Arrow's (im)possibility theorem

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Definition paragraph

Arrow's (im)possibility theorem is one of the most famous and important contributions in economics. It concerns the difficulty to aggregate a set of individual preferences, given as rankings of a set of available alternatives, into a unique social preferences ranking via a social welfare function, or into a unique social choice. Arrow proves that in a specific framework, it is impossible to find a social welfare function which simultaneously satisfies four conditions: universal domain, weak Pareto principle, independence of irrelevant alternatives, and no dictator. Our notice presents this theorem, one of its proofs, and, we hope, invites the reader to discover social choice theory.

1 Introduction

Kenneth Arrow (1921-2017) was one of the greatest economists of the twentieth century. In 1972, with J. Hicks, he received the Sveriges Bank Prize in Economic Sciences in Memory of Alfred Nobel aged 51 and remains its youngest recipient.

He made fundamental contributions in the fields of general equilibrium, mathematical programming, risk and insurance economics, organizational theory, and social choice and welfare. By his own admission ([Stotsky, 2014]) if he had not existed "it wouldn't have made that much difference" to general equilibrium theory, whereas in social choice theory his contribution is not questioned, a fact that made him say: "So that, I'm proud of". On the origins of his work on social choice, one can refer to [Kelly, 1987, Suppes, 2005], and [Arrow, 2014].

Before formally explaining and proving this theorem, we will in turn discuss the question it originated from, the framework and the method followed by Arrow, as well as the scope of this result.

As the English poet Donne pointed out in 1624: "No man is an island". Indeed, it is essential to note that in every human society life is lived in the company of others, and that we must constantly make decisions in common that will affect us all. Obviously, in the highly unlikely case where everyone agrees unanimously there is no problem. The question arises in case of disagreement between individuals about choices that will affect them all. Taking up the idea of Knight's tripartition [Knight, 1944], Arrow considers that human societies have developed three alternative solutions to the problems of collective choice: authority, custom, and consensus. Within the later category, Arrow distinguishes between voting (in the political field) and the market (in the economic field). The question that Arrow asks is: how is it possible to arrive at aggregate judgements at the level of society in a transparent and indisputable way? In other words, how can we guarantee the sovereignty and rationality of individuals to make a collective decision? One can reformulate the question more specifically as follows: from individual preferences P_i dealing with different alternatives (i.e a ranking) how does one construct a function of social preference (i.e a ranking function) $\mathsf{P} = f(\mathsf{P}_1, \mathsf{P}_2, ..., \mathsf{P}_n)$ which is faithful to these preferences and rational? This question concerns as much the choice of a program on television for a family as the determination of the objectives of a company from the preferences of its shareholders, or the choice of a common policy for a set of countries.

To answer this question, Arrow did not resort to a historical or comparative approach but rather retained an analytical approach. His idea was to adopt an axiomatic to simultaneously study all possible aggregation methods in a unified framework. His objective was thus to study all the *social preference functions*, also called, with Arrow himself, *social welfare functions* (when society only wants the best option —and not a ranking— it is a *social choice function*). Arrow started his reasoning with the ideas that individuals are rational, that their preferences are ordinal (i.e the intensity of intrapersonal comparisons is meaningless), not comparable, and that they are not modified by the aggregation process. Arrow then defines what he calls "reasonable conditions" for the construction of the social welfare function, while noting that these can naturally be disputed. The number of these conditions may vary according to the way in which Arrow's construction is exposed, they are usually presented as four: universal domain, weak Pareto principle, independence of irrelevant alternatives, and no dictator. We will naturally return hereafter to the meaning of each of these principles. It is based on them that Arrow provides the answer to the question he posed (and which we call since "Arrow's (im)possibility theorem"): if there are at least three alternatives and two individuals then there is no social welfare function that satisfies these four principles.

