

Fast waves and incompressible models

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Fast waves and incompressible models

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Low Mach number flows

Compressible Euler equations :

$$\partial_t \rho + \operatorname{div} \rho \mathbf{u} = 0 \quad \rho = \rho_* \rho$$

$$\partial_t \rho \mathbf{u} + \operatorname{div} \rho \mathbf{u} \otimes \mathbf{u} + \nabla p = 0 \quad \mathbf{u} = u_* \mathbf{u}$$

$$\partial_t p + \mathbf{u} \cdot \nabla p + \rho a^2 \operatorname{div} \mathbf{u} = 0 \quad p = \rho_* (a_*)^2 p$$

$$x_i = L_* x_i; \quad t = L_* / u_* t \quad \varepsilon = u_* / a_*$$

$$\partial_t \rho + \operatorname{div} \rho \mathbf{u} = 0$$

$$\partial_t \rho \mathbf{u} + \operatorname{div} \rho \mathbf{u} \otimes \mathbf{u} + \frac{1}{\varepsilon^2} \nabla p = 0$$

$$\partial_t p + \mathbf{u} \cdot \nabla p + \rho a^2 \operatorname{div} \mathbf{u} = 0$$



Low Mach number flows

The incompressible limit

$$f = f_0 + M_* f_1 + M_*^2 f_2$$

- $\mathcal{O}(1/M_*^2)$: $\nabla p_0 = 0$
 - if $\partial_t p_0 = 0 \rightarrow \operatorname{div} \mathbf{u}_0 = 0$
 - if $D\rho_0/Dt = 0 \rightarrow \rho_0 = \text{constant}$
- $\mathcal{O}(1/M_*)$ same analysis
- $\mathcal{O}(1)$ $\rho_0 D\mathbf{u}_0/Dt + \nabla p_2 = 0$

Incompressible Euler equations

$$\begin{aligned} \rho D\mathbf{u}/Dt + \nabla p &= 0 \\ \operatorname{div} \mathbf{u} &= 0 \end{aligned}$$

Low Mach number flows

The acoustic limit

Incompressible limit is **not** the unique low Mach limit of compressible eqs

- hidden assumption in incompressible asymptotic analysis
- time scale $t_* = L_*/u_*$: large time scale
- choose instead $t_* = L_*/a_*$: short time scale

scaling becomes

$$\left\{ \begin{array}{l} \frac{1}{\varepsilon} \partial_t \rho + \operatorname{div} \rho \mathbf{u} = 0 \\ \frac{1}{\varepsilon} \partial_t \rho \mathbf{u} + \operatorname{div} \rho \mathbf{u} \otimes \mathbf{u} + \frac{1}{M_*^2} \nabla p = 0 \\ \frac{1}{\varepsilon} \partial_t p + \mathbf{u} \cdot \nabla p + \rho a^2 \operatorname{div} \mathbf{u} = 0 \end{array} \right.$$



First example : Low Mach number flows

Superposition incompressible + acoustics

Asymptotic analysis of the acoustic limit

$$f = f_0 + M_* f_1 + M_*^2 f_2$$

- $\mathcal{O}(1/M_*^2)$: $\nabla p_0 = 0$
- $\mathcal{O}(1/M_*)$
 - $\partial_t p_0 = \partial_t p_0 = 0$
 - $\rho_0 \partial_t \mathbf{u}_0 + \nabla p_1 = 0$
- $\mathcal{O}(1)$: $\partial_t p_1 + \rho_0 a_0^2 \nabla \cdot \mathbf{u}_0 = 0$

Linear Acoustic equations

$$\rho_0 \partial_t \mathbf{u} + \nabla p = 0$$

$$\partial_t p + \rho_0 a_0^2 \operatorname{div} \mathbf{u} = 0$$



Incompressible + Acoustic superposition

Provisional conclusion (Intuition):

- General solution = Slow (incompressible) + fast (Acoustic) component
- Can we prove it ?
- Does acoustic-acoustic interactions are able to modify the dynamics of the incompressible
- Can we understand it ? component ?



