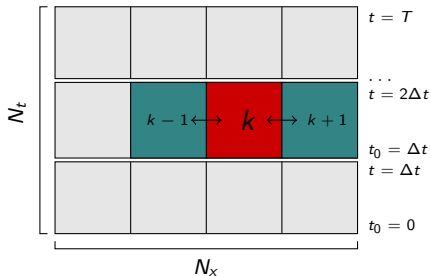
IMPLEMENTATION OF THE ALGORITHM

MESH AND ELEMENT NUMBERING



IMPLEMENTATION OF THE ALGORITHM

MESH AND ELEMENT NUMBERING



IMPLEMENTATION OF THE ALGORITHM

BILINEAR OPERATOR

$$A_{TDG} \equiv \underbrace{\int_{\mathcal{F}_h^\Omega} + \int_{\mathcal{F}_h^T} + \int_{\mathcal{F}_h^0}}_{\mathcal{A}_{TDG}^\Omega} + \underbrace{\int_{\mathcal{F}_h^I} + \int_{\mathcal{F}_h^D}}_{\mathcal{A}_{TDG}^I}$$

IMPLEMENTATION OF THE ALGORITHM

BILINEAR OPERATOR

$$A_{TDG} \equiv \underbrace{\int_{\mathcal{F}_h^\Omega} + \int_{\mathcal{F}_h^T} + \int_{\mathcal{F}_h^0}}_{\mathcal{A}_{TDG}^\Omega} + \underbrace{\int_{\mathcal{F}_h^I} + \int_{\mathcal{F}_h^D}}_{\mathcal{A}_{TDG}^I}$$

GLOBAL SPACE-TIME MATRIX

$$\mathbf{M} = \Delta_x \mathbf{M}_\Omega + \Delta_t \mathbf{M}_I$$

IMPLEMENTATION OF THE ALGORITHM

BILINEAR OPERATOR

$$A_{TDG} \equiv \underbrace{\int_{\mathcal{F}_h^\Omega} + \int_{\mathcal{F}_h^T} + \int_{\mathcal{F}_h^0}}_{A_{TDG}^\Omega} + \underbrace{\int_{\mathcal{F}_h^I} + \int_{\mathcal{F}_h^D}}_{A_{TDG}^I}$$

GLOBAL SPACE-TIME MATRIX

$$\mathbf{M} = \Delta_x M_\Omega + \Delta_t M_I$$

M_Ω - block-diagonal

IMPLEMENTATION OF THE ALGORITHM

BILINEAR OPERATOR

$$A_{TDG} \equiv \underbrace{\int_{\mathcal{F}_h^\Omega} + \int_{\mathcal{F}_h^T} + \int_{\mathcal{F}_h^0}}_{A_{TDG}^\Omega} + \underbrace{\int_{\mathcal{F}_h^I} + \int_{\mathcal{F}_h^D}}_{A_{TDG}^I}$$

GLOBAL SPACE-TIME MATRIX

$$\mathbf{M} = \Delta_x M_\Omega + \Delta_t M_I$$

M_Ω - block-diagonal M_I - sparse

IMPLEMENTATION OF THE ALGORITHM

GLOBAL MATRIX INVERSION

$$\mathbf{M}^{-1} \equiv \left[\Delta_x \mathbf{M}_\Omega + \Delta_t \mathbf{M}_I \right]^{-1}$$

IMPLEMENTATION OF THE ALGORITHM

GLOBAL MATRIX INVERSION

$$\begin{aligned}
 \mathbf{M}^{-1} &\equiv \left[\Delta_x \mathbf{M}_\Omega + \Delta_t \mathbf{M}_I \right]^{-1} \\
 &= \\
 &\left[\mathbf{I} + \kappa \left(\mathbf{M}_\Omega^{-1} \mathbf{M}_I \right) \right]^{-1} \left[\Delta_x \mathbf{M}_\Omega \right]^{-1}
 \end{aligned}$$

IMPLEMENTATION OF THE ALGORITHM

GLOBAL MATRIX INVERSION

$$\begin{aligned}\mathbf{M}^{-1} &\equiv \left[\Delta_x \mathbf{M}_\Omega + \Delta_t \mathbf{M}_I \right]^{-1} \\ &= \\ &\left[\mathbf{I} + \kappa (\mathbf{M}_\Omega^{-1} \mathbf{M}_I) \right]^{-1} \left[\Delta_x \mathbf{M}_\Omega \right]^{-1}\end{aligned}$$

TAYLOR EXPANSION (IN CONDITION OF SMALL ENOUGH κ)

$$\mathbf{M}^{-1} = \left[\sum_{n=0}^{\infty} (-1)^n \kappa^n (\mathbf{M}_\Omega^{-1} \mathbf{M}_I)^n \right] \left[\Delta_x \mathbf{M}_\Omega \right]^{-1}$$

