Implementation of Control Strategies for Sterile Insect Techniques

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Abstract

In this paper, we propose a sex-structured entomological model that serves as a basis for design of control strategies relying on releases of sterile male mosquitoes (*Aedes spp*) and aiming at elimination of the wild vector population in some target locality. We consider different types of releases (constant and periodic impulsive), providing sufficient conditions to reach elimination. However, the main part of the paper is focused on the study of the periodic impulsive control in different situations. When the size of wild mosquito population cannot be assessed in real time, we propose the so-called *open-loop* control strategy that relies on periodic impulsive releases of sterile males with constant release size. Under this control mode, global convergence towards the mosquito-free equilibrium is proved on the grounds of sufficient condition that relates the size and frequency of releases. If periodic assessments (either synchronized with the releases or more sparse) of the wild population size are available in real time, we propose the so-called *closed-loop* control strategy, under which the release size is adjusted in accordance with the wild population size estimate. Finally, we propose a mixed control strategy that combines open-loop and closed-loop strategies. This control mode renders the best result, in terms of overall time needed to reach elimination and the number of releases to be effectively carried out during the whole release campaign, while requiring for a reasonable amount of released sterile insects.

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1. Introduction

Since decades, the control of vector-borne diseases has been a major issue in Southern countries. It recently became a major issue in Northern countries too. Indeed, the rapid expansion of air travel networks connecting regions of
5   endemic vector-borne diseases to Northern countries, and the rapid invasion and establishment of mosquitoes population, like *Aedes albopictus*, in Northern hemisphere have amplified the risk of Zika, Dengue, or Chikungunya epidemics[1].

For decades, chemical control was the main tool to control or eradicate mosquitoes. Taken into account resistance development and the impact of in-
10   secticides on the biodiversity, other alternatives have been developed, such as biological control tools, like the Sterile Insect Technique (SIT).

Sterile Insect Technique (SIT) is a promising control method that has been first studied by E. Knipling and collaborators and first experimented successfully in the early 1950's by eradicating screw-worm population in Florida. Since then,
15   SIT has been applied on different pest and disease vectors (see [1] for an overall presentation of SIT and its applications).

The classical SIT relies on massive releases of males sterilized by ionizing radiations. However, another technique, called the *Wolbachia* technique, is under consideration. *Wolbachia* [2] is a symbiotic bacterium that infects many
20   Arthropods, including some mosquito species in nature. These bacteria have many particular properties, including one that is very useful for vector control: the cytoplasmic incompatibility (CI) property [3, 4]. CI can be used for two different control strategies:

- *Incompatible Insect Technique (IIT):* males infected with CI-inducing *Wol-*
25   *bachia* produce altered sperms that cannot successfully fertilize uninfected eggs. This can result in a progressive reduction of the target population. Thus, IIT can be seen as equivalent to classical SIT.

- *Population Replacement (PR):* in this case, males and females, infected with CI-inducing *Wolbachia*, are released in a susceptible (uninfected) pop-
30   ulation, such that *Wolbachia*-infected females will produce more offspring

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[1]See, for instance, the most recent distribution map of *Aedes albopictus* provided by ECDC (European Centre for Disease Prevention and Control, `https://ecdc.europa.eu/en/publications-data/aedes-albopictus-current-known-distribution-june-2018`)

than uninfected females. Because *Wolbachia* is maternally inherited, this will result in a population replacement by *Wolbachia*-infected mosquitoes (such replacements or invasions have been observed in natural population, see [5] for the example of Californian *Culex pipiens*). Recent studies have shown that PR may be very interesting with *Aedes aegypti*, shortening their lifespan (see for instance [6]), or more interesting, cutting down their competence for dengue virus transmission [7]. However, it is also acknowledged that *Wolbachia* infection can have fitness costs, so that the introgression of *Wolbachia* into the field can fail [6].

Based on these biological properties, classical SIT and IIT (see [8, 9, 10, 11, 12, 13] and references therein) or population replacement (see [14, 15, 16, 17, 18, 19, 20, 6, 21] and references therein) have been modeled and studied theoretically in a large number of papers, in order to derive results to explain the success or failure of these strategies using discrete, continuous or hybrid modeling approaches, temporal and spatio-temporal models. More recently, the theory of monotone dynamical systems [22] has been applied efficiently to study SIT [23, 13] or population replacement [24, 25, 26] systems.

In this paper, we derive and study a dynamical system to model the release and elimination process for SIT/IIT. We analyze and compare constant continuous/periodic impulsive releases and derive conditions relating the sizes and frequency of the releases that are sufficient to ensure successful elimination. Such conditions enable the design of SIT-control strategies with constant or variable number of sterile males to be released that drive the wild population of mosquitoes towards elimination. Among all the previous strategies, we are also able to derive the best strategy, meaning the one that needs to release the least amount of sterile males to reach elimination. This can be of utmost importance for field applications.

The outline of the paper is as follows. In Section 2, we first develop and briefly study a simple entomological model that describes the natural evolution of mosquitoes. Then, in Section 3, we introduce a constant continuous SIT-control and determine the size of constant releases that ensures global elimination of wild mosquitoes in the target locality. In Section 4, periodic impulsive SIT-control with constant impulse amplitude is considered, and a sufficient condition relating the size and frequency of periodic releases is derived to ensure global convergence towards the mosquito-free equilibrium. This condition enables the design of open-loop (or feedforward) strategies that ensure mosquito elimination in finite time and without assessing the size of wild mosquito population. Alternatively, Section 5 is focused on the design of closed-loop (or feedback) SIT-control strategies, which are achievable when periodic measurements (either synchronized with releases or more sparse) of the wild population size are available in real time. Notice that such estimates may be obtained in practice e.g. by use of Mark-Release-Recapture (MRR) technique [27]. In such situation, the release amplitude is computed on the basis of these measurements. Thorough analysis of the feedback SIT-control implementation mode leads to another sufficient condition to reach mosquito elimination. This condition re-

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lates not only the size and frequency of periodic releases but also the frequency of sparse measurements. Finally, in Section 6 we propose a mixed control strategy for periodic impulsive SIT-control. The latter is essentially based on the use of the smallest of the release values proposed by the previous open-loop and closed-loop strategies. It turns out that this control mode renders the best result from multiple perspectives: in terms of overall time needed to reach elimination and of peak-value of the input control, but also in terms of total amount of released sterile insects and of number of releases to be effectively carried out during a whole SIT-control campaign. The paper ends with numerical simulations highlighting the key features and outcomes of periodic impulsive SIT-control strategies (Section 7) followed by discussion and conclusions.

2. A sex-structured entomological model

We consider the following 2-dimensional system to model the dynamics of mosquito populations. It involves two state variables, the number of males M and the number of females F.

$$
\begin{cases}
\dot{M} = r\rho F e^{-\beta(M+F)} - \mu_M M, & \text{(1a)} \\
\dot{F} = (1-r)\rho F e^{-\beta(M+F)} - \mu_F F. & \text{(1b)}
\end{cases}
$$

All the parameters are positive, and listed in Table 1. The model assumes that all females are equally able to mate. It includes direct and/or indirect competition effect at different stages (larvae, pupae, adults), through the parameter β. The latter may be seen as the ratio, $\frac{\sigma}{K}$, between σ, a quantity characterizing the transition between larvae and adults under density dependence and larval competition, and a carrying capacity K, typically proportional to the breeding sites capacity. The primary sex ratio in offspring is denoted by $r \in (0, 1)$, and ρ represents the mean number of eggs that a single female can deposit in average per day. Last, μ_M and μ_F represent, respectively, the mean death rate of male and female adult mosquitoes. As a rule, it is observed that in general the male mortality is larger, and we assume throughout the paper that:

$$
\mu_M \geq \mu_F. \tag{2}
$$

Parameter	Description	Unit
r	Primary sex ratio	$--$
ρ	Mean number of eggs deposited per female per day	day^{-1}
μ_M, μ_F	Mean death rates for males & females per day	day^{-1}
β	Characteristic of the competition effect per individual	$--$

Table 1: Parameters of the sex-structured entomological model (1)

Existence and uniqueness of the solutions of the Cauchy problem for dynamical system (1) follow from standard theorems, ensuring continuous differentiability of the latter with respect to time. System (1) is *dissipative*: there exists a bounded positively invariant set $\mathcal{D}$ with the property that, for any bounded set in $E \subset \mathbb{R}_+^2$, there exists $t^* = t(\mathcal{D}, E)$ such that $\big(M(0), F(0)\big) \in E$ implies $\big(M(t), F(t)\big) \in \mathcal{D}$ for all $t > t^*$. The set $\mathcal{D}$ is called an *absorbing set*. In our case, it may be taken, e.g., as:

$$\mathcal{D} = \{(M, F) \ : \ 0 \leq M \leq C, 0 \leq F \leq C\} \tag{3}$$

for some $C > 0$.

Remark 1. *Population models of the form $\dot{N} = B(N)N - \mu N$ for several birth rate functions, including $B(N) = e^{-\beta N}$, have been studied in* [28]. *Maturation delay can also be included* [28].

Obviously $E_0^* = (0, 0)$ is a trivial equilibrium of system (1), called the *mosquito-free equilibrium*. Being the state to which one desires to drag the system by adequate releases of sterile insects, it will play a central role in the sequel. Denote for future use

$$\mathcal{N}_F := \frac{(1 - r)\rho}{\mu_F}, \qquad \mathcal{N}_M := \frac{r\rho}{\mu_M}. \tag{4}$$

These positive constants represent *basic offspring numbers* related to the wild female and male populations, respectively. The first of them governs the number of equilibria, as stated by the following result, whose proof presents no difficulty and is left to the reader.

Theorem 1 (Equilibria of the entomological model)**.**

- If $\mathcal{N}_F \leq 1$, then system (1) possesses E_0^* as unique equilibrium.

- If $\mathcal{N}_F > 1$, then system (1) also possesses a unique positive *equilibrium* $E^* := (M^*, F^*)$, namely

$$F^* = \frac{\mathcal{N}_F}{\mathcal{N}_F + \mathcal{N}_M} \frac{1}{\beta} \ln \mathcal{N}_F, \qquad M^* = \frac{\mathcal{N}_M}{\mathcal{N}_F + \mathcal{N}_M} \frac{1}{\beta} \ln \mathcal{N}_F.$$

Notice that the total population at the nonzero equilibrium is given by $M^* + F^* = \dfrac{1}{\beta} \ln \mathcal{N}_F$. It depends upon the basic offspring number and the competition parameter β. As an example, mechanical control through reduction of the breeding sites induces an increase of β and consequently a decrease of the population at equilibrium. Analogously, altering biological parameters may modify the basic offspring number, and therefore the size of the population.

The stability of the equilibria is addressed by the following result.

Theorem 2 (Stability properties of the entomological model)**.**

- If $\mathcal{N}_F \leq 1$, then the (unique) equilibrium E_0^* is Globally Asymptotically Stable (GAS) for system (1).

- If $\mathcal{N}_F > 1$, then E_0^* is unstable for system (1), and E^* is GAS in $\mathcal{D} \setminus \{(M, 0), M \in \mathbb{R}_+\}$.

Figure 1 shows the convergence of all trajectories to the positive equilibrium in the viable case, when $\mathcal{N}_F > 1$ (the pertinent case for the applications we have in mind).
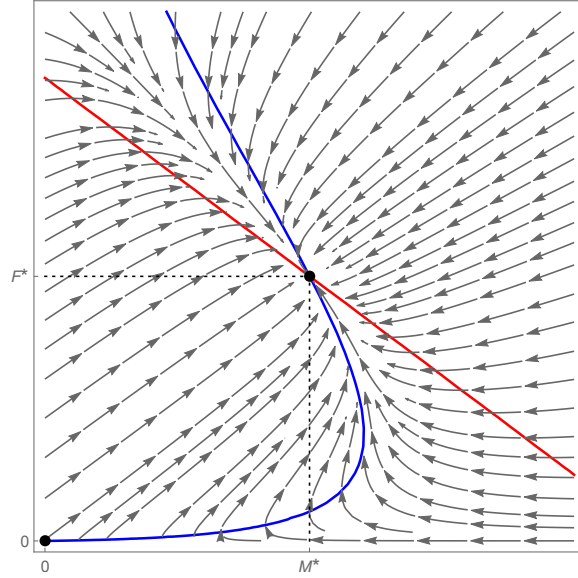

Figure 1: Phase portrait of model (1) when $\mathcal{N}_F > 1$. The positive equilibrium appears at the intersection of the two curves on which $\dot{F}$ (in red) and $\dot{M}$ (in blue) vanish.