Incompressible + Acoustic superposition

Very complex answer depending on

- Initial data
 - well-prepared initial data (initial data “close” to incompressible flow)
 - general initial data
- State Law
 - barotropic flow ($p = p(\rho)$)
 - $p = p(\rho, s)$: the sound speed (at first order) is NOT a constant
- Dissipative or not (Euler or NS)
- Boundary conditions
 - Whole space
 - Periodic BC
 - Closed vessel (slip bc or no-slip)



Incompressible + Acoustic superposition

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Slow and fast limits of hyperbolic PDEs

Let $\mathbf{W} \in \mathbf{R}^N$ solution of the hyperbolic system with a **large operator**

$$\begin{cases} \partial_t \mathbf{W} + \sum_j [A_j(\mathbf{W}, \varepsilon) + \frac{1}{\varepsilon} C_j] \partial_{x_j} \mathbf{W} = 0 \\ \mathbf{W}(0, \mathbf{x}, \varepsilon) = \mathbf{W}_0(\mathbf{x}, \varepsilon) \end{cases}$$

What is the behavior of the solutions when $\varepsilon \rightarrow 0$?

Let \mathbf{n} be a arbitrary direction, then some eigenvalues of $\sum_j n_j (A_j + \frac{1}{\varepsilon} C_j)$ are of the form $a_k + \frac{1}{\varepsilon} c_k \rightarrow \pm\infty$ while the others (kernel of $\sum_j n_j C_j$) are simply a_k

What is the behavior of the solutions when Slow and Fast waves co-exist ?

Singular limit of hyperbolic PDEs : Slow limit

$$\partial_t \mathbf{W} + \sum_j A_j(\mathbf{W}, \varepsilon) \partial_{x_j} \mathbf{W} + \frac{1}{\varepsilon} \sum_j C_j \partial_{x_j} \mathbf{W} = 0$$

$\mathbb{L} \mathbf{W} = \sum_j C_j \partial_{x_j} \mathbf{W}$ has to be $\mathcal{O}(\varepsilon)$

Look for the solution as $\mathbf{W} = \mathbf{W}_0 + \varepsilon \mathbf{W}_1$ with $\mathbb{L} \mathbf{W}_0 = 0$, obtain :

$$\partial_t \mathbf{W}_0 + \sum_j A_j(\mathbf{W}, \varepsilon) \partial_{x_j} \mathbf{W}_0 + \mathbb{L} \mathbf{W}_1 = \mathcal{O}(\varepsilon)$$

and the solutions converge to \mathbf{W}_0 defined by :

$$\begin{cases} \mathbb{L} \mathbf{W}_0 = 0 \\ \partial_t \mathbf{W}_0 + \mathbb{P} \sum_j A_j(\mathbf{W}_0, 0) \partial_{x_j} \mathbf{W}_0 = 0 \end{cases}$$

\mathbb{P} projection on the kernel of \mathbb{L}



But the system has also a fast limit

$$\partial_t \mathbf{W} + \sum_j A_j(\mathbf{W}, \varepsilon) \partial_{x_j} \mathbf{W} + \frac{1}{\varepsilon} \sum_j C_j \partial_{x_j} \mathbf{W} = 0$$

Let us do the simple change of variable : $t = \varepsilon \tau$:

$$\frac{1}{\varepsilon} \partial_\tau \mathbf{W} + \sum_j A_j(\mathbf{W}, \varepsilon) \partial_{x_j} \mathbf{W} + \frac{1}{\varepsilon} \sum_j C_j \partial_{x_j} \mathbf{W} = 0$$

and when $\varepsilon \rightarrow 0$ the limiting form becomes :

$$\partial_\tau \mathbf{W} + \sum_j C_j \partial_{x_j} \mathbf{W} = 0$$

Solution are fast waves moving at velocity $\frac{1}{\varepsilon}$



Singular limit of hyperbolic PDEs

Let $\mathbf{W} \in \mathbf{R}^N$ solution of the hyperbolic system with a **large operator**

$$\begin{cases} \partial_t \mathbf{W} + \sum_j [A_j(\mathbf{W}, \varepsilon) + \frac{1}{\varepsilon} C_j] \partial_{x_j} \mathbf{W} = 0 \\ \mathbf{W}(0, \mathbf{x}, \varepsilon) = \mathbf{W}_0(\mathbf{x}, \varepsilon) \end{cases}$$

What is the behavior of the solutions when $\varepsilon \rightarrow 0$?