IMPLEMENTATION OF THE ALGORITHM

GLOBAL MATRIX INVERSION

$$\begin{aligned} \mathbf{M}^{-1} &\equiv \left[\Delta_x \mathbf{M}_\Omega + \Delta_t \mathbf{M}_I \right]^{-1} \\ &= \\ &\left[\mathbf{I} + \kappa (\mathbf{M}_\Omega^{-1} \mathbf{M}_I) \right]^{-1} \left[\Delta_x \mathbf{M}_\Omega \right]^{-1} \end{aligned}$$

TAYLOR EXPANSION (IN CONDITION OF ENOUGH SMALL $\kappa \equiv \frac{\Delta_t^d}{\Delta_x^d}$)

$$\mathbf{M}^{-1} = \left[\sum_{n=0}^{\infty} (-1)^n \kappa^n (\mathbf{M}_\Omega^{-1} \mathbf{M}_I)^n \right] \left[\Delta_x \mathbf{M}_\Omega \right]^{-1}$$

block-diagonal

IMPLEMENTATION OF THE ALGORITHM

GLOBAL MATRIX INVERSION

$$\begin{aligned}\mathbf{M}^{-1} &\equiv \left[\Delta_x \mathbf{M}_\Omega + \Delta_t \mathbf{M}_I \right]^{-1} \\ &= \\ &\left[\mathbf{I} + \kappa (\mathbf{M}_\Omega^{-1} \mathbf{M}_I) \right]^{-1} \left[\Delta_x \mathbf{M}_\Omega \right]^{-1}\end{aligned}$$

TAYLOR EXPANSION (IN CONDITION OF ENOUGH SMALL κ)

$$\mathbf{M}^{-1} = \left[\sum_{n=0}^{\infty} (-1)^n \kappa^n (\mathbf{M}_\Omega^{-1} \mathbf{M}_I)^n \right] \left[\Delta_x \mathbf{M}_\Omega \right]^{-1}$$

block-diagonal $\times$ sparse

IMPLEMENTATION OF THE ALGORITHM

APPROXIMATE INVERSION ($\kappa = 10^{-2}$)

n	$\Delta_x = 10^{-2}$	$\Delta_x = 2 \cdot 10^{-2}$	$\Delta_x = 5 \cdot 10^{-2}$	$\Delta_x = 10^{-1}$
3	1.4166E-05	4.3741E-05	2.8780E-04	2.5772E-03
4	3.1623E-07	1.2656E-06	5.3868E-05	1.2674E-03
5	2.8903E-07	9.1744E-07	4.1029E-05	1.3010E-03

FULL INVERSION ($\kappa = 10^{-2}$)

n	$\Delta_x = 10^{-2}$	$\Delta_x = 2 \cdot 10^{-2}$	$\Delta_x = 5 \cdot 10^{-2}$	$\Delta_x = 10^{-1}$
·	2.2540E-07	8.9583E-07	5.5811E-05	1.3004E-03

IMPLEMENTATION OF THE ALGORITHM

APPROXIMATE INVERSION ($\kappa = 10^{-2}$)

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FULL INVERSION ($\kappa = 10^{-2}$)

n	$\Delta_x = 10^{-2}$	$\Delta_x = 2 \cdot 10^{-2}$	$\Delta_x = 5 \cdot 10^{-2}$	$\Delta_x = 10^{-1}$
·	2.2540E-07	8.9583E-07	5.5811E-05	1.3004E-03

NUMERICAL RESULTS

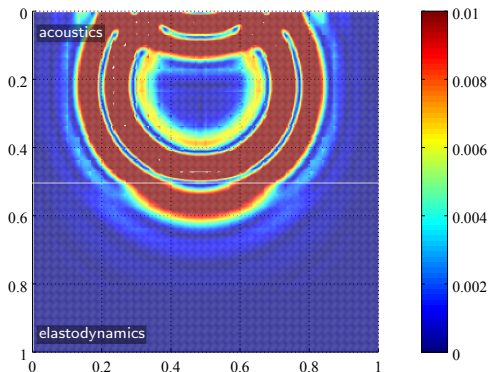
2D + TIME SIMULATIONS

NUMERICAL RESULTS

2D ELASTO-ACOUSTIC SYSTEM

FIGURE 9: 2D Elasto-acoustic system.