This is a result of major importance because of its universal scope. It includes all possible and imaginable forms of social welfare function in this framework. The four principles that form Arrow's "reasonable conditions" appear undemanding and natural. The result indicates, in particular, that all the voting rules are imperfect if Arrow's framework and axioms are considered satisfactory. For example, the majority rule respects the four principles, but is not a social choice function since it can lead to incoherent, because cyclical, social choices as Condorcet had already pointed out in the eighteenth century. As a consequence, the impact of this theorem on economic analysis is considerable since it opens both a new field of analysis with the axiomatic approach of social choice, and renews the foundations of the theory of welfare. But beyond that, this result influences part of political philosophy as well as political science.

2 Arrow's theorem

There are a large number of different demonstrations of Arrow's theorem. One even gets the impression that finding a new one has become a recurrent exercise in the theory of social choice. Today, the two main routes taken are those based on the notion of "decisive coalition" following [Arrow, 1951], [Arrow, 1963], and [Sen, 2017], and that of "pivotal player" following [Barberá, 1980]. As [Sen, 2014] notes, what matters is not so much the speed with which this theorem is obtained but the ease with which, without thorough mathematical knowledge, it can be understood. In its initial version, Arrow's theorem only assumes basic knowledge in logic. The presentation we retain does the same and will distinguish the formal setup (which is composed of the data, the problem and the principles) and the proof of the theorem.

2.1 The formal setup

2.1.1 Data and problem

We are given a finite set of $N \geq 3$ alternatives $\mathcal{A} = \{a, b, c \dots\}$ (denoted here by lower case letters of the beginning of the alphabet), and a finite set of n players or voters $\mathcal{V} = \{1, 2 \dots, n\}$ denoted here by (small) integers or lower case letters beyond h in the alphabet $(i, j, \dots)$.

Each voter has a complete set of *preferences* among the alternatives, i.e. a total strict (non reflexive) order. Thus "a is preferred to b by player i" is denoted $a \succ_i b$. Player i's complete preferences order is called P_i. In this short presentation, we restrict players to strict orders for simplicity. Thus, "a and b are indifferent to player i" (which would be denoted $a \approx_i b$ or $a \succeq_i b$ and $b \succeq_i a$) is not allowed. It is not difficult to extend the results to either strict or non strict player orders. It makes statements and proofs longer, but not more difficult. The set of all possible (strict) orders over $\mathcal{A}$ is denoted $P_s \subset P$ the set of total orders.

A *profile* is a set of preferences for each voter, i.e. an ordered set

$$\Pi = \{\mathsf{P}_1, \mathsf{P}_2, \dots, \mathsf{P}_n\} \in P_s^n = \mathcal{P}\,.$$

The objective is to construct a *social welfare function*, SWF, (or *social preferences function*) $f : \mathcal{P} \to P$ from profiles to preferences assigning a complete system of *social preferences* $\mathsf{P}_\Pi = f(\Pi) \in P$ to every profile Π. The social preferences will be simply denoted $\geq$ if no ambiguity results, or with a subscript such as $\geq_\Pi$ to denote the social preferences under the profile Π, for a given SWF. If $a \geq b$ and $b \geq a$, we write $a \simeq b$. To mean that $a \geq b$ but $b \ngeq a$, we write $a > b$.

Two remarks are in order:

1. While it is possible, although unnecessary, to restrict individual preferences to being strict, it is very desirable to allow for non-strict social preferences, as purely symmetric players' preferences, such as in Condorcet's paradox, should lead to a tie in the social preferences.

2. This way of posing the problem hides what is sometimes considered as a separate axiom called *Universal domain*, i.e. the fact that the domain of the SWF be all of $\mathcal{P}$ as opposed to some subset of it. Restricting this domain was at the root of a slight mistake in Arrow's original publications [Arrow, 1950, Arrow, 1951], but also a possible path to obviating the (im)possibility theorem (see [Sen and Pattanaik, 1969]).

2.1.2 Desirable properties

We describe here a set of properties that a SWF might exhibit. While all of them seem desirable, the first two will always be part of the *axioms*, i.e. *properties that we definitely assume the SWF to share*.