How Slow and Fast waves co-exist ?

Why do we think that we can split the fast and slow phenomena ?



An Explicit linear example I

Consider the **linear** system

$$\frac{\partial r}{\partial t} + \mathbf{a} \cdot \nabla r + \frac{1}{\varepsilon} \operatorname{div} \mathbf{u} = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{a} \cdot \nabla \mathbf{u} + \frac{1}{\varepsilon} \nabla r = 0$$



Warm-up : Explicit linear example II

Compact form :

$$\partial_t \mathbf{v} + \mathbb{H} \mathbf{v} + \frac{1}{\varepsilon} \mathbb{L} \mathbf{v} = 0$$

Notations :

$$\mathbf{v} = \begin{pmatrix} r \\ \mathbf{u} \end{pmatrix} \quad \mathbb{L} \mathbf{v} = \begin{pmatrix} \nabla \cdot \mathbf{u} \\ \nabla r \end{pmatrix}$$

$\mathbb{H} \mathbf{v} = \mathbf{a} \cdot \nabla \mathbf{v}$ is a constant velocity linear advection operator
In Fourier space

$$\frac{\partial \hat{\mathbf{v}}(\mathbf{k})}{\partial t} + i[\hat{\mathbb{H}}(\mathbf{k}) + \frac{1}{\varepsilon} \hat{\mathbb{L}}(\mathbf{k})] \hat{\mathbf{v}}(\mathbf{k}) = 0 \quad \text{for } \mathbf{k} \in \mathbb{Z}^2 \quad (1)$$

where the matrix $\hat{\mathbb{H}}(\mathbf{k}) + 1/\varepsilon \hat{\mathbb{L}}(\mathbf{k})$ is equal to :

$$\begin{pmatrix} \mathbf{a} \cdot \mathbf{k} & k_1/\varepsilon & k_2/\varepsilon \\ k_1/\varepsilon & \mathbf{a} \cdot \mathbf{k} & 0 \\ k_2/\varepsilon & 0 & \mathbf{a} \cdot \mathbf{k} \end{pmatrix} \quad (2)$$



This matrix is diagonalizable, its eigenvectors are :

$$s_1(\mathbf{k}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -k_1/|\mathbf{k}| \\ -k_2/|\mathbf{k}| \end{pmatrix}, \quad s_2(\mathbf{k}) = \frac{1}{|\mathbf{k}|} \begin{pmatrix} 0 \\ -k_2 \\ k_1 \end{pmatrix} \quad (3)$$

$$, \quad s_3(\mathbf{k}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ k_1/|\mathbf{k}| \\ k_2/|\mathbf{k}| \end{pmatrix}$$

with associated eigenvalues $\lambda_1 = \mathbf{a} \cdot \mathbf{k} - \frac{|\mathbf{k}|}{\varepsilon}$, $\lambda_2 = \mathbf{a} \cdot \mathbf{k}$ and $\lambda_3 = \mathbf{a} \cdot \mathbf{k} + \frac{|\mathbf{k}|}{\varepsilon}$.

Note : $\hat{\mathbb{L}}s_2(\mathbf{k}) = 0$; in physical space $s_2(\mathbf{k})$ corresponds to constant density ($\nabla r = 0$) and div free vectors ($\nabla \cdot \mathbf{u} = 0$)