Proof of Theorem 2.
- Assume first $\mathcal{N}_F < 1$. Rewriting equation (1b) as follows:

$$\dot{F} = \left((1-r)\rho e^{-\beta(M+F)} - \mu_F\right) F \leq \left((1-r)\rho - \mu_F\right) F$$

one deduces that $\dot{F} < -\varepsilon F$ for some positive ε. The state variable F being nonnegative, it then converges to 0. Using now equation (1a), we deduce that M converges to 0 too, and the GAS of E_0^* follows.
- Assume $\mathcal{N}_F = 1$. From equation (1b), $F = 0$ iff $\dot{F} = 0$, otherwise $F > 0$. We also derive

$$\dot{F} \leq \mu_F \left(\mathcal{N}_F e^{-\beta F} - 1\right) F \leq \mu_F \left(e^{-\beta F} - 1\right) F. \qquad (5)$$

Let $0 < \delta$. As long as $F \geq \delta$, then $\dot{F} < -\mu_F \left(1 - e^{-\beta \delta}\right) F$ and F is (strictly) decreasing. When $F < \delta$, using an asymptotic expansion of the right-hand side

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of (5), we obtain that $\dot{F} \leq -\mu_F \beta F^2 + O(F^3)$, such that 0 is LAS within $[0, \delta)$. Altogether, we infer that F converges to 0 and so does M. Thus E_0^* is GAS when $\mathcal{N}_F = 1$.

• Assume now that $\mathcal{N}_F > 1$. Let us compute the Jacobian matrix related to entomological system (1), page 4:

$$J(M, F) = \begin{pmatrix} -\beta r \rho F e^{-\beta(M+F)} - \mu_M & r\rho(1 - \beta F)e^{-\beta(M+F)} \\ -\beta(1 - r)\rho F e^{-\beta(M+F)} & (1 - r)\rho(1 - \beta F)e^{-\beta(M+F)} - \mu_F \end{pmatrix}.$$

so that

$$J(E_0^*) = \begin{pmatrix} -\mu_M & r\rho \\ 0 & (1 - r)\rho - \mu_F \end{pmatrix},$$

from which we deduce that E_0^* is unstable, as $\mathcal{N}_F > 1$.

For the positive equilibrium E^*, using the fact that $e^{-\beta(M^* + F^*)} = \dfrac{1}{\mathcal{N}_F}$, we have:

$$J(E^*) = \begin{pmatrix} -\dfrac{\beta r \rho}{\mathcal{N}_F} F^* - \mu_M & \dfrac{r\rho}{\mathcal{N}_F}(1 - \beta F^*) \\ -\dfrac{\beta(1 - r)\rho}{\mathcal{N}_F} F^* & -\dfrac{\beta(1 - r)\rho}{\mathcal{N}_F} F^* \end{pmatrix}.$$

Obviously $\texttt{trace}\{J(E^*)\} < 0$ and

$$\det J(E^*) = \frac{\beta}{\mathcal{N}_F}(1 - r)\rho F^* \left(\mu_M + \frac{r\rho}{\mathcal{N}_F}\right) > 0$$

so that E^* is LAS when $\mathcal{N}_F > 1$.

Using Dulac criterion [29], we now show that system (1) has no closed orbits wholly contained in the attracting set $\mathcal{D}$ defined in (3). Indeed, setting

$$\psi_1(F) := \frac{1}{F}, \qquad f_1(M, F) := r\rho F e^{-\beta(M+F)} - \mu_M M,$$

$$g_1(M, F) := (1 - r)\rho F e^{-\beta(M+F)} - \mu_F F,$$

let us study the sign of the function

$$D_1(M, F) := \frac{\partial}{\partial M}\left(\psi_1(F) f_1(M, F)\right) + \frac{\partial}{\partial F}\left(\psi_1(F) g_1(M, F)\right).$$

We have

$$\frac{\partial}{\partial M}\left(\psi_1(F) f_1(M, F)\right) = -\beta r \rho e^{-\beta(M+F)} - \frac{\mu_M}{F},$$

$$\frac{\partial}{\partial F}\left(\psi_1(F) g_1(M, F)\right) = -\beta(1 - r)\rho e^{-\beta(M+F)},$$

and thus

$$D_1(M, F) = -\beta \rho e^{-\beta(M+F)} - \frac{\mu_M}{F} < 0$$

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for all $(M, F) \in \mathcal{D}$ such that $F > 0$. Therefore, Dulac criterion [29] applies, demonstrating that system (1) possesses no nonconstant periodic solutions. Thus, using the fact that E^* is LAS, by the Poincaré-Bendixson theorem, all trajectories in $\mathcal{D} \setminus \{(M, 0) \ : \ M \geq 0\}$ converge towards E^*.

Convergence towards E_0^* clearly occurs in absence of females, i.e. when $F(0) = 0$. (Notice that for this reason, the point E_0^* cannot be repulsive.) Consider on the contrary a trajectory such that $F(0) > 0$. As $\dot{F} \geq -\mu_F F$, this induces that $F(t) \geq 0$ for any $t \geq 0$. We will show that convergence to E_0^* is impossible, so convergence towards E^* occurs. First of all, one deduces from (1) and the continuity of F that

$$M(t) = e^{-\mu_M t} M(0) + r\rho \int_0^t e^{-\mu_M(t-s)} F(s) e^{-\beta(M(s)+F(s))} \ ds > 0$$

for any $t > 0$. The ratio $\dfrac{F}{M}$ is therefore well defined and remains *positive* along this trajectory. It is moreover continuously differentiable, and

$$\frac{d}{dt}\left(\frac{F}{M}\right) = \frac{F}{M}\left(\mu_M - \mu_F + \rho e^{-\beta(M+F)}\left(1 - r - r\frac{F}{M}\right)\right) > (\mu_M - \mu_F)\frac{F}{M}$$

if $\dfrac{F}{M} \leq \dfrac{1-r}{r}$. From (2), it is deduced immediately that there exists for this trajectory a real number $T \geq 0$, such that

$$\forall\, t \geq T, \qquad \frac{F}{M} > \frac{1-r}{r}.$$

Then it holds for any $t \geq T$ that

$$\dot{F} = \left((1-r)\rho e^{-\beta(M+F)} - \mu_F\right)F \geq \left((1-r)\rho e^{-\frac{\beta}{1-r}F} - \mu_F\right)F\,.$$

The right-hand side of the previous formula is a continuous function of F which is *positive* on $(0, \frac{1-r}{\beta}\ln\mathcal{N}_F)$ and *negative* on $(\frac{1-r}{\beta}\ln\mathcal{N}_F, +\infty)$. As $F(t) > 0$ for any $t \geq 0$ (see above), one deduces that

$$\liminf_{t \to +\infty} F \geq \frac{1-r}{\beta}\ln\mathcal{N}_F > 0. \tag{6}$$

As the compact set $\mathcal{D}$ is absorbing, the trajectory is ultimately uniformly bounded. We deduce from this and the uniform bound (6), the existence of certain $T' \geq T$ (whose precise value depends upon the considered trajectory) and $\delta > 0$, such that

$$\forall\, t \geq T', \qquad Fe^{-\beta F} \geq \delta > 0.$$

Now, we have for any $t \geq T'$

$$\dot{M} \geq r\rho\delta e^{-\beta M} - \mu_M M,$$

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which is strictly positive in a neighborhood of $M = 0$. The trajectory under study therefore stays at a positive distance from the point E_0^*, and, being convergent, has to converge to the other equilibrium, namely E^*. This shows that any trajectory departing with $F(0) > 0$ converges towards E^*, and finally concludes the proof of Theorem 2. □

3. Elimination with constant releases of sterile insects

We now extend system (1), in order to incorporate *continuous, constant* releases driven by an equation for M_S, the number of *sterile* males:

$$
\begin{cases}
\dot{M} = r\rho \dfrac{FM}{M + \gamma M_S} e^{-\beta(M+F)} - \mu_M M, & \text{(7a)} \\[2ex]
\dot{F} = (1-r)\rho \dfrac{FM}{M + \gamma M_S} e^{-\beta(M+F)} - \mu_F F, & \text{(7b)} \\[2ex]
\dot{M_S} = \Lambda - \mu_S M_S. & \text{(7c)}
\end{cases}
$$

The positive constants μ_S and γ represent, respectively, the mortality rate of sterile insects, and their relative reproductive efficiency or fitness (compared to the wild males), which is usually smaller than 1. The nonnegative quantity Λ is the number of sterile insects released at the beginning of each release period (so that it is a "number of released mosquitoes per time unit"). It is taken constant over time in the present section. The other parameters are the same as for model (1), see Table 1.

The mortality of the sterile males is usually larger than that of wild males, so in complement to (2), we also have:

$$
\mu_S \geq \mu_M. \tag{8}
$$

Assuming t large enough, we may suppose $M_S(t)$ at its equilibrium value $M_S^* := \dfrac{\Lambda}{\mu_S}$ in (7c), and the previous system then reduces to

$$
\begin{cases}
\dot{M} = r\rho \dfrac{FM}{M + \gamma M_S^*} e^{-\beta(M+F)} - \mu_M M, & \text{(9a)} \\[2ex]
\dot{F} = (1-r)\rho \dfrac{FM}{M + \gamma M_S^*} e^{-\beta(M+F)} - \mu_F F. & \text{(9b)}
\end{cases}
$$

System (9) is dissipative too, with all trajectories converging towards the same set $\mathcal{D}$ introduced in (3). It admits the same mosquito-free equilibrium E_0^*.

We are interested here in the issues of existence and stability of positive equilibria. Driven by the application in view, we assume that the mosquito population is viable (that is $\mathcal{N}_F > 1$, see Theorem 1), and focus on conditions sufficient for its elimination.

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3.1. Existence of positive equilibria

The mosquito-free equilibrium E_0^* is always an equilibrium of system (9). The following result is concerned with possible supplementary equilibria.

Theorem 3 (Existence of positive equilibria for the SIT entomological model with constant releases). *Assume $\mathcal{N}_F > 1$. Then*

- *there exists $\Lambda^{crit} > 0$ such that system (7) admits two positive distinct equilibria if $0 < \Lambda < \Lambda^{crit}$, one positive equilibrium if $\Lambda = \Lambda^{crit}$, and no positive equilibrium if $\Lambda > \Lambda^{crit}$;*

- *the value of Λ^{crit} is uniquely determined by the formula*

$$\Lambda^{crit} := 2\,\frac{\mu_S}{\beta\gamma}\,\frac{\phi^{crit}(\mathcal{N}_F)}{1 + \frac{\mathcal{N}_F}{\mathcal{N}_M}}, \tag{10}$$

where $\phi = \phi^{crit}(\mathcal{N}_F)$ is the unique positive solution to the equation

$$1 + \phi\left(1 + \sqrt{1 + \frac{2}{\phi}}\right) = \mathcal{N}_F \exp\left(-\frac{2}{1 + \sqrt{1 + \frac{2}{\phi}}}\right). \tag{11}$$

Theorem 3 provides a characterization of the constant release rate above which no positive equilibrium may appear. We prove in the next section (Section 3.2) that in such a situation, convergence towards the mosquito-free equilibrium E_0^* occurs, which ensures elimination of the wild population.

Proof of Theorem 3.
Clearly, nullity of M at equilibrium is equivalent to nullity of F. In order to find possible nonzero equilibria, let (M^*, F^*) with $M^* > 0, F^* > 0$ be one of them. The populations at equilibrium have to fulfill:

$$r\rho\frac{F^*}{M^* + \gamma M_S^*}e^{-\beta(M^*+F^*)} = \mu_M, \qquad (1-r)\rho\frac{M^*}{M^* + \gamma M_S^*}e^{-\beta(M^*+F^*)} = \mu_F.$$

In particular, we have, for $\mathcal{N}_F, \mathcal{N}_M$ defined in (4),

$$\frac{M^*}{M^* + \gamma M_S^*}e^{-\beta(M^*+F^*)} = \frac{1}{\mathcal{N}_F}, \qquad \frac{F^*}{M^* + \gamma M_S^*}e^{-\beta(M^*+F^*)} = \frac{1}{\mathcal{N}_M}, \tag{12}$$

which implies the relation:

$$\frac{F^*}{M^*} = \frac{\mathcal{N}_F}{\mathcal{N}_M}.$$

Injecting this value in the first equation of (12), the number of males M^* at equilibrium has to fulfil the equation

$$\frac{M^*}{M^* + \gamma M_S^*}e^{-\beta\left(1 + \frac{\mathcal{N}_F}{\mathcal{N}_M}\right)M^*} = \frac{1}{\mathcal{N}_F}$$

10

or again

$$1 + \frac{\gamma M_S^*}{M^*} = \mathcal{N}_F e^{-\beta\left(1+\frac{\mathcal{N}_F}{\mathcal{N}_M}\right)M^*}. \tag{13}$$

The study of equation (13) is done through the following result, whose proof is given in Appendix.

Lemma 1. *Let $\mathcal{N}_F > 1$, then equation* (11) *admits a unique positive root, denoted ϕ^{crit}. Moreover, for any a, c positive, the equation*

$$f(x) := 1 + \frac{a}{x} - \mathcal{N}_F e^{-cx} = 0 \tag{14}$$

admits two positive distinct roots if $0 < ac < 2\phi^{crit}$; one positive root if $ac = 2\phi^{crit}$; no positive root otherwise.