1. **Unanimity rule** (Or "Pareto efficiency"): a pair of alternatives being ordered the same way by all players should be ordered accordingly by the social preferences:
$$\forall a, b \in \mathcal{A}\,, [\forall i \in \mathcal{V}\,,\ a \succ_i b] \Rightarrow a > b\,.$$

2. **IIA** (For Independence of Irrelevant Alternatives): the relative preference between two alternatives in the social preferences should only depend on the preferences between these two alternatives in the players' preferences. If the relative rankings of two alternatives a and b are the same in a profile Π as in a profile Π', then a and b must have the same social relative rankings in P_Π and $\mathsf{P}_{\Pi'}$, irrespective of the rankings of any other alternatives.

3. **Non-dictatorship** A player is a *dictator* if the social order reflects his own preferences, irrespective of other players' preferences. Non-dictatorship says that there should not be a dictator.

   What is really wanted is the stronger assumption of *anonymity*: all voters should be equal, swapping the preferences among them should not change the resulting social preferences. But it turns out that the weaker requirement of non dictatorship suffices to prove the impossibility theorem.

4. **Neutrality** The conclusions of the social choice should not depend on which alternatives are considered. If the preferences between a and b in Π are the same as the preferences between a' and b' in Π', then the social preference between a and b in P_Π should be the same as that between a' and b' in $\mathsf{P}_{\Pi'}$. Stated otherwise: *all alternatives are equal*

   While some authors have posited neutrality as an axiom, this has been attacked by [Samuelson, 1977] where an example is given where neutrality violates the common sense of ethics. Yet we will show that it is implied by the unanimity rule and IIA.

2.2 Theorem and proof

We aim to prove Arrow's (im)possibility theorem:

Theorem 1 ([Arrow, 1951, Arrow, 1963]) *The three axioms of Pareto efficiency, IIA, and non-dictatorship are inconsistent.*

We shall proceed via the equivalent statement:

Proposition 1 *If a social welfare function satisfies the axioms of Pareto efficiency and IIA, it necessarily is a dictatorship of one of the players.*

The proof goes in three steps: 1/ The neutrality lemma (demonstration that Pareto efficiency and IIA imply neutrality), 2/ use of the definition of decisive coalitions, and strongly decisive coalitions which are, in essence "joint dictatorship" of a coalition, and of Blau's lemma which states that a decisive coalition is necessarily strongly decisive, 3/ Sen's partitioning lemma which shows that a decisive coalition of two or more players always have a decisive strict sub-coalition, leading to the conclusion.

Lemma 1 (Neutrality) *Unanimity rule and IIA imply neutrality.*

Proof (This proof essentially follows [Geanakoplos, 2005].) Let Π be such that $a \geq_\Pi b$ (resp. $a >_\Pi b$). Let $(a', b') \neq (a, b)$ and Π' be such that the relative preferences of a' and b' in Π' be the same as those of a and b in Π. (Π and Π' may coincide.) Assume first that $a' \neq b$, $b' \neq a$. (But we allow either $a' = a$ or $b' = b$.) Define another profile Π'' by modifying Π as follows: if $a' \neq a$, place alternative a' immediately above a and, if $b' \neq b$, place b' immediately below b,

This does not modify the relative preferences of a and b, and in this profile Π'', a' and b' have the same relative preferences as have a and b in Π and therefore the same as they have in Π'. By unanimity $a' \geq_{\Pi''} a$ and $b \geq_{\Pi''} b'$. By IIA, it still holds that $a \geq_{\Pi''} b$ (resp. $a >_{\Pi''} b$). By transitivity, it follows from these three preferences that $a' \geq_{\Pi''} b'$ (resp. $a' >_{\Pi''} b$) and by IIA again, $a' \geq_{\Pi'} b'$ (resp. $a' >_{\Pi'} b'$). Finally, if $a' = b$ or $b' = a$ or both, use a third alternative c and the same reasoning for the sequence of pairs $(a, b) \to (c, b) \to (c, a) \to (b, a) \to (a', b')$ to reach the same conclusion. ∎