Explicit linear example III

$$\hat{v}(\mathbf{k}, t) = \left\{ \begin{array}{l} \frac{1}{\sqrt{2}}(\hat{r}(\mathbf{k}, 0) - \frac{k_1}{|\mathbf{k}|}\hat{u}(\mathbf{k}, 0) - \frac{k_2}{|\mathbf{k}|}\hat{v}(\mathbf{k}, 0))e^{-i(\mathbf{a}\cdot\mathbf{k}-|\mathbf{k}|/\varepsilon)t} s_1(\mathbf{k}) \\ + \frac{1}{|\mathbf{k}|}(-k_2\hat{u}(\mathbf{k}, 0) + k_1\hat{v}(\mathbf{k}, 0))e^{-i\mathbf{a}\cdot\mathbf{k}t} s_2(\mathbf{k}) \\ + \frac{1}{\sqrt{2}}(\hat{r}(\mathbf{k}, 0) + \frac{k_1}{|\mathbf{k}|}\hat{u}(\mathbf{k}, 0) + \frac{k_2}{|\mathbf{k}|}\hat{v}(\mathbf{k}, 0))e^{-i(\mathbf{a}\cdot\mathbf{k}+|\mathbf{k}|/\varepsilon)t} s_3(\mathbf{k}) \end{array} \right.$$



Explicit linear example IV

Fast oscillatory component $\hat{\mathbf{v}}_f(\mathbf{k}, t, t/\varepsilon)$

$$\frac{1}{\sqrt{2}} \left\{ \begin{array}{l} (\hat{r}(\mathbf{k}, 0) - \frac{k_1}{|\mathbf{k}|} \hat{u}(\mathbf{k}, 0) - \frac{k_2}{|\mathbf{k}|} \hat{v}(\mathbf{k}, 0)) e^{-i(\mathbf{a} \cdot \mathbf{k} - \frac{|\mathbf{k}|}{\varepsilon})t} s_1(\mathbf{k}) \\ + \\ (\hat{r}(\mathbf{k}, 0) + \frac{k_1}{|\mathbf{k}|} \hat{u}(\mathbf{k}, 0) + \frac{k_2}{|\mathbf{k}|} \hat{v}(\mathbf{k}, 0)) e^{-i(\mathbf{a} \cdot \mathbf{k} + \frac{|\mathbf{k}|}{\varepsilon})t} s_3(\mathbf{k}) \end{array} \right. \quad (4)$$



Explicit linear example V

Slow component belonging to the kernel of \mathbb{L}

$$\hat{\mathbf{v}}_s(\mathbf{k}, \tau) = \frac{1}{|\mathbf{k}|} (-k_2 \hat{u}(\mathbf{k}, 0) + k_1 \hat{v}(\mathbf{k}, 0)) e^{-i\mathbf{a} \cdot \mathbf{k} t} s_2(\mathbf{k})$$

This component belongs to the kernel of \mathbb{L} and satisfies the incompressible system

$$\begin{cases} \frac{\partial \mathbf{v}_s}{\partial t} + \mathbb{H} \mathbf{v}_s = 0 \\ \mathbb{L} \mathbf{v}_s = 0 \end{cases}$$



Explicit linear example VI

What is the behavior of the solutions when $\varepsilon \rightarrow 0$?

For any ε the solution is composed of a superposition of fast and slow waves.

Does the solution converge toward something when $\varepsilon \rightarrow 0$?

- In a point-wise : **NO** : faster and faster oscillations
- In a weak sense (average or distribution) **YES**

$$e^{\pm i \left(\frac{|\mathbf{k}|}{\varepsilon} \right) t} \rightarrow 0$$

thus the oscillatory part of the solution $\rightarrow 0$

and the solutions converge (weakly) toward \mathbf{v}_0 that satisfies the incompressible system :

$$\begin{cases} \frac{\partial \mathbf{v}_0}{\partial t} + \mathbb{H} \mathbf{v}_0 = 0 \\ \mathbb{L} \mathbf{v}_0 = 0 \end{cases}$$

Is it true also for non-linear systems ?

Can we discard the fast component of the solution ?