Using Lemma 1 with the two positive constants

$$a := \gamma M_S^* = \gamma \frac{\Lambda}{\mu_S}, \qquad c := \beta\left(1 + \frac{\mathcal{N}_F}{\mathcal{N}_M}\right),$$

one deduces that equation (13) admits exactly one positive root when the root of (11) is equal to $\phi^{crit} = \frac{ac}{2} = \frac{1}{2}\beta\gamma\left(1 + \frac{\mathcal{N}_F}{\mathcal{N}_M}\right)\frac{\Lambda^{crit}}{\mu_S}$, which implies (10) and thus achieves the proof of Theorem 3. □

3.2. Asymptotic stability of the equilibria

Assume $\mathcal{N}_F > 1$. We first study the asymptotic stability of the mosquito-free equilibrium E_0^* in the case where it is the unique equilibrium, that is when $\Lambda > \Lambda^{crit}$.

Theorem 4 (Stability of the mosquito-free equilibrium of the SIT entomological model with constant releases). *If system* (7) *admits no positive equilibrium (that is, if $\Lambda > \Lambda^{crit}$), then the mosquito-free equilibrium E_0^* is globally exponentially stable.*

Proof of Theorem 4. The Jacobian matrix $J(M, F)$ of the reduced system (9) is defined by its four coefficients

$$J_{21} := \frac{(1-r)\rho F}{M + \gamma M_S^*}e^{-\beta(M+F)}\left(1 - \beta M - \frac{M}{M + \gamma M_S^*}\right), \ J_{11} := \frac{r}{1-r}J_{21} - \mu_M ,$$

$$J_{12} := \frac{r\rho M}{M + \gamma M_S^*}e^{-\beta(M+F)}(1 - \beta F), \ J_{22} := \frac{1-r}{r}J_{12} - \mu_F .$$

Its value at the mosquito-free equilibrium E_0^* is just $\mathtt{diag}\{-\mu_M; -\mu_F\}$, which guarantees local asymptotic stability at this point.

We use again Dulac criterion to show that system (9) has no closed orbits wholly contained in the set $\mathcal{D}$. We set

$$\psi_2(M, F) := \frac{M + \gamma M_S^*}{MF},$$

$$f_2(M, F) := r\rho \frac{FM}{M + \gamma M_S^*} e^{-\beta(M+F)} - \mu_M M,$$

$$g_2(M, F) := (1 - r)\rho \frac{FM}{M + \gamma M_S^*} e^{-\beta(M+F)} - \mu_F F,$$

and then study the sign of the function

$$D_2(M, F) := \frac{\partial}{\partial M}\Big(\psi_2(M, F)f_2(M, F)\Big) + \frac{\partial}{\partial F}\Big(\psi_2(M, F)g_2(M, F)\Big).$$

As

$$\frac{\partial}{\partial M}\Big(\psi_2(M, F)f_2(M, F)\Big) = -\beta r\rho e^{-\beta(M+F)} - \frac{\mu_M}{F},$$

$$\frac{\partial}{\partial F}\Big(\psi_2(M, F)g_2(M, F)\Big) = -\beta(1 - r)\rho e^{-\beta(M+F)},$$

one has

$$D_2(M, F) = -\beta\rho e^{-\beta(M+F)} - \frac{\mu_M}{F} < 0,$$

for all $(M, F) \in \mathcal{D}$ such that $F > 0$. Thus, by the Poincaré-Bendixson theorem, since E_0^* is the only asymptotically stable equilibrium, all trajectories in $\mathcal{D}$ approach the equilibrium E_0^*. This concludes the proof of Theorem 4. $\qquad\square$

On the other hand, when $\Lambda < \Lambda^{crit}$ is not large enough and system (9) admits two distinct positive equilibria $E_1^* < E_2^*$, one may show by studying the spectrum of the Jacobian matrices that $E_0^* = (0, 0)$ and $E_2^* = (M_2^*, F_2^*)$ are locally asymptotically stable. It is likely that this case presents bistability and that $E_1^* = (M_1^*, F_1^*)$ is unstable, with the basin of attraction of E_0^* containing the interval $[0, E_1^*) := \{(M, F) \in \mathbb{R}_+^2 \ : \ 0 \le M < M_1^*, \ 0 \le F < F_1^*\}$, and the basin of attraction of E_2^* containing the interval $(E_2^*, \infty) := \{(M, F) \in \mathbb{R}_+^2 \ : \ M > M_2^*, \ F > F_2^*\}$. This is at least what is suggested by the vector field illustrating this situation presented in Figure 2. It is worth noting that when $\Lambda \to \Lambda^{crit}$ from below, we have $E_1^* \to E_2^*$ and the two positive equilibria merge and vanish.
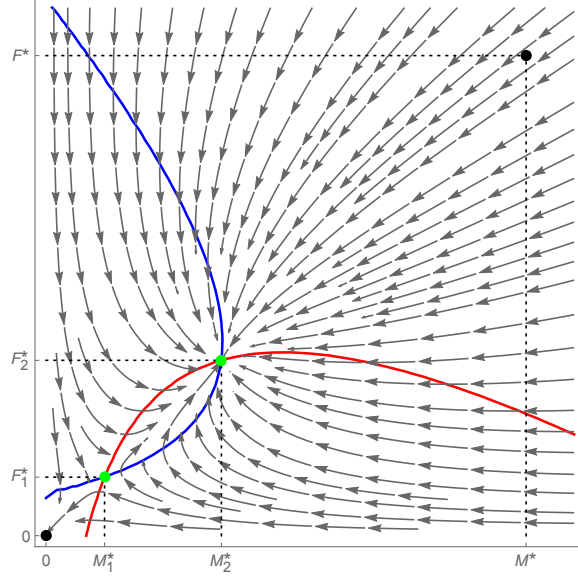
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Figure 2: Trajectories of system (9) related to each equilibria: bi-stable case. The two positive equilibria E_1^*, E_2^* (green points) are located at the intersection of the two curves where $\dot{F}$ (in red) and $\dot{M}$ (in blue) vanish. The isolated black points denote the initial equilibria $E_0^* = (0,0), E^* = (M^*, F^*)$ of the system (1), present when no control is applied.

4. Elimination with periodic impulsive releases of sterile insects

We now consider *periodic impulsive* releases $\Lambda(t)$, modeled by the following variant of system (7):

$$
\begin{cases}
\dot{M} = r\rho\dfrac{FM}{M + \gamma M_S}e^{-\beta(M+F)} - \mu_M M, & \text{(15a)} \\[2ex]
\dot{F} = (1-r)\rho\dfrac{FM}{M + \gamma M_S}e^{-\beta(M+F)} - \mu_F F, & \text{(15b)} \\[2ex]
\dot{M}_S = -\mu_S M_S \ \text{ for any } \ t \in \bigcup_{n\in\mathbb{N}}\big(n\tau, (n+1)\tau\big), & \text{(15c)} \\[2ex]
M_S(n\tau^+) = \tau\Lambda_n + M_S(n\tau^-), \qquad n = 1, 2, 3, \ldots & \text{(15d)}
\end{cases}
$$

where $M_S(n\tau^{\pm})$ denote the right and left limits of the function $M_S(t)$ at time $n\tau$. In other terms, system (15) evolves according to (15a), (15b), (15c) on the union of *open* intervals $(n\tau, (n+1)\tau)$; while M_S is submitted to jumps at each point $n\tau$, accounting for the released sterile males. We choose in this section Λ_n *constant*, and drop consequently the subindex n. For such release schedule, it is clear that the function M_S converges when $t \to +\infty$ towards the periodic solution

$$
M_S^{\text{per}}(t) = \frac{\tau\Lambda e^{-\mu_S\left(t-\lfloor\frac{t}{\tau}\rfloor\tau\right)}}{1 - e^{-\mu_S\tau}}. \tag{16}
$$

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We therefore introduce now the following *periodic system*:

$$
\begin{cases}
\dot{M} = r\rho \dfrac{FM}{M + \gamma M_S^{\mathrm{per}}(t)} e^{-\beta(M+F)} - \mu_M M, & \text{(17a)} \\[3ex]
\dot{F} = (1-r)\rho \dfrac{FM}{M + \gamma M_S^{\mathrm{per}}(t)} e^{-\beta(M+F)} - \mu_F F. & \text{(17b)}
\end{cases}
$$

Existence and uniqueness of continuously differentiable solutions of system (17) on the interval $[0, +\infty)$ may be shown by standard arguments, as well as the forward invariance of the positive orthant. Notice that the mosquito-free equilibrium E_0^* previously introduced is still an equilibrium of (17). We are interested here in studying the conditions under which E_0^* is globally asymptotically stable. For future use, we note that the *mean value* of $1/M_S^{\mathrm{per}}$ corresponding to (16) verifies:

$$
\left\langle \frac{1}{M_S^{\mathrm{per}}} \right\rangle := \frac{1}{\tau} \int_0^\tau \frac{1}{M_S^{\mathrm{per}}(t)} dt = \frac{1 - e^{-\mu_S \tau}}{\tau^2 \Lambda} \int_0^\tau e^{\mu_S t} dt = \frac{2\big(\cosh(\mu_S \tau) - 1\big)}{\mu_S \tau^2 \Lambda}.
$$

$$
(18)
$$

Theorem 5 (Sufficient condition for elimination by periodic impulses). *For any given $\tau > 0$, assume that Λ is chosen such that*

$$
\Lambda \geq \Lambda_{per}^{crit}
$$
$$
:= \frac{\cosh(\mu_S \tau) - 1}{\mu_S \tau^2} \frac{1}{e\beta\gamma} \min\left\{ 2\mathcal{N}_M, 2\mathcal{N}_F, \max\{r, 1-r\} \max\left\{ \frac{\mathcal{N}_M}{r}, \frac{\mathcal{N}_F}{1-r} \right\} \right\} (19)
$$

Then every solution of system (17) converges globally exponentially to the mosquito-free equilibrium E_0^*.

Notice that in (19) and in the sequel, $e = e^1$. The previous result provides a simple sufficient condition for stabilization of the mosquito-free equilibrium, through an adequate choice of the amplitude of the releases, Λ, for given period τ.

Remark 2. *When $r = 1 - r$ and $\mathcal{N}_F > \mathcal{N}_M$ (which is the case of the application we are interested in), the expression of Λ_{per}^{crit} simplifies as follows:*

$$
\Lambda_{per}^{crit} = \frac{2\big(\cosh(\mu_S \tau) - 1\big)}{\mu_S \tau^2} \frac{\mathcal{N}_F}{e\beta\gamma}.
$$

The function $\tau \mapsto \dfrac{2\cosh(\mu_S \tau) - 1}{\mu_S \tau^2}$ is increasing and tends towards μ_S when $\tau \to 0$. Making $\tau \to 0_+$, we derive the following sufficient condition for stabilization:

$$
\Lambda_{per}^{crit} \geq \frac{\mu_S \mathcal{N}_F}{e\beta\gamma},
$$

to be compared to $\Lambda^{crit} = 2\dfrac{\mu_S}{\beta\gamma} \dfrac{\phi^{crit}(\mathcal{N}_F)}{1 + \frac{\mathcal{N}_F}{\mathcal{N}_M}}$ (see Theorem 3).

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Proof of Theorem 5. First rewrite (17) as

$$\dot{M} = \left(r\rho \frac{F}{M + \gamma M_S^{\text{per}}} e^{-\beta(M+F)} - \mu_M \right) M, \tag{20a}$$

$$\dot{F} = \left((1-r)\rho \frac{M}{M + \gamma M_S^{\text{per}}} e^{-\beta(M+F)} - \mu_F \right) F, \tag{20b}$$

in order to emphasize the factorization of M and F.

- 1. Notice that, for any $M, F \geq 0$ and any $t \geq 0$,

$$\frac{M}{M + \gamma M_S^{\text{per}}} e^{-\beta(M+F)} \leq \frac{M}{M + \gamma M_S^{\text{per}}} e^{-\beta M} \leq \frac{\alpha}{M + \gamma M_S^{\text{per}}} \leq \frac{\alpha}{\gamma M_S^{\text{per}}}, \tag{21}$$

where we write for simplicity

$$\alpha := \max\left\{ xe^{-\beta x} \ : \ x \geq 0 \right\} = \frac{1}{e\beta}. \tag{22}$$

Integrating (20b) between $n\tau$ and $t > n\tau$ leads to

$$\begin{aligned}
F(t) &\leq e^{\int_{n\tau}^t \left((1-r)\rho \frac{M}{M + \gamma M_S^{\text{per}}} e^{-\beta(M+F)} - \mu_F \right) ds} F(n\tau) \\
&\leq e^{\int_{n\tau}^t \left((1-r)\rho \frac{\alpha}{\gamma} \frac{1}{M_S^{\text{per}}(s)} - \mu_F \right) ds} F(n\tau).
\end{aligned}$$

Thus, taking $t = (n+1)\tau$, for any $n \in \mathbb{N}$, we deduce that

$$F\big((n+1)\tau\big) \leq e^{\left((1-r)\rho \frac{\alpha}{\gamma} \left\langle \frac{1}{M_S^{\text{per}}} \right\rangle - \mu_F \right) \tau} F(n\tau).$$

Therefore, the sequence $\{F(n\tau)\}_{n\in\mathbb{N}}$ decreases towards 0, provided that

$$(1-r)\rho \frac{\alpha}{\gamma} \left\langle \frac{1}{M_S^{\text{per}}} \right\rangle < \mu_F,$$

that is

$$\left\langle \frac{1}{M_S^{\text{per}}} \right\rangle < \frac{\gamma}{\alpha} \frac{\mu_F}{(1-r)\rho} = e\beta\gamma \frac{1}{\mathcal{N}_F}, \tag{23}$$

This is sufficient to ensure that F converges towards 0, and this induces the same behavior for M: condition (23) implies that E_0^* is GAS.