Definition 1 *A coalition $\mathcal{S} \subset \mathcal{A}$ is called* (a, b)-decisive *in a given SWF f if, when in a profile Π its members agree to set $a \succ b$ while all other players choose $b \succ a$ it results that $a > b$ in the preferences $f(\Pi)$. It is* strongly decisive *if the conclusion $a > b$ holds irrespective of the preferences of the players outside the coalition.*

Proposition 2 *It follows from the neutrality lemma that for a SWF satisfying Pareto optimality and IIA, if a coalition is (a, b)-decisive, it is decisive for all pairs of alternatives.*

Lemma 2 ([Blau, 1957]) *Under Pareto efficiency and IIA, decisive coalitions are strongly decisive.*

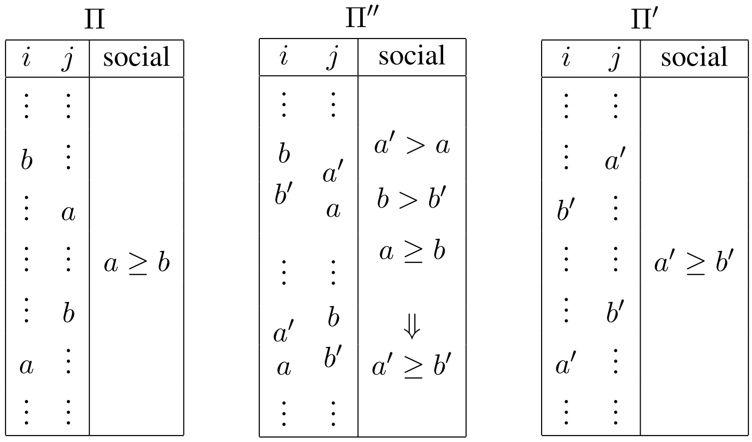

Figure 1: Proof of the neutrality lemma. Π and Π' are given. In Π'', $a' > a$ and $b > b'$ are by unanimity, $a \geq b$ by IIA, $a' \geq b'$ by transitivity, conclusion in Π' by IIA.

Proof Let $\mathcal{S}$ be a decisive coalition. Let Π be such that $\forall i \in \mathcal{S}$, $a \succ_i b$ and $\forall j \in \mathcal{S}^c$, $b \succ_j a$. By decisiveness, $a >_\Pi b$. Let c be another alternative. Consider a profile Π' where $\forall i \in \mathcal{S}$, $a \succ_i c \succ_i b$, and $\forall j \in \mathcal{S}^c$, c is on top of the preferences, and other preferences are arbitrary. Then, by decisiveness of $\mathcal{S}$ and the neutrality lemma, $a >_{\Pi'} c$, and by unanimity $c >_{\Pi'} b$. Hence by transitivity, $a >_{\Pi'} b$. Finally, construct a profile Π'' from Π' by moving c anywhere in all individual preferences. This does not modify any relative preferences of a and b in the individual preferences, and therefore by IIA not either in the social preferences, hence $a >_{\Pi''} b$ irrespective of the orderings in $\mathcal{S}^c$. ∎

Lemma 3 (Partitioning lemma. [Sen, 1970]) *Let $\mathcal{S}$ be a decisive coalition of cardinal greater than one, and $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ be a non-trivial partition of it. (i.e., $\mathcal{S}_1 \neq \emptyset$, $\mathcal{S}_2 \neq \emptyset$, $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$.) Then either $\mathcal{S}_1$ or $\mathcal{S}_2$ is decisive.*

Proof Let the premises of the lemma hold. Let $a \neq b \neq c$ be three alternatives, and a profile Π be such that,

- $\forall i \in \mathcal{S}_1\,,\ a \succ_i b \succ_i c$,
- $\forall j \in \mathcal{S}_2\,,\ b \succ_j c \succ_j a$,
- $\forall k \notin \mathcal{S}\,,\ c \succ_k a \succ_k b$.