- How to deal with non-linear interactions of the fast waves :
non linear system contain quadratic terms e.g : $Q(U, U) = (\mathbf{v} \cdot \nabla)\mathbf{v}$

$$\mathbf{W} = \mathbf{W}_{\text{Slow}} + \mathbf{W}_{\text{Fast}}$$

thus

$$Q(\mathbf{W}, \mathbf{W}) = Q(\mathbf{W}_{\text{Slow}}, \mathbf{W}_{\text{Slow}}) + Q(\mathbf{W}_{\text{Slow}}, \mathbf{W}_{\text{Fast}}) + Q(\mathbf{W}_{\text{Fast}}, \mathbf{W}_{\text{Slow}}) + Q(\mathbf{W}_{\text{Fast}}, \mathbf{W}_{\text{Fast}})$$

Can we prove that non-linear interaction of fast waves : $Q(\mathbf{W}_{\text{Fast}}, \mathbf{W}_{\text{Fast}})$ is not important for the slow dynamics of the system ?

Is it true also for non-linear systems ?

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- How to deal with non-linear interactions of the fast waves :
non linear system contain quadratic terms e.g : $Q(U, U) = (\mathbf{v} \cdot \nabla)\mathbf{v}$

$$\mathbf{W} = \mathbf{W}_{\text{Slow}} + \mathbf{W}_{\text{Fast}}$$

thus

$$Q(\mathbf{W}, \mathbf{W}) = Q(\mathbf{W}_{\text{Slow}}, \mathbf{W}_{\text{Slow}}) + Q(\mathbf{W}_{\text{Slow}}, \mathbf{W}_{\text{Fast}}) + Q(\mathbf{W}_{\text{Fast}}, \mathbf{W}_{\text{Slow}}) + Q(\mathbf{W}_{\text{Fast}}, \mathbf{W}_{\text{Fast}})$$

Can we prove that non-linear interaction of fast waves : $Q(\mathbf{W}_{\text{Fast}}, \mathbf{W}_{\text{Fast}})$ is not important for the slow dynamics of the system ?

Counter-example : Turbulence and Reynolds stresses !



Some notations

The variables : $\mathcal{V}^\varepsilon = (p^\varepsilon, \mathbf{v})^t$

The equations : $\partial_t \mathcal{V}^\varepsilon + \mathbb{H}(\mathcal{V}^\varepsilon, \mathcal{V}^\varepsilon) + \frac{1}{\varepsilon} \mathbb{L} \mathcal{V}^\varepsilon = \mathcal{O}(\varepsilon)$

$\mathbb{H}(\mathcal{V}, \mathcal{V})$ is a non-linear operator (at most quadratic)

$$\mathbb{H}(\mathcal{V}, \mathcal{V}) = \begin{pmatrix} (\mathbf{v} \cdot \nabla) p \\ (\mathbf{v} \cdot \nabla) \mathbf{v} \end{pmatrix}$$

$\mathbb{L} \mathcal{V}$ is the constant coefficient linear operator

$$\mathbb{L} \mathcal{V} = \begin{pmatrix} \nabla \cdot \mathbf{v} \\ \nabla p \end{pmatrix}$$



The proof strategy

S. Schochet, E. Grenier, P.L.Lions-N.Masmoudi, B. Desjardins...

- 1 Introduce a filtered variable $\tilde{\psi}^\varepsilon = \mathcal{F}\psi^\varepsilon$ to remove the oscillations
- 2 Prove that the filtered variable $\tilde{\psi}^\varepsilon \rightarrow \tilde{\psi}^0$ satisfying some equation $\partial_t \tilde{\psi}^0 + \mathcal{H}(\tilde{\psi}^0, \psi^0) = 0$ with \mathcal{H} time-independent.
- 3 Prove that the original variable $\psi^\varepsilon \rightarrow \mathcal{F}^{-1}\tilde{\psi}^0$
- 4 Since $\mathcal{F}^{-1}\tilde{\psi}^0 \rightarrow P\tilde{\psi}^0$ where P is the L^2 projection on the kernel of \mathbb{L}

Result

$$\psi^\varepsilon \rightarrow \bar{\psi} = P\tilde{\psi}^0 \text{ and } \bar{\psi} \text{ satisfies :}$$

$$\partial_t \bar{\psi} + P\mathcal{H}(\bar{\psi}, \psi^0) = 0$$



The wave operator \mathbb{L}

$$\mathbb{L}\mathcal{V} = \begin{pmatrix} \nabla \cdot \mathbf{v} \\ \nabla p \end{pmatrix}$$