- 2. The same argument may be conducted from (20a) rather than (20b), leading to:

$$\frac{F}{M + \gamma M_S^{\text{per}}} e^{-\beta(M+F)} \leq \frac{F}{M + \gamma M_S^{\text{per}}} e^{-\beta F} \leq \frac{\alpha}{M + \gamma M_S^{\text{per}}} \leq \frac{\alpha}{\gamma M_S^{\text{per}}} \tag{24}$$

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Global asymptotic stability is thereby guaranteed if

$$\left\langle \frac{1}{M_S^{\mathrm{per}}} \right\rangle < \frac{\gamma}{\alpha} \frac{\mu_M}{r\rho} = e\beta\gamma \frac{1}{\mathcal{N}_M}. \tag{25}$$

• 3. Define the positive definite function

$$\mathcal{V}(M, F) := \frac{1}{2}(M^2 + F^2) \tag{26}$$

and write its derivative along the trajectories of (17) as

$$\dot{\mathcal{V}} = M\dot{M} + F\dot{F} = -\mu_M M^2 - \mu_F F^2 + \rho \frac{FM(rM + (1-r)F)}{M + \gamma M_S^{\mathrm{per}}} e^{-\beta(M+F)}. \tag{27}$$

On the one hand, we have

$$-\mu_M M^2 - \mu_F F^2 \le -\min\{\mu_M, \mu_F\}(M^2 + F^2) = -2\min\{\mu_M, \mu_F\}\mathcal{V}.$$

On the other hand,

$$
\begin{aligned}
\frac{FM(rM + (1-r)F)}{M + \gamma M_S^{\mathrm{per}}} e^{-\beta(M+F)} &\le \max\{r, 1-r\} \frac{FM(M+F)}{M + \gamma M_S^{\mathrm{per}}} e^{-\beta(M+F)} \\
&\le \max\{r, 1-r\}\alpha \frac{FM}{M + \gamma M_S^{\mathrm{per}}} \\
&\le \max\{r, 1-r\}\alpha \frac{1}{M + \gamma M_S^{\mathrm{per}}}\mathcal{V} \\
&\le \max\{r, 1-r\}\alpha \frac{1}{\gamma M_S^{\mathrm{per}}}\mathcal{V}.
\end{aligned}
$$

Coming back to (27), we deduce that

$$\dot{\mathcal{V}} \le \left(\max\{r, 1-r\}\alpha \frac{1}{\gamma M_S^{\mathrm{per}}} - 2\min\{\mu_M, \mu_F\} \right)\mathcal{V}.$$

One may conclude that E_0^* is GAS provided that

$$\max\{r, 1-r\}\rho \frac{\alpha}{\gamma} \left\langle \frac{1}{M_S^{\mathrm{per}}} \right\rangle < 2\min\{\mu_M, \mu_F\},$$

that is,

$$\left\langle \frac{1}{M_S^{\mathrm{per}}} \right\rangle < 2\frac{\gamma}{\alpha} \frac{\min\{\mu_M, \mu_F\}}{\max\{r, 1-r\}\rho} = 2e\beta\gamma \frac{1}{\max\{r, 1-r\}} \min\left\{ \frac{r}{\mathcal{N_M}}, \frac{1-r}{\mathcal{N}_F} \right\}. \tag{28}$$

• 4. Finally, putting together the sufficient conditions in (23), (25) and (28) yields the following sufficient condition for global asymptotic stability of E_0^*:

$$\left\langle \frac{1}{M_S^{\mathrm{per}}} \right\rangle \quad < \quad e\beta\gamma \max\left\{ \frac{1}{\mathcal{N}_M}, \frac{1}{\mathcal{N}_F}, \frac{2}{\max\{r, 1-r\}} \min\left\{ \frac{r}{\mathcal{N}_M}, \frac{1-r}{\mathcal{N}_F} \right\} \right\}.$$

Expressing the mean value as a function of Λ with the help of (18), one establishes that E_0^* is GAS if

$$\Lambda \quad > \quad \frac{2}{e\beta\gamma} \frac{\cosh(\mu_S\tau)-1}{\mu_S\tau^2} \frac{1}{\max\left\{ \frac{1}{\mathcal{N}_M}, \frac{1}{\mathcal{N}_F}, \frac{2}{\max\{r,1-r\}} \min\left\{ \frac{r}{\mathcal{N}_M}, \frac{1-r}{\mathcal{N}_F} \right\} \right\}}$$

$$= \quad \frac{2}{e\beta\gamma} \frac{\cosh(\mu_S\tau)-1}{\mu_S\tau^2} \min\left\{ \mathcal{N}_M, \mathcal{N}_F, \frac{\max\{r,1-r\}}{2\min\left\{ \frac{r}{\mathcal{N}_M}, \frac{1-r}{\mathcal{N}_F} \right\}} \right\}$$

$$= \quad \frac{2}{e\beta\gamma} \frac{\cosh(\mu_S\tau)-1}{\mu_S\tau^2} \min\left\{ \mathcal{N}_M, \mathcal{N}_F, \frac{\max\{r,1-r\}}{2} \max\left\{ \frac{\mathcal{N}_M}{r}, \frac{\mathcal{N}_F}{1-r} \right\} \right\},$$

which is exactly the formula (19). This concludes the proof of Theorem 5. □

Remark 3. *A rough upper bound estimate for Λ_{per}^{crit} can be obtained using the result from the constant continuous release case: if Λ is chosen such that $\Lambda > \Lambda^{crit} := 2\frac{\mu_S}{\beta\gamma} \frac{\phi^{crit}(\mathcal{N}_F)}{1 + \frac{\mathcal{N}_F}{\mathcal{N}_M}}$, then E_0^* is GAS for the constant continuous release system (7). Thus, using a comparison principle, a sufficient condition to ensure global asymptotic stability of E_0^* is to choose*

$$\underline{M}_S^{per} \geq \frac{\Lambda^{crit}}{\mu_S},$$

where $\underline{M}_S^{per} = \min_{t\in[0,\tau]} M_S^{per}(t) = \tau\Lambda \frac{e^{-\mu_S\tau}}{1 - e^{-\mu_S\tau}}$. Thus, we derive that, for a given τ, if

$$\Lambda \geq \Lambda^{crit} \frac{e^{\mu_S\tau}-1}{\mu_S\tau}, \tag{29}$$

then E_0^* is GAS. When $\tau \to 0^+$, we recover the result for the constant continuous release (cf. Theorem 3).

5. Elimination by feedback control

We now assume that measurements are available, providing real time estimates of the number of wild males and females $M(t), F(t)$, at least for any $t = n\tau, n \in \mathbb{N}$. One thus has the possibility to choose the number $\tau\Lambda_n$ of mosquitoes released at time $n\tau$ in view of this information: this is a *closed-loop control* option. We study in the sequel this strategy.

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5.1. Principle of the method

The principle of the stabilization method that we introduce now is based on two steps. The first one (Section 5.1.1) consists in solving the stabilization problem under the hypothesis that one can directly actuate on M_S. The second one (Section 5.1.2) consists in showing how to realize, through adequate choice of Λ_n, the prescribed behavior of M_S defined in Step 1. The formal statement and proof are provided later, in Section 5.2.

5.1.1. Step 1 – Setting directly the sterile population level

We first suppose to be capable of directly controlling the quantity M_S. We will rely on the following key property.

Proposition 1. *Let k be a real number such that*

$$0 < k < \frac{1}{\mathcal{N}_F}. \tag{30}$$

Then every solution of (7a)-(7b) such that

$$\frac{M(t)}{M(t) + \gamma M_S(t)} \leq k, \qquad t \geq 0 , \tag{31}$$

converges exponentially to E_0^*.

The idea behind formula (30) is quite natural: it suffices to impose a fixed upper bound k on the ratio $\dfrac{M}{M + \gamma M_S^*}$ in order to make the 'apparent' basic offspring number $k\mathcal{N}_F$ smaller than 1, and consequently to render inviable the wild population. Notice that this condition corresponds exactly to the stability of the system linearized around the origin. It may be excessively demanding for *large* population sizes, as it ignores the effects of competition modeled by the exponential term. We shall come back to this point in Section 6 and introduce saturation.

Proof of Proposition 1. From equations (7a) and (7b), we have, for any solution that fulfils (31):

$$\dot{M} = r\rho \frac{FM}{M + \gamma M_S} e^{-\beta(M+F)} - \mu_M M$$

$$\leq r\rho \frac{FM}{M + \gamma M_S} - \mu_M M \leq -\mu_M M + r\rho kF \tag{32a}$$

and

$$\dot{F} = (1-r)\rho \frac{FM}{M + \gamma M_S} e^{-\beta(M+F)} - \mu_F F \leq ((1-r)\rho k - \mu_F)F. \tag{32b}$$

The linear autonomous system

$$\begin{pmatrix} \dot{M}' \\ \dot{F}' \end{pmatrix} = \begin{pmatrix} -\mu_M & r\rho k \\ 0 & -\mu_F + (1-r)\rho k \end{pmatrix} \begin{pmatrix} M' \\ F' \end{pmatrix} \tag{33}$$

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is *monotone* [22] (it involves a Metzler matrix) and may thus serve as a comparison system for the evolution of (7a)-(7b). Thus, it is deduced that

$$0 \leq M(t) \leq M'(t), \qquad 0 \leq F(t) \leq F'(t), \qquad t \geq 0,$$

where (M', F') is the solution of (33) generated by the same initial values as the underlying solution (M, F) of (7a)-(7b).

On the other hand, system (33) is asymptotically stable when (30) holds. In other words, $M'(t)$ and $F'(t)$ converge to E_0^* asymptotically. In consequence, $M(t)$ and $F(t)$ also converge to E_0^* asymptotically when (30) is in force. This achieves the proof of Proposition 1. □

5.1.2. Step 2 – Shaping an impulsive control compliant with Step 1

We now want to ensure that condition (31) is fulfilled, through an adequate choice of the impulse amplitude Λ_n. In virtue of (15c)-(15d), the value of M_S on the interval $\big(n\tau, (n+1)\tau\big]$ is given by

$$M_S(t) = M_S(n\tau^+)e^{-\mu_S(t-n\tau)} = \big(\Lambda_n\tau + M_S(n\tau)\big)e^{-\mu_S(t-n\tau)}, \qquad (34)$$

and we would like to choose Λ_n in such a way that (31) stays in force. However, instead of computing the (nonlinear) evolution of $M(t)$ on the interval $\big(n\tau, (n+1)\tau\big]$, we will impose, rather than (31), the stronger condition

$$\gamma M_S(t) \geq \left(\frac{1}{k} - 1\right) M'(t), \qquad t \geq 0 \qquad (35)$$

where $M'(t)$ refers to the *super-solution* of $M(t)$ introduced in the proof of Proposition 1. (Notice that the conservatism introduced in this step remains reasonable when the original nonlinear system evolves in region where $\beta(M + F) \ll 1$.) Due to its linearity, system (33) may be solved explicitly on $\big(n\tau, (n+1)\tau\big]$ using the following result.

Lemma 2. *The solution of system (33) on $\big(n\tau, (n+1)\tau\big]$ with initial values $\big(M'(n\tau), F'(n\tau)\big) = \big(M(n\tau), F(n\tau)\big)$ is given by*

$$\begin{pmatrix} M'(t) \\ F'(t) \end{pmatrix} = P \begin{pmatrix} M(n\tau) \\ F(n\tau) \end{pmatrix} \qquad (36a)$$

where

$$P := \begin{pmatrix} e^{-\mu_M(t-n\tau)} & \dfrac{r\rho k}{\mu_M - \mu_F + (1-r)\rho k}\left(e^{-(\mu_F-(1-r)\rho k)(t-n\tau)} - e^{-\mu_M(t-n\tau)}\right) \\ 0 & e^{-(\mu_F-(1-r)\rho k)(t-n\tau)} \end{pmatrix}$$

$$(36b)$$

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The proof of Lemma 2 presents no difficulty and is left to the reader.