Because $\mathcal{S}$ is decisive, it holds that $b >_\Pi c$. If $a >_\Pi c$, the sub-coalition $\mathcal{S}_1$ is decisive, at least in profile Π. If, to the contrary, $c \geq_\Pi a$, by transitivity, it follows

Π

S	S^c	social
$\vdots$	$\vdots$	
a	$\vdots$	
$\vdots$	b	$a > b$
$\vdots$	$\vdots$	
b	a	
$\vdots$	$\vdots$	

Π'

S	S^c	social
$\vdots$	c	
a	$\vdots$	$a > c$
c	$\vdots$	$c > b$
$\vdots$		$\Downarrow$
b	$\vdots$	$a > b$
$\vdots$	$\vdots$	

Figure 2: Proof of Blau's lemma. The social preference shown in Π is by decisiveness of S. In Π', $a > c$ is by decisiveness, $c > b$ by unanimity, the conclusion $a > b$ by transitivity. By IIA, it is not changed if c is moved around.

that $b >_{\Pi} a$. Then $\mathcal{S}_2$ is decisive in profile Π. But by IIA, the rankings of c relative to a and b is irrelevant. Therefore the conclusion holds in any profile. ∎

Corollary 1 *A SWF enjoying Pareto efficiency and IIA is necessarily dictatorial.*

Proof The grand coalition $\mathcal{V}$ is decisive by Pareto efficiency. Partition it, and recursively partition the resulting decisive sub-coalitions until it remains a single player in this sub-coalition. Any preference it has results in the same preference in the social preferences. Hence it is a dictator. ∎

3 Conclusion

Arrow's spectacular result, which is his doctoral dissertation, must result in "engagement rather than resignation," as Sen argued in his speech when he received the Sveriges Bank Prize in Economic Sciences in Memory of Alfred Nobel in 1998. Arrow defended this idea in the conclusion of his 1972 speech when he himself received this award. We follow this idea throughout this notice by using parentheses for the term "(im)possibility". Arrow himself following an advice of Koopmans on this point, calls "possibility theorem" a result generally known as "impossibility theorem".

Thus, since this result of 1951 two paths have mainly been followed: some have modified the Arrovian framework (for illustration [Sen, 1969] proved that if the social choice function is only quasi-transitive —i.e the indifference is no longer transitive— there is no more dictator but at least one player with a veto) and others have modified the principles (as e.g. [Black, 1948] who proved that when individual preferences are unimodal and there is an odd number of individuals, the majority rule leads to a transitive order at the collective level). Moreover, in recent years, other economists have been studying the different properties of all possible rules of social choice by trying to demonstrate whether some dominate them. All these contributions are extremely interesting and important

However to finish this conclusion we wish to draw the reader's attention to one of the most beautiful spin-off of Arrow's theorem, namely the Gibbard-Satterthwaite theorem, which is at the heart of mechanism design. The first is related to the second in the sense that on the one hand Arrow from the beginning posed the question of the veracity of the individual preferences announced by the agents; and on the other hand, in the sense that Gibbard used it in its original proof.

3.1 Social choice and manipulation

The problem of *social choice* is concerned with choosing one best "social alternative" given the preferences of the players. Therefore, a social choice function (SCF) is a mapping $F : \mathcal{P} \to \mathcal{A}$ from profiles to alternatives. Typically the problem of electing a committee chairman or a political President.

A result closely related to Arrow's theorem, another impossibility theorem, is credited to Gibbard [Gibbard, 1973] and Satterthwaite [Satterthwaite, 1975]. It is concerned with preventing "strategic playing". A player i has an opportunity to *play strategically* if, by pretending another preferences ordering than its "true" one, it may get a better result, according to its true ordering.