Numerical velocity $v(x, 0.567)$. Dirichlet boundaries

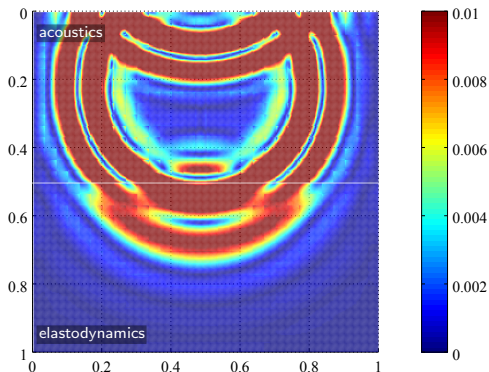


NUMERICAL RESULTS

2D ELASTO-ACOUSTIC SYSTEM

FIGURE 9: 2D Elasto-acoustic system.

Numerical velocity $v(x, 0.633)$. Dirichlet boundaries

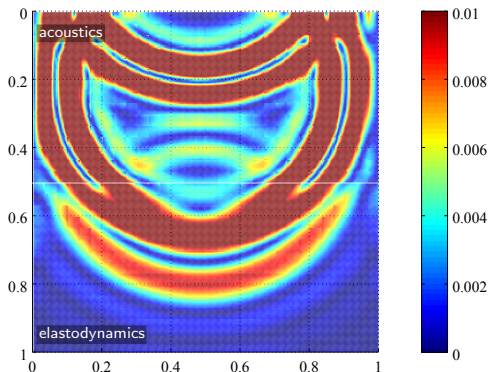


NUMERICAL RESULTS

2D ELASTO-ACOUSTIC SYSTEM

FIGURE 9: 2D Elasto-acoustic system.

Numerical velocity $v(x, 0.700)$. Dirichlet boundaries



NUMERICAL RESULTS

2D ACOUSTIC AND ELASTODYNAMIC SYSTEMS

FIGURE 10: 2D Acoustic system.

Convergence of velocity v_F in function of cell size h

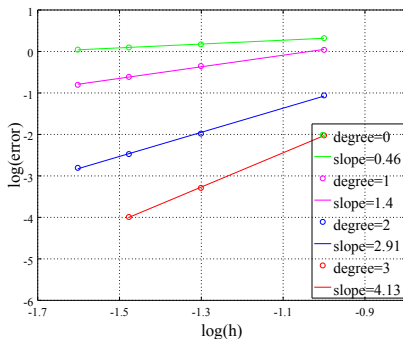
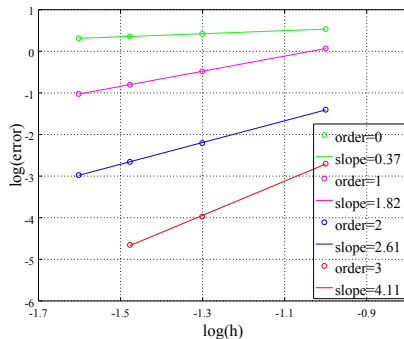


FIGURE 11: 2D Elastodynamic system.

Convergence of velocity v_S in function of cell size h



CONCLUSION

CONCLUSION



Implementation of the method for 2D acoustic, elastic, elasto-acoustic systems

Numerical validation of prototype Matlab code

Implementation of an algorithm of the approximate inversion

Development of 2D TDG acoustic propagator in the framework of Elasticus code

CONCLUSION

ON-GOING WORK AND PERSPECTIVES



Development of 2D TDG elastic propagator in the framework of Elasticus code; optimization

Exploration of space-time tent pitching meshes

Analysis of performance

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THANK YOU FOR ATTENTION!

IMPLEMENTATION OF THE ALGORITHM

POLYNOMIAL BASIS (ORDER 3)

p=0, ndof=2

$$\phi_1^v = 0$$

$$\phi_2^v = 1$$

$$\phi_1^p = -c_F$$

$$\phi_2^p = 0$$

p=1, ndof=4

$$\phi_3^v = x$$

$$\phi_4^v = c_F t$$

$$\phi_3^p = -c_F^2 t$$

$$\phi_4^p = -c_F x$$

p=2, ndof=6

$$\phi_5^v = -\frac{x^2}{2} - \frac{c_F^2 t^2}{2}$$

$$\phi_6^v = -c_F x t$$

$$\phi_5^p = c_F^2 x t$$

$$\phi_6^p = c_F \left(\frac{x^2}{2} + \frac{c_F^2 t^2}{2} \right)$$

p=3, ndof=8

$$\phi_7^v = -\frac{x^3}{6} - \frac{x c_F^2 t^2}{2}$$

$$\phi_8^v = -\frac{c_F^3 t^3}{6} - \frac{x^2 c_F t}{2}$$

$$\phi_7^p = c_F \left(\frac{c_F^3 t^3}{6} + \frac{x^2 c_F t}{2} \right)$$

$$\phi_8^p = c_F \left(\frac{x^3}{6} + \frac{x c_F^2 t^2}{2} \right)$$