All the components of the matrix in (36a) are nonnegative provided that μ_F, μ_M, ρ and k are chosen such that $\mu_F - \mu_M - (1-r)\rho k \leq 0$. It is worthwhile to recall that $\mu_F \leq \mu_M$ (see (2), page 4); therefore, the former condition is indeed verified for any positive ρ and k.

We now come back to the control synthesis. Using (34) and (36a), condition (35) is *equivalent*, on any interval $\big(n\tau, (n+1)\tau\big]$, with the condition

$$\gamma\big(\Lambda_n\tau + M_S(n\tau)\big)e^{-\mu_S(t-n\tau)} = \gamma M_S(t) \geq \left(\frac{1}{k} - 1\right) M'(t)$$

$$= \frac{1-k}{k}\left(e^{-\mu_M(t-n\tau)}M(n\tau)\right.$$

$$\left. + \frac{r\rho k}{\mu_M - \mu_F + (1-r)\rho k}\left(e^{-(\mu_F - (1-r)\rho k)(t-n\tau)} - e^{-\mu_M(t-n\tau)}\right) F(n\tau)\right) \quad (37)$$

This condition is equivalent to

$$\Lambda_n\tau \geq -M_S(n\tau) + \frac{1-k}{\gamma k}e^{(\mu_S - \mu_M)s}\left(M(n\tau)\right.$$

$$\left. + \frac{r\rho k}{\mu_M - \mu_F + (1-r)\rho k}\left(e^{(\mu_M - \mu_F + (1-r)\rho k)s} - 1\right) F(n\tau)\right) \quad (38)$$

for any $s \in [0,\tau]$. In virtue of the relationships (2) and (8), the right-hand side of previous inequality (38) is *increasing* in s. Therefore, condition (38) has to be checked only for $s = \tau$.

5.2. Stabilization result

5.2.1. Synchronized measurements and releases

We now state and prove the stabilization result suggested by the previous considerations.

Theorem 6 (Sufficient condition for stabilization by impulsive feedback control)**.** *For a given $k \in \left(0, \frac{1}{\mathcal{N}_F}\right)$, assume that for any $n \in \mathbb{N}$:*

$$\tau\Lambda_n \geq \left| K\begin{pmatrix} M(n\tau) \\ F(n\tau) \end{pmatrix} - M_S(n\tau) \right|_+ \quad (39a)$$

$$K := \frac{1}{\gamma}\left(\begin{array}{c} \frac{1-k}{k}e^{(\mu_S - \mu_M)\tau} \\ \frac{r\rho(1-k)}{\mu_M - \mu_F + (1-r)\rho k}\left(e^{(\mu_S - \mu_F + (1-r)\rho k)\tau} - e^{(\mu_S - \mu_M)\tau}\right) \end{array}\right)^{\mathsf{T}} \quad (39b)$$

Then every solution of system (15) converges exponentially towards E_0^*, with a convergence rate bounded from below by a value independent of the initial condition.

If moreover

$$\tau\Lambda_n \leq K\begin{pmatrix} M(n\tau) \\ F(n\tau) \end{pmatrix} \quad (39c)$$

then the series of impulses $\sum\limits_{n=0}^{+\infty} \Lambda_n$ converges.

In (39a), the notation $|z|_+ := \max\{0, z\}$ represents the *positive part* of the real number z. Notice that the row vector K defined in (39b) has positive components.

Implementing the previous control law necessitates the measurement of $M(n\tau)$, $F(n\tau)$ (or their upper estimates), and of $M_S(t)$ (or its lower estimate). A possibility to have (39a) fulfilled, is to ignore the population of sterile males already present at time $n\tau$ and to take simply the *linear* control law

$$\tau \Lambda_n = K \begin{pmatrix} M(n\tau) \\ F(n\tau) \end{pmatrix}.$$

Notice that this expression corresponds to the value in the right-hand side of (39c).

On the other hand, (39a) means that the release of sterile males at time $t = n\tau$ is not (really) necessary if the sterile males population is large enough, more precisely if $M_S(n\tau) \geq K \begin{pmatrix} M(n\tau) \\ F(n\tau) \end{pmatrix}$. Using this result, one may avoid unnecessary releases, thereby reducing the overall cumulative number of released males and the underlying cost of SIT control.

Proof of Theorem 6. When $\big(M(n\tau), F(n\tau)\big) = (0,0)$, an impulsion Λ_n has no effect on the evolution of (M, F): the origin is an equilibrium point of system (15). We now consider the case $\big(M(n\tau), F(n\tau)\big) \neq (0,0)$.

• 1. Assume first that (39a) is fulfilled with a *strict* inequality. By construction, one has:

$$\forall t \in \big(n\tau, (n+1)\tau\big], \qquad \gamma M_S(t) > \frac{1-k}{k} M'(t) \tag{40}$$

where (M', F') stands for solution of (33) departing from $\big(M(n\tau), F(n\tau)\big)$ at time $n\tau$.

We will first establish that this implies:

$$\forall t \in \big[n\tau, (n+1)\tau\big], \qquad M(t) \leq M'(t), \quad F(t) \leq F'(t). \tag{41}$$

For this, let t_0 be any element of $\big[n\tau, (n+1)\tau\big)$ such that $M(t_0) \leq M'(t_0)$, $F(t_0) \leq F'(t_0)$ *with at least one equality*. Let us show the existence of t_1 such that $t_0 < t_1 < (n+1)\tau$ and

$$\forall t \in (t_0, t_1), \qquad M(t) < M'(t), \ F(t) < F'(t). \tag{42}$$

Indeed, due to (40) and by definition of t_0, one has

$$\gamma M_S(t_0) > \frac{1-k}{k} M'(t_0) \geq \frac{1-k}{k} M(t_0),$$

where we write by convention $M_S(t_0) := M_S(n\tau^+)$ when $t_0 = n\tau$. By continuity of the functions $M(t)$ and $M_S(t)$ on the open interval $\big(n\tau, (n+1)\tau\big)$, there thus exists t_1 such that $t_0 < t_1 < (n+1)\tau$ and

$$\forall t \in (t_0, t_1), \qquad \gamma M_S(t) > \frac{1-k}{k} M(t).$$

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In such conditions, it can be shown as in Proposition 1 that $\big(M'(t), F'(t)\big) \geq \big(M(t), F(t)\big)$ for any $t \in (t_0, t_1)$, and even that $\big(M'(t), F'(t)\big) > \big(M(t), F(t)\big)$, because the functions defining the right-hand sides of (15a) and (15b) take on strictly smaller values than those defining the right-hand sides of (33). Therefore, for any $t_0 \in \{n\tau^+\} \cup \big(n\tau, (n+1)\tau\big)$, there exists $t_1 > t_0$ such that (42) holds.

From (42) and the fact that $\big(M(n\tau), F(n\tau)\big) = \big(M'(n\tau), F'(n\tau)\big)$, one deduces that (42) is true for $t_1 = (n+1)\tau$, and therefore that (41) is true. Finally, putting together (40) and (41) yields the following key property:

$$\forall\, t \in \big(n\tau, (n+1)\tau\big], \qquad \gamma M_S(t) > \frac{1-k}{k} M(t). \tag{43}$$

• 2. Assume now that (39a) is fulfilled (with the original non-strict inequality). Considering values of Λ_n converging from above towards the quantity in the right-hand side of this inequality and relying on the continuity of the flow with respect to Λ_n, yields instead of (43) the non-strict inequality:

$$\forall t \in (n\tau, (n+1)\tau], \qquad \gamma M_S(t) \geq \frac{1-k}{k} M(t). \tag{44}$$

• 3. Let us now study F. In view of (44), we have that for any $t \in \big(n\tau, (n+1)\tau\big]$ it holds that

$$\frac{M(t)}{M(t) + \gamma M_S(t)} e^{-\beta(M(t)+F(t))} \leq \frac{M(t)}{M(t) + \gamma M_S(t)} \leq k.$$

Therefore,

$$\dot{F} = (1-r)\rho \frac{FM}{M + \gamma M_S} e^{-\beta(M+F)} - \mu_F F \leq \big((1-r)\rho k - \mu_F\big)F.$$

Due to (39b), there exists $\varepsilon > 0$ such that

$$\mu_F - (1-r)\rho k > \varepsilon$$

and then $\dot{F} \leq -\varepsilon F$. This property ensures that $F(t)$ *decreases* with time, and converges exponentially towards 0. It is then deduced from (15a) that $M(t)$ also converges exponentially towards 0: overall, $\big(M(t), F(t)\big)$ converges towards E_0^*.

• 4. Last, choose now Λ_n fulfilling (39a) and (39c). From the property of exponential stability previously demonstrated, there exist $C, \varepsilon > 0$ such that

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$M(t) < Ce^{-\varepsilon t}$ and $F(t) < Ce^{-\varepsilon t}$ for any $t \geq 0$. We then deduce that

$$
\begin{aligned}
\Lambda_n \quad \leq \quad & \frac{1}{\gamma\tau}\left(\frac{1-k}{k}e^{(\mu_S-\mu_M)\tau}M(n\tau)\right. \\
& \left. +\frac{r\rho(1-k)}{\mu_M-\mu_F+(1-r)\rho k}\left(e^{(\mu_S-\mu_F+(1-r)\rho k)\tau}-e^{(\mu_S-\mu_M)\tau}\right)F(n\tau)\right) \\
\leq \quad & \frac{C}{\gamma\tau}\left(\frac{1-k}{k}e^{(\mu_S-\mu_M)\tau}\right. \\
& \left. +\frac{r\rho(1-k)}{\mu_M-\mu_F+(1-r)\rho k}\left(e^{(\mu_S-\mu_F+(1-r)\rho k)\tau}-e^{(\mu_S-\mu_M)\tau}\right)\right)e^{-n\varepsilon\tau},
\end{aligned}
$$

and one gets by summation

$$
\begin{aligned}
\sum_{n=0}^{+\infty}\Lambda_n \leq \frac{C}{\gamma\tau}\left(\frac{1-k}{k}e^{(\mu_S-\mu_M)\tau}\right. \\
\left. +\frac{r\rho(1-k)}{\mu_M-\mu_F+(1-r)\rho k}\left(e^{(\mu_S-\mu_F+(1-r)\rho k)\tau}-e^{(\mu_S-\mu_M)\tau}\right)\right)\frac{1}{1-e^{-\varepsilon\tau}}.
\end{aligned}
$$

This shows the convergence of the series and concludes the proof of Theorem 6.

$\square$

5.2.2. Sparse measurements

The feedback control approach requires to assess the size of mosquito population at every time $t \in \tau\mathbb{N}$. As mentioned in the Introduction, rough estimates of a wild population are achievable through direct capture and counting, or through more sophisticated methods such as Mark-Release-Recapture [27]. However, this protocol is long and costly. We now show how it is possible to reduce its frequency and to complete measurements only with a period $p\tau$ for some $p \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$. The values of the $(p-1)$ intermediate releases are computed using the last sampled information.

The following result adapts in consequence the control laws given in Theorem 6 to sparse measurements.

Theorem 7 (Stabilization by impulsive control with sparse measurements). *Let $p \in \mathbb{N}^*$. For a given $k \in \left(0, \frac{1}{\mathcal{N}_F}\right)$, assume for any $n \in \mathbb{N}$, $m = 0, 1, \ldots, p-1$,*

$$
\tau\Lambda_{np+m} \geq \left| K_p\begin{pmatrix} M(n\tau) \\ F(n\tau) \end{pmatrix} - M_S(np\tau)e^{-m\mu_S\tau} - \sum_{i=0}^{m-1}\Lambda_{np+i}e^{-\mu_S(m-i)\tau}\right|_+ \quad \text{(45a)}
$$

$$
K_p := \frac{e^{\mu_S\tau}}{\gamma}\left(\frac{\frac{1-k}{k}e^{-(m+1)\mu_M\tau}}{\frac{r\rho(1-k)}{\mu_M-\mu_F+(1-r)\rho k}\left(e^{-(\mu_F-(1-r)\rho k)(m+1)\tau}-e^{-\mu_M(m+1)\tau}\right)}\right)^{\top} \quad \text{(45b)}
$$

Then every solution of system (15) converges exponentially towards E_0^*, with a convergence speed bounded from below by a value independent of the initial condition.

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If moreover

$$\tau \Lambda_{np+m} \leq K_p \begin{pmatrix} M(n\tau) \\ F(n\tau) \end{pmatrix} , \tag{45c}$$

then the series of impulses $\sum\limits_{n=0}^{+\infty} \Lambda_n$ *converges.*

Notice that Theorem 7 represents an *extension* of Theorem 6, recovered in the case $p = 1$ (and thus $m = 0$): in this case, (45a) boils down to (39a).