Definition 2

1. *Given a SCF F, player i has an opportunity to* play strategically *(or to* manipulate *F) at some profile Π in which its ordering is P_i, if there exists a different ordering P'_i such that, in the profile Π' obtained by replacing P_i by P'_i in Π, one has $F(\Pi') \succ_i F(\Pi)$ (in the ordering P_i).*

2. *A SCF F is called* strategy-proof *if no player ever has an opportunity to play strategically whatever the profile:*

$$\forall i \in \mathcal{V}\,,\ \forall \Pi = (\mathsf{P}_i, \mathsf{P}_{-i}) \in \mathcal{P}\,,\ \forall \mathsf{P}'_i \in P_s\,, \quad F(\Pi) \succeq_i F(\mathsf{P}'_i, \mathsf{P}_{-i})\,.$$

The impossibility result is sated in a way similar to Proposition 1:

Theorem 2 ([Gibbard, 1973, Satterthwaite, 1975]) *If a SCF is strategy-proof and an onto function, it is dictatorial.*

3.2 Analysis

All proofs of theorem 2 are significantly longer than that of Arrow's theorem above. There are parallel proofs of both (see, e.g. [Reny, 2001]) and even at least one proof of a more general result encompassing both (see [Eliaz, 2004]), but all too long to be detailed here. Here we only want to stress the parallel between the underlying hypotheses in the two theorems, as they do not look similar at first reading.

We need to introduce a rather specific notion of monotonicity.

Definition 3 *A SCF is called* monotonic *if, the following holds: let* Π *and* Π' *be two profiles. If for all* $i \in \mathcal{V}$*, the subsets of* $\mathcal{A}$*:* $\{a \mid F(\Pi) \succ_i a\}$ *as defined by* Π' *contain the subsets* $\{a \mid F(\Pi) \succ_i a\}$ *as defined by* Π*, then* $F(\Pi') = F(\Pi)$*.*

The point we wish to stress in that definition is that it contains a hidden notion of IIA. Indeed, the result should hold irrespective of the relative orders of the alternatives in the subsets quoted and in their complementary subsets in $\mathcal{A}$. Only the rankings with respect to the social choice are relevant.

We may also extend here the unanimity rule or Pareto efficiency:

Definition 4 *A SCF* F satisfies the unanimity rule *(is* Pareto efficient*) if, whenever the same alternative is the preferred alternative of all players, it becomes the social choice:*

$$[\exists a \in \mathcal{A} : \ \forall i \in \mathcal{V}\,,\ \forall b \neq a \in \mathcal{A}\,,\ a \succ_i b] \Rightarrow F(\Pi) = a\,.$$

The similarity between the hypotheses of the two theorems follows from the following two propositions:

Proposition 3 ([Muller and Satterthwaite, 1977]) *If a SCF is strategy-proof and onto, it is monotonic and Pareto efficient.*

Proposition 4 *If a SCF is monotonic and Pareto efficient, it is dictatorial.*

The second proposition above is the difficult one to prove. The similarity resides in the first one [Muller and Satterthwaite, 1977] whose proof is easy:

Proof of Proposition 3. Let Π and Π' be related as in Definition 3, and suppose that $F(\Pi) = a \neq F(\Pi')$. Make the change from Π to Π' one player at a time in numeric order. Denote by Π_k the profile obtained after changing P_k to P'_k. (And $\Pi_0 = \Pi$). At some point, we have that $F(\Pi_{i-1}) = a \neq b = F(\Pi_i)$. By strategy-proofness, it follows that $a \succ_i b$ in P_i while $b \succ_i a$ in P'_i. But by hypothesis, if $a \succ_i b$ in P_i it is a fortiori true in Π'. A contradiction. Therefore the SCF is monotonic.

Because F is assumed to be an onto function, for any given $a \in \mathcal{A}$, there exists a profile Π such that $F(\Pi) = a$. Build Π' by moving a at the top of the preferences of all players. By monotonicity, it still holds that $F(\Pi') = a$. Now, get Π'' by shuffling at will the preferences of all players *below* a, leaving a at their top position. Because of the definition of monotonicity, and specifically its IIA-like character, it still holds that $F(\Pi'') = a$. Therefore the SCF is Pareto efficient. ∎

Cross References

Anarchy

Capitalism

Economic analysis of law

Electoral systems

Gibbard-Satterthwaite theorem

Heterarchy

Liberty

Ordoliberalism

Political competition

Political economy

Politicians

Power indices

Public goods

Public interest

Rationality

Simple majority

Voting power indices

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