Proof of Theorem 7. The demonstration comes from a slight adaptation of the proof of Theorem 6. Indeed, it suffices to verify that, under the conditions in Theorem 7, property (37) holds on the interval $(np\tau, (n + 1)p\tau]$, of length $p\tau$. Let $m \in \{0, 1, \ldots, p - 1\}$. One has for any $s \in (0, \tau]$ that

$$M_S\big(s + (np + m)\tau\big) = \Big(\Lambda_{np+m}\tau + M_S\big((np + m)\tau\big)\Big)e^{-\mu_S s}$$

$$= \Big(\Lambda_{np+m}\tau + \Lambda_{np+m-1}\tau e^{-\mu_S \tau} + \ldots$$

$$+ \Lambda_{np}\tau e^{-m\mu_S \tau} + M_S(np\tau)e^{-m\mu_S \tau}\Big)e^{-\mu_S s}.$$

Inequality (37) is thus true on $\big((np + m)\tau, (np + m + 1)\tau\big]$ *if and only if* it is imposed that, for any $m \in \{0, 1, \ldots, p - 1\}$ and any $s \in (0, \tau]$, $\gamma\Big(\Lambda_{np+m}\tau +$

$\Lambda_{np+m-1}\tau e^{-\mu_S \tau} + \cdots + \Lambda_{np}\tau e^{-m\mu_S \tau} + M_S(np\tau)e^{-m\mu_S \tau}\Big)e^{-\mu_S s} \geq \frac{1-k}{k}\Big(e^{-\mu_M(s+m\tau)}$

$M(n\tau) + \frac{r\rho k}{\mu_M - \mu_F + (1-r)\rho k}\Big(e^{-(\mu_F - (1-r)\rho k)(s+m\tau)} - e^{-\mu_M(s+m\tau)}\Big)F(n\tau)\Big)$, that is,

$$\Lambda_{np+m}\tau e^{m\mu_S \tau} + \Lambda_{np+m-1}\tau e^{(m-1)\mu_S \tau} + \cdots + \Lambda_{np}\tau + M_S(np\tau)$$

$$\geq \frac{1 - k}{\gamma k}\Big(e^{(\mu_S - \mu_M)(s+m\tau)}M(n\tau)$$

$$+ \frac{r\rho k}{\mu_M - \mu_F + (1-r)\rho k}\Big(e^{(\mu_S - \mu_F + (1-r)\rho k)(s+m\tau)} - e^{(\mu_S - \mu_M)(s+m\tau)}\Big)F(n\tau)\Big) \tag{46}$$

In virtue of the relationships (2) and (8), the right-hand side of (46) is an *increasing* function of s. Therefore, (46) is more restrictive when taken at $s = \tau$. This yields (45a) and shows the first part of the result. The convergence of the series of impulses is demonstrated similarly to Theorem 6. $\qquad\square$

6. Mixed impulsive control strategies

The results obtained in the previous sections for open-loop and closed-loop SIT control allow us to compare several SIT release strategies. Here, we consider only periodic impulsive control, which is more realistic than continuous control.

The open-loop approach (developed in Section 4), is based on the determination of a sufficient size of sterile males to be released, in order to eradicate the wild population. This choice is made according to (19). Under this approach,

even though the previous formula is 'tight', the same amount of sterile insects is used during the whole release campaign.

On the contrary, the closed-loop control approach (exposed in Section 5) is based on estimates of the wild population and thereby it enables fitting the release sizes. As evidenced by (39a), under this approach the released volume is essentially chosen as *proportional* to the measured population. However, this condition is certainly too demanding for large values of M, F (see the comments preceding Lemma 2). Taking advantage of the apparent complementarity of the two approaches, we propose here *mixed impulsive control strategies*, combining the two previous modes. They gather the advantages of both approaches, guaranteeing convergence to the mosquito-free equilibrium with releases that remain bounded (like the periodic impulsive control strategies, Section 4) and vanishing with the wild population (like the feedback control strategies, Section 5).

Theorem 8. *Let $p \in \mathbb{N}^*$. Assume that, for any $n \in \mathbb{N}$, Λ_n is chosen at least equal to the* smallest *of the right-hand side of* (45a) *and of a positive constant $\bar{\Lambda}$ that verifies one of the following cases:*

- **Case 1.**

$$\bar{\Lambda} = 2 \frac{(\cosh(\mu_S \tau) - 1)}{\mu_S \tau^2} \frac{1}{e \beta \gamma} \mathcal{N}_F \qquad if \ k \in \left(0, \frac{\mu_F}{(1-r)\rho} \right). \qquad (47)$$

- **Case 2.**

$$\bar{\Lambda} = \frac{(\cosh(\mu_S \tau) - 1)}{\mu_S \tau^2} \frac{1}{e \beta \gamma} \max\{r, 1-r\} \max \left\{ \frac{\mathcal{N}_M}{r}, \frac{\mathcal{N}_F}{1-r} \right\}$$

$$if \ k \in \left(0, 2 \frac{\mu_M}{\rho} \frac{1-r}{r^2} \left(\sqrt{1 + \frac{\mu_F}{\mu_M} \left(\frac{r}{1-r} \right)^2} - 1 \right) \right). \qquad (48)$$

Then every solution of system (15) *converges globally exponentially to E_0^*.*

The interest of the previous result is of course to consider the *smallest* of the two values of $\bar{\Lambda}$ and of the value provided by the closed-loop control law: it results in *saturated* control laws.

The main issue of the proof (presented below) is to establish convergence in the occurrence of infinitely many switches between the two modes. The demonstration is based on the use of *common Lyapunov functions*, that decrease along the trajectories of the system, regardless of the mode in use. Different Lyapunov functions are required for the two cases.

Remark 4. *Notice that*

$$2 \frac{\mu_M}{\rho} \frac{1-r}{r^2} \left(\sqrt{1 + \frac{\mu_F}{\mu_M} \left(\frac{r}{1-r} \right)^2} - 1 \right)$$

$$< 2 \frac{\mu_M}{\rho} \frac{1-r}{r^2} \frac{1}{2} \frac{\mu_F}{\mu_M} \left(\frac{r}{1-r} \right)^2 = \frac{\mu_F}{(1-r)\rho}, \qquad (49)$$

so the condition on k contained in (48) *is more restrictive than the one in* (47).

Remark 5. *The values of $\bar{\Lambda}$ that appear in (47) and (48) are two of the three that appear in (19), corresponding to (23) and (28) in the proof of Theorem 5, page 14. See the proof for more explanations.*

Proof of Theorem 8. For simplicity, we consider here the case where $p = 1$. The case with $p > 1$ is treated in a similar way.

● 1. For the **Case 1**, consider the evolution of F. As shown in the proof of Theorem 5, item 1, it holds that

$$F\big((n+1)\tau\big) \leq e^{-\varepsilon\tau} F(n\tau) \tag{50}$$

for a certain $\varepsilon > 0$ (independent of n) when Λ_n is at least equal to $\bar{\Lambda}$ given in (47). On the other hand, it is shown in the proof of Theorem 6, item 3, that F also decreases exponentially according to (50) when Λ_n is chosen according to (39a) (which is (45a) in the case $p = 1$). Therefore, regardless of the mode commutations, $F(t)$ converges exponentially towards zero for every trajectory. As substantiated in the proof of Theorem 6, this is sufficient to deduce the convergence of $M(t)$ towards zero. Thereby, Theorem 8 is proved in the **Case 1**.

● 2. For the **Case 2**, let $\mathcal{V}$ be the positive definite function $\mathcal{V}(M, F) := \frac{1}{2}(M^2 + F^2)$ introduced in (26), page 16. It was shown in the proof of Theorem 5, item 3, that property (50) also holds for some $\varepsilon > 0$ when Λ_n is chosen according to (39a).

On the other hand, when Λ_n is taken smaller than the value in (19), due to Theorem 6, one has for all $t \in \big(n\tau, (n+1)\tau\big]$, see (37), that

$$\gamma M_S(t) \geq \left(\frac{1}{k} - 1\right) M(t), \qquad \text{that is:} \quad \frac{M(t)}{M(t) + \gamma M_S(t)} \leq k. \tag{51}$$

Therefore, on the same interval, it holds:

$$\begin{aligned}
\dot{\mathcal{V}} &= M\dot{M} + F\dot{F} \\
&= \rho \frac{FM(rM + (1-r)F)}{M + \gamma M_S} e^{-\beta(M+F)} - \mu_M M^2 - \mu_F F^2 \\
&\leq \rho k F(rM + (1-r)F)e^{-\beta(M+F)} - \mu_M M^2 - \mu_F F^2 \\
&\leq \rho k F(rM + (1-r)F) - \mu_M M^2 - \mu_F F^2 \\
&= -\Big(\mu_M M^2 - \rho k r MF + (\mu_F - \rho k(1-r))F^2\Big).
\end{aligned}$$

The discriminant of the previous quadratic form is

$$\Delta' = r^2 \rho^2 k^2 + 4\mu_M(1-r)\rho k - 4\mu_M\mu_F, \tag{52}$$

which is negative when k is taken according to (48). In such case, $\dot{\mathcal{V}}$ is negative definite. One concludes that $\mathcal{V}$ decreases exponentially to zero, and this ensures the global exponential stability of the mosquito-free equilibrium E_0^*. The result is thus also proved in the **Case 2**. This achieves the proof of Theorem 8. $\quad\square$

7. Numerical illustrations

We present here several numerical simulations, aimed at illustrating and comparing the different impulsive release strategies developed in the previous sections. The values of the vital characteristics of the mosquitoes which are used in the simulations are summarized in Table 2.

Parameter	Value	Description
ρ	4.55	Number of eggs a female can deposit
r	0.5	$r : (1 - r)$ expresses the primary sex ratio in offspring
σ	0.05	Regulates the larvae development into adults under density dependence and larval competition
K	140	Carrying capacity
μ_M	0.04	Mean mortality rate of wild adult male mosquitoes
μ_F	0.03	Mean mortality rate of wild adult female mosquitoes
μ_S	0.04	Mean mortality rate of sterile adult male mosquitoes
γ	1	Fitness of sterile adult male mosquitoes

Table 2: *Aedes spp* parameters values

The carrying capacity K in the Table corresponds to 1 hectare. With the above numbers, we have here for the global competition coefficient $\beta = \dfrac{\sigma}{K} = 3.57 \times 10^{-4}$, and for the basic offspring numbers $\mathcal{N}_F \approx 75.83$ and $\mathcal{N}_M \approx 56.87$. At equilibrium, the mosquito population is thus $E^* = (M^*, F^*)$ with $M^* \approx 6,925$, $F^* \approx 5,194$ individuals per hectare. Let us remind that the basic offspring numbers represent the average number of offspring produced over the lifetime of an individual. As reflected in Theorem 1, they are closely related to the viability of the species, while for its part, the coefficient β affects directly the value of the equilibrium population.

The SIT control starts at time $t = 100$. All simulations run as long as $\log_{10} F(t)$ remains larger than a given threshold value, here -1, below which we assume that elimination has been reached.

● *Constant impulsive releases.* We here illustrate the release method exposed in Theorem 5. For open-loop periodic impulsive releases carried out every 7 (resp. 14) days, we consider the smallest value given in (19), page 14, to estimate the number of sterile males to release, that is, $7 \times \Lambda_{per}^{crit} = 7 \times 1,573 = 11,011$ (resp. $14 \times \Lambda_{per}^{crit} = 14 \times 1,604 = 22,456$) sterile males per hectare and per week (resp. every two weeks).

The corresponding simulations are given in Figure 3. In Table 3, we summarize the cumulative number of sterile males as well as the number of releases needed to reach nearly "elimination". While, as expected, the total number of released sterile males is lower for $\tau = 7$, there is no gain in terms of treatment duration. Thus, taking into account the cost of each release and also the risk of failure during the transport, it seems preferable to consider the lower number of releases, and thus to choose $\tau = 14$.
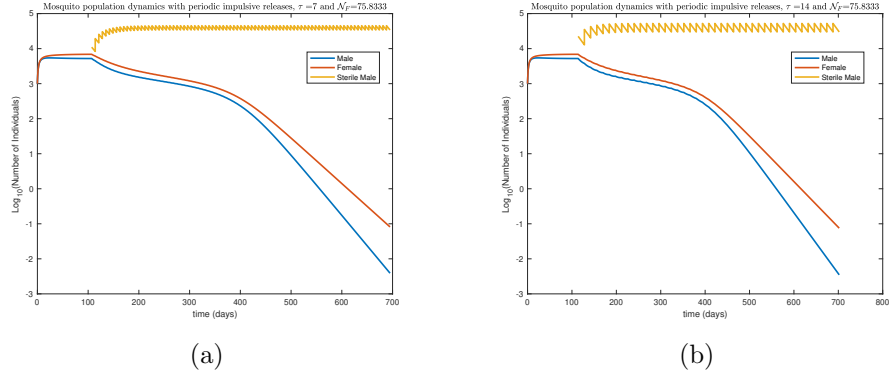
Figure 3: Open-loop periodic impulsive SIT control of system (15) with a period of: (a) 7 days, (b) 14 days.

Period (days)	Cumulative Number of released sterile males	Nb of Weeks to reach elimination
$\tau = 7$	**924, 627**	84
$\tau = 14$	942, 869	84

Table 3: Cumulative number of released sterile males for each open-loop periodic SIT control treatment.

The closed-loop approach can be used to reduce the cumulative number of released sterile insects and the number of effective releases. Further on, we will consider several sub-cases.

• *Impulsive feedback control.* We now show application of the algorithms given in Theorems 6 and 7. Here and in the study of all feedback methods, we consider measurements of the wild population every τ days or every $p\tau$ days for $p = 4$. Also, in order to display the tradeoff between *treatment duration* and *control effort*, we investigate two values of k, namely

$$k\mathcal{N}_F = 0.2 \quad \text{and} \quad k\mathcal{N}_F = 0.99. \tag{53}$$

With the smaller value $k = 0.2/\mathcal{N}_F$, the control effort is larger and one expects faster convergence toward $E_0 = (0,0)$, at the price of larger releases of sterile males, i.e. higher costs. On the contrary, for larger $k = 0.99/\mathcal{N}_F$, the control effort is smaller and convergence should be slower, with smaller total number of released insects.

The size Λ_n of the n-th release is taken equal to the right-hand side of formula (39a) for $p = 1$ (of (45a) for $p = 4$): if, at the moment of the estimate, the size of the sterile male population is sufficiently large, Λ_n may be small or even null.

Simulations presented in Figures 4 (page 30) and 5 (page 31) clearly show that the choice of k and p, as well as the period τ of the releases play an

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important role in the convergence of the wild population to E_0^*. Tables 4 and
5 provide the total cumulative number of released sterile males, the number of
weeks of SIT treatment needed to reach elimination, and the number of *effective*
(that is nonzero) releases. For instance, when $(\tau, p) = (14, 4)$ and $k = \dfrac{0.2}{\mathcal{N}_F}$ is
relatively small, elimination of wild mosquitoes can be achieved in 56 weeks,
with only 17 effective releases, as shown in Fig. 6(b), page 32. However, this
option requires to release significant number of sterile insects per hectare (close
to 2.9×10^6 for the whole treatment).

For the larger $k = \dfrac{0.99}{\mathcal{N}_F}$ and with $(\tau, p) = (7, 1)$ (see Figure 5(a)), the
convergence is slower: more than 240 weeks of SIT treatment are required to
reach nearly elimination. For $p = 4$ (see Figure 5(b)), the wild population is
close to extinction after 58 weeks of SIT treatment. However, based on Table
5, it seems that the choice $(\tau, p) = (14, 4)$ leads to the best result in terms of
timing (62 weeks) and also in terms of cumulative size encompassing 20 effective
releases.

The parameter k is of main importance: when $p = 4$, while the number of
weeks to reach elimination is quite similar for both values of τ, the cumulative
number of released sterile males is clearly smaller when k is closer to $1/\mathcal{N}_F$.

p Period	Cumulative Nb of released sterile males		Nb of weeks to reach elimination		Nb of nonzero releases
	1	4	1	4	4
$\tau = 7$	$2,251,052$	$4,363,430$	64	54	34
$\tau = 14$	$2,390,676$	$2,896,835$	64	56	17

Table 4: Cumulative number of released sterile males and number of releases for each closed-loop periodic SIT control treatment when $k = \dfrac{0.2}{\mathcal{N}_F}$. See Figure 4, page 30.

p Period	Cumulative Nb of released sterile males		Nb of weeks needed to reach elimination		Nb of nonzero releases
	1	4	1	4	4
$\tau = 7$	$794,807$	$1,221,593$	240	58	37
$\tau = 14$	$909,344$	$1,043,107$	130	62	20

Table 5: Cumulative number of released sterile males and number of releases for each closed-loop periodic SIT control treatment when $k = \dfrac{0.99}{\mathcal{N}_F}$. See Figure 5, page 31

• *Mixed control.* We now consider mixed control strategies as exposed in Section
6 (Theorem 8). In Figures 7 and 8 (pages 33 and 34, respectively) are shown the
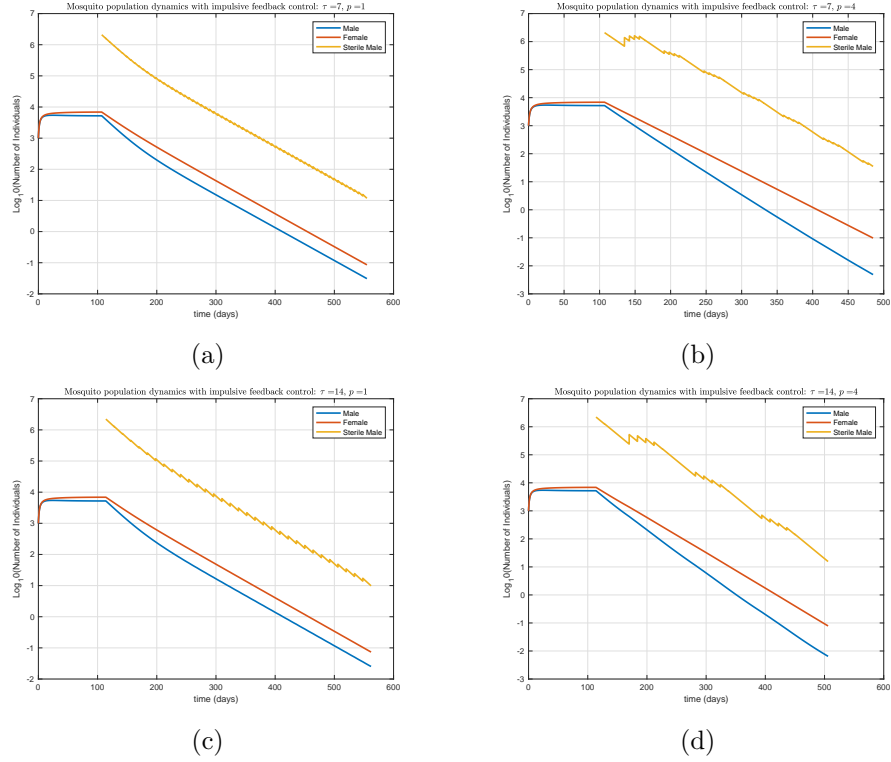simulations obtained with the same two underlying values of k given in (53).

Figure 4: Closed-loop periodic impulsive SIT control of system (15) with $k = \dfrac{0.2}{\mathcal{N}_F}$: (a) 7 days, $p = 1$; (b) 7 days, $p = 4$; (c) 14 days, $p = 1$; (d) 14 days, $p = 4$. See Table 4, page 29

	Cumulative Nb of released sterile males		Nb of weeks needed to reach elimination		Nb of nonzero releases
Period $\diagdown$ p	1	4	1	4	4
$\tau = 7$	$450,668$	$534,849$	72	65	53
$\tau = 14$	$465,187$	$499,497$	72	66	25

Table 6: Cumulative number of released sterile males and number of releases for each mixed open/closed-loop periodic SIT control treatment when $k = \dfrac{0.2}{\mathcal{N}_F}$. See Figure 7, page 33.

Except for the case with $(\tau, p) = (7, 1)$ and $k = \dfrac{0.99}{\mathcal{N}_F}$ (see Table 6, page 30), where the convergence to E_0^* is slow, it turns out that the mixed open/closed-loop control strategies derive the best results, not only in terms of releases number but also in terms of overall cumulative number of sterile males to be released.

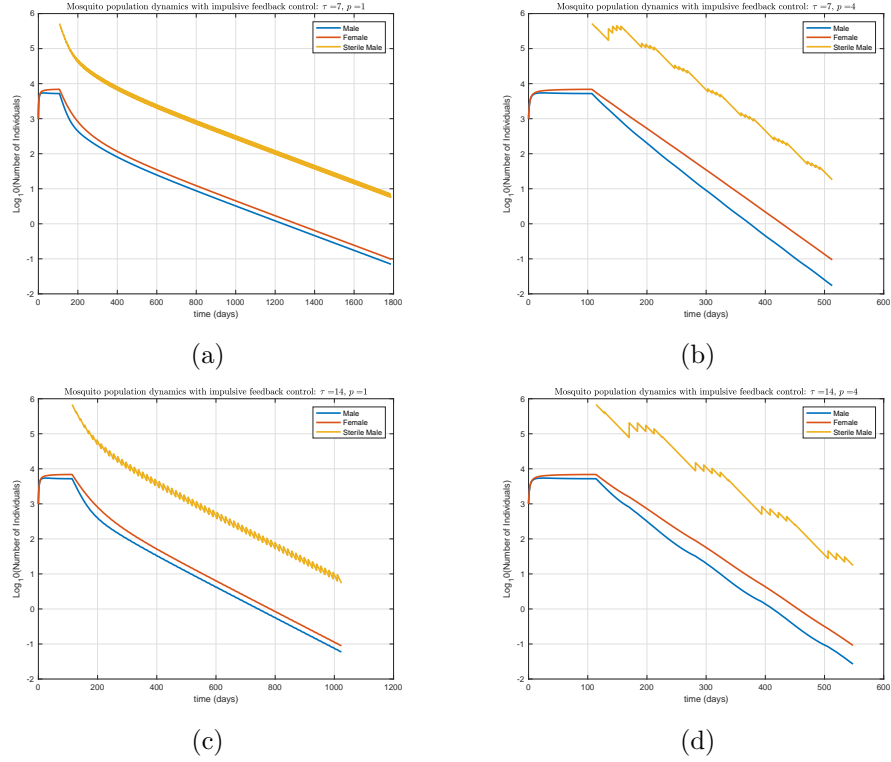According to Tables 6 and 7 (pages 30 and 31, respectively), for both values

(a)

(b)

(c)

(d)

Figure 5: Closed-loop periodic impulsive SIT control of system (15) with $k = \dfrac{0.99}{\mathcal{N}_F}$: (a) 7 days, $p = 1$; (b) 7 days, $p = 4$; (c) 14 days, $p = 1$; (d) 14 days, $p = 4$. See Table 5, page 29

	Cumulative Nb of released sterile males		Nb of weeks needed to reach elimination		Nb of nonzero releases
p / Period	1	4	1	4	4
$\tau = 7$	$457,489$	$450,077$	246	69	53
$\tau = 14$	$427,701$	$449,059$	136	74	28

Table 7: Cumulative number of released sterile males and number of releases for each mixed open/closed-loop periodic SIT control treatment when $k = \dfrac{0.99}{\mathcal{N}_F}$. See Figure 8, page 34.

of k, the best solution would be to release sterile insects every 2 weeks with population assessments carried out by measurements every 4 weeks ($p = 4$). In addition, and thanks to (45), Figure 9 displays the release sizes Λ_n for each mixed strategy. It clearly shows that during the first releases, $\Lambda_n = \Lambda_{per}^{crit}$. Further, when the wild population drops below a certain threshold, the feedback control occurs or not, depending on the (estimated) size of the sterile male
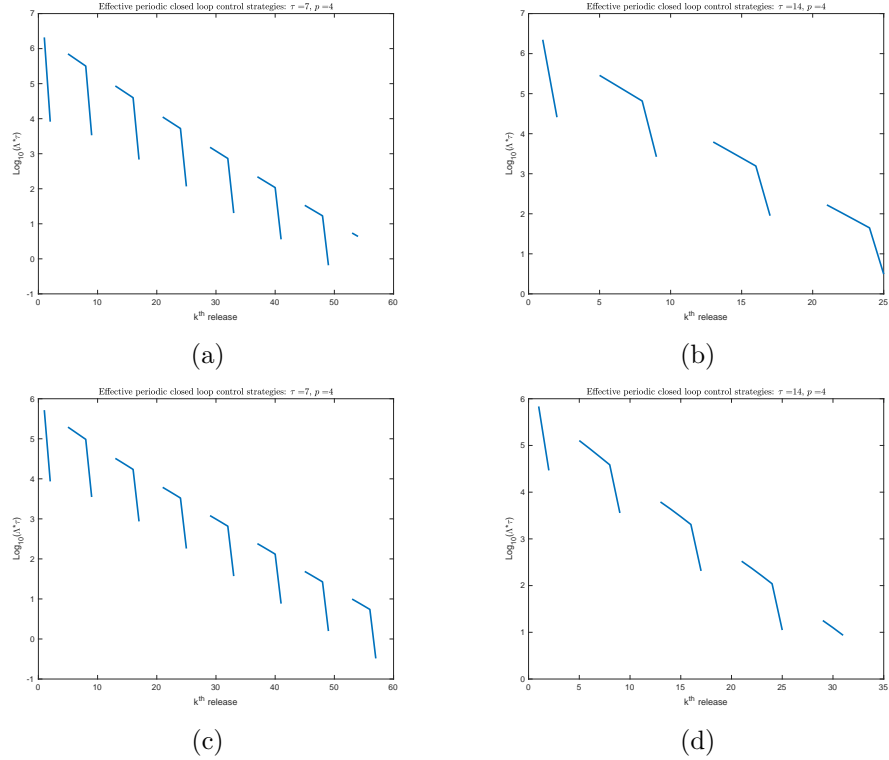
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Figure 6: Size of the release, Λ_n at time $t = n\tau$ for closed-loop SIT control: (a,b) $k = \dfrac{0.2}{\mathcal{N}_F}$; (c,d) $k = \dfrac{0.99}{\mathcal{N}_F}$. The discontinuities indicate "no release".

population. That is why in Tables 6 and 7, we derive the number of effective releases (only for the case $p = 4$) which confirms that the best combination is $(\tau, p) = (14, 4)$, regardless of the value of k.

For the mixed open/closed-loop periodic impulsive SIT control, the choice of k does not matter compared to the closed-loop control only. Our preliminary results thereby indicate that a mixed SIT control option with ($\tau = 14, p = 4$) leads to the best strategy in terms of the total number of released sterile males and also in terms of effective releases number.

8. Conclusion

In this work, we studied various strategies to control mosquito population using SIT: open-loop and closed-loop periodic impulsive control strategies, as well as their combination (mixed open/closed-loop strategy). For the open-loop strategy (that is usually considered during field experiments) we found the minimal number of sterile males to be released every τ days in order to reach
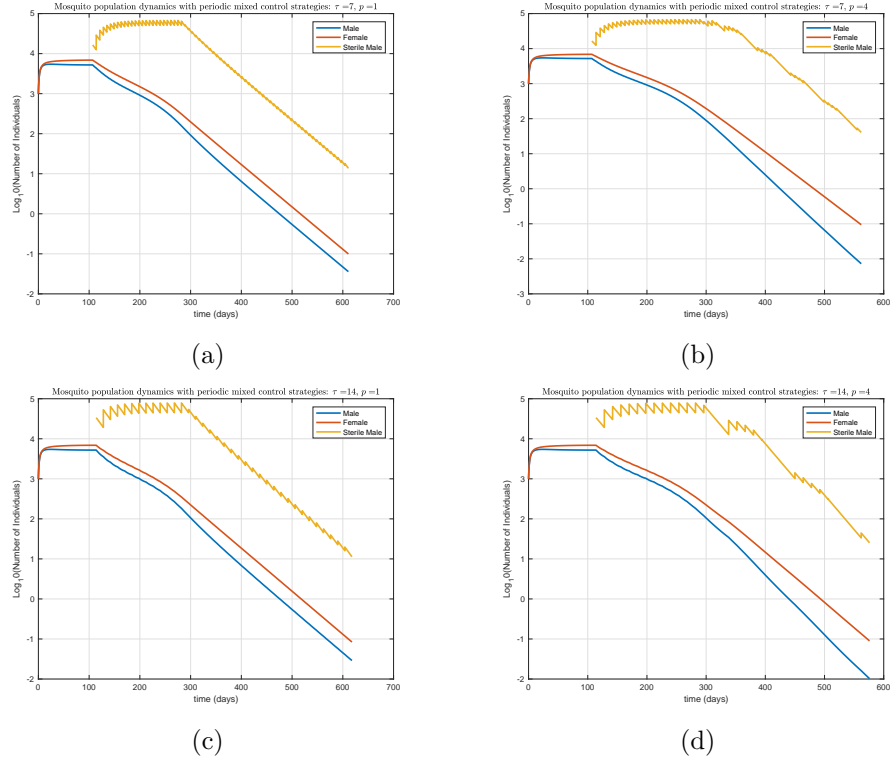
Figure 7: Combination of open/closed-loop periodic impulsive SIT control of system (15) with $k = \dfrac{0.2}{\mathcal{N}_F}$: (a) 7 days, $p = 1$; (b) 7 days, $p = 4$; (c) 14 days, $p = 1$; (d) 14 days $p = 4$. See Table 6, page 30.

elimination of wild mosquitoes. This number is constant and relatively low. The question of determining a stopping time for the release campaigns is not simple, but clearly of primordial importance, as premature ending ruins the preceding efforts.

On the contrary, the feedback SIT-control commences with relatively abundant releases and their amplitude steadily declines with the wild population size until fading away and vanishing when the system converges towards the desired mosquito-free state. This closed-loop control strategy requires to assess the current size of the wild population (using MRR experiments, for instance).

Finally, we proposed a mixed control strategy, combining open-loop and closed-loop strategies. This control input mode renders the best result, and turns out rather meaningful from the experimental standpoint: the control input is launched at the open-loop mode during first weeks (initial phase) and then is shifted to the closed-loop mode (final phase), once the size of wild population exhibits steady decline. With this approach, the gain in terms of release pick-value, number of nonzero releases, and overall cumulative volume is clearly
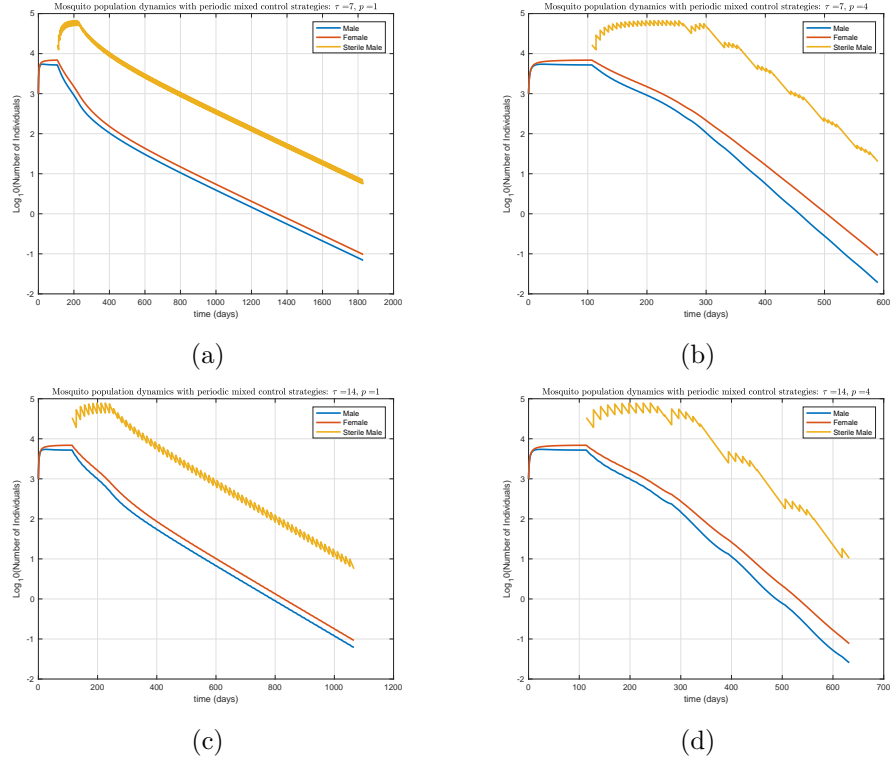
33

Figure 8: Combination of open/closed-loop periodic impulsive SIT control of system (15), with $k = \dfrac{0.99}{\mathcal{N}_F}$: (a) 7 days, $p = 1$, (b) 7 days, $p = 4$, (c) 14 days, $p = 1$, (d) 14 days $p = 4$. See Table 7, page 31.

visible. This is due to the fact that initial phase of control action is done at the open-loop mode, i.e. by performing releases of sterile males regardless of the current size of wild population, a pretreatment which induces an essential decline of the wild population before switching to the closed-loop control mode. Even considering the simulations displayed in terms of cost, the mixed control seems to be definitively the best choice when a release is carried out every two weeks, based on a population estimate acquired every four weeks.

Knowledge of the cost of each stage of the SIT control (mass rearing, sterilization either by irradiation or using *Wolbachia*, transportation to the target locality, wild population measurements with MRR techniques, and other necessary supplies) will allow to estimate more precisely and optimize the treatment cost, and thus to make the most appropriate choices from an economical point of view.

As a last remark, we notice that, from a mathematical point of view, the use of closed-loop methods, as well as the fact that the proof of their effectiveness is based on argument of monotonicity, are certainly able to guarantee robust-
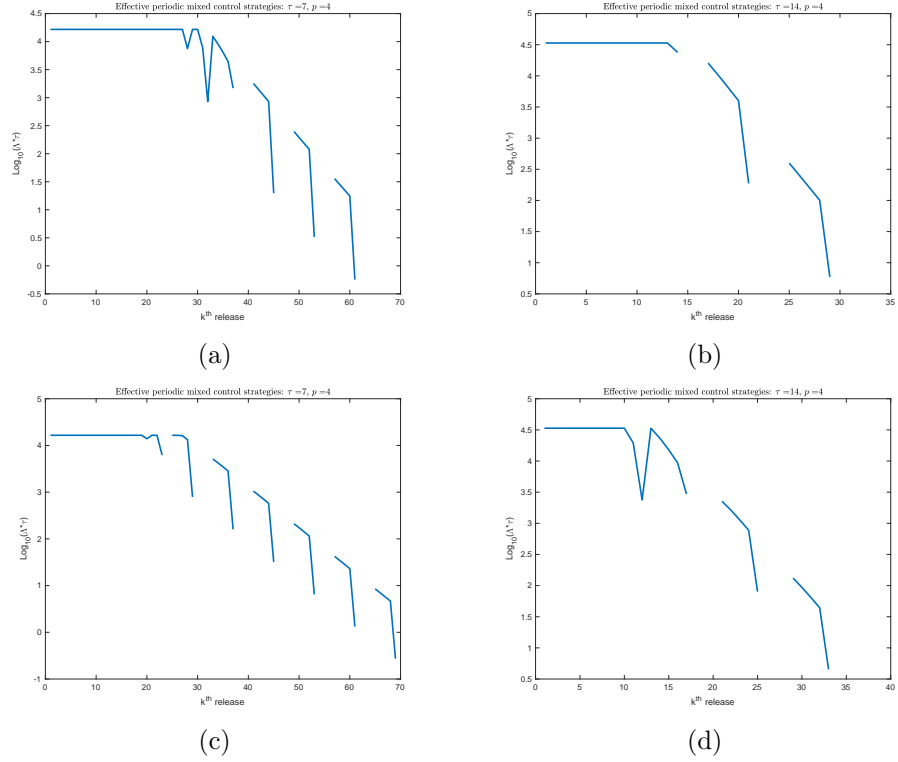
34

(a)

(b)

(c)

(d)

Figure 9: Size of the release, Λ_n, at time $t = n\tau$ for mixed open/closed-loop SIT control: (a,b) $k = \dfrac{0.2}{\mathcal{N}_F}$; (c,d) $k = \dfrac{0.99}{\mathcal{N}_F}$. The discontinuities indicate "no release".

ness of the proposed closed-loop algorithms with respect to several uncertainties present in the problem under study. In particular, it is believed that the framework developed here could most certainly be extended to consider the effects of modeling and measurement errors, as well as imprecision and delay in the control-loop.

Acknowledgments

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Appendix: Proof of Lemma 1, page 11

First, it is easy to check that function f, defined in (14), page 11, is first decreasing and then increasing, and, thus, may solely have no root, one root or two zeros.

On the other hand, the number of roots of $f(x) = 0$ is clearly *non-increasing* with respect to $a > 0$: it has two roots for 'small values' of a, no root for 'large values' of a, and exactly one root for a certain critical value a^{crit} separating the two previous regions. This critical value is characterized by the fact that it possesses a double root $x^{crit} > 0$, such that $f(x^{crit}) = f'(x^{crit}) = 0$, that is:

$$1 + \frac{a^{crit}}{x^{crit}} = \mathcal{N}_F e^{-cx^{crit}}, \qquad \frac{a^{crit}}{(x^{crit})^2} = \mathcal{N}_F c e^{-cx^{crit}}. \qquad \text{(A-1)}$$

Eliminating the exponential term in the previous formulas yields the second-order polynomial equation in $\dfrac{1}{x^{crit}}$

$$\left(\frac{1}{x^{crit}}\right)^2 - \frac{c}{x^{crit}} - \frac{c}{a^{crit}} = 0.$$

Its unique positive root is

$$\frac{1}{x^{crit}} = \frac{c + \sqrt{c^2 + 4\dfrac{c}{a^{crit}}}}{2} = \frac{c}{2}\left(1 + \sqrt{1 + \frac{4}{a^{crit}c}}\right),$$

that is:

$$x^{crit} := \frac{2}{c}\frac{1}{1 + \sqrt{1 + \dfrac{4}{a^{crit}c}}}.$$

Introducing this expression back in (A-1), leads to

$$1 + \frac{a^{crit}c}{2}\left(1 + \sqrt{1 + \frac{4}{a^{crit}c}}\right) = \mathcal{N}_F \exp\left(-\frac{2}{1 + \sqrt{1 + \frac{4}{a^{crit}c}}}\right).$$

Thus $\phi^{crit} := \dfrac{a^{crit}c}{2}$ is solution of (11), page 10, such that, at the critical point,
the parameters $a^{crit}, \mathcal{N}_F, c$ are interrelated.

For positive values of a smaller than a^{crit}, the equation $f(x) = 0$ has two roots, and no root whenever $a > a^{crit}$. This achieves the proof of Lemma 1.