- $L^2(\Omega) \times (L^2(\Omega))^2 = \text{Ker}\mathbb{L} \oplus \text{Im}\mathbb{L}$
 $\text{Ker}\mathbb{L} = \{(p, \mathbf{v}); p = \text{cte}, \nabla \cdot \mathbf{v} = 0\}$
 $\text{Im}\mathbb{L} = \{(p, \mathbf{v}); \int p = 0, \exists \Phi \mathbf{v} = \nabla \Phi\}$

- Spectrum of \mathbb{L} on $\text{Im}\mathbb{L}$

Let $\{\psi_k, k \geq 1\}$ the eigenvectors of the Laplace operator

$$-\Delta\psi_k = \lambda_k^2\psi_k \quad \lambda_k > 0$$

then the eigenvectors of \mathbb{L} are :

$$\Phi_k^\pm = \begin{bmatrix} \psi_k \\ \pm \frac{\nabla\psi_k}{i\lambda_k} \end{bmatrix} \quad \text{with} \quad \mathbb{L}\Phi_k^\pm = \pm i\lambda_k\Phi_k^\pm$$

The solution operator \mathcal{L} of the wave equation

Let $\mathcal{L}(t)$ be the group ($\mathcal{L}(t), t \in \mathbf{R}$) defined by

$$\mathcal{L}(t) = \exp(-\mathbb{L}t) \quad (5)$$

In other words

$$\mathcal{V}(t, \mathbf{x}) = \mathcal{L}(t)\mathcal{V}_0(\mathbf{x}) \quad \text{means that} \quad \frac{\partial \mathcal{V}}{\partial t} + \mathbb{L}\mathcal{V} = 0 \quad \text{with} \quad \mathcal{V}(t = 0, \mathbf{x}) = \mathcal{V}_0(\mathbf{x})$$

Using the expression of the spectrum of \mathbb{L} we can have an explicit representation of the solution operator $\mathcal{L}(t)$: Let P be the L^2 projection on $\text{Ker}\mathbb{L}$

Expression of the solution at time t

$$\text{if } \mathcal{V}(0) = P\mathcal{V}(0) + \sum_{k, \pm} a_k^{\pm} \Phi_k^{\pm} \quad \text{then} \quad \mathcal{V}(t) = P\mathcal{V}(0) + \sum_{k, \pm} \pm a_k^{\pm} e^{\pm i\lambda_k t} \Phi_k^{\pm}$$

$a_k^- = (a_k^+)^*$ conjugate (real functions)

Step 1 : Filtered variable $\tilde{\mathcal{V}}^\varepsilon$

$$\partial_t \mathcal{V}^\varepsilon + \mathbb{H}(\mathcal{V}^\varepsilon, \mathcal{V}^\varepsilon) + \frac{1}{\varepsilon} \mathbb{L} \mathcal{V}^\varepsilon = \mathcal{O}(\varepsilon)$$

introduce the filtered variable $\tilde{\mathcal{V}}^\varepsilon = \mathcal{L}(-t/\varepsilon) \mathcal{V}^\varepsilon$

with

$$\mathcal{L}(t) = \exp(-\mathbb{L}t)$$

From the definition of \mathcal{L} , we deduce that

$$\begin{aligned} \frac{\partial \tilde{\mathcal{V}}^\varepsilon}{\partial t} &= \frac{\mathbb{L}}{\varepsilon} \tilde{\mathcal{V}}^\varepsilon + \mathcal{L}(-t/\varepsilon) \frac{\partial \mathcal{V}^\varepsilon}{\partial t} \\ &= \frac{\mathbb{L}}{\varepsilon} \tilde{\mathcal{V}}^\varepsilon - \mathcal{L}(-t/\varepsilon) \mathbb{H}(\mathcal{L}(t/\varepsilon) \tilde{\mathcal{V}}^\varepsilon, \mathcal{L}(t/\varepsilon) \tilde{\mathcal{V}}^\varepsilon) - \mathcal{L}(-t/\varepsilon) \frac{\mathbb{L}}{\varepsilon} \mathcal{L}(t/\varepsilon) \tilde{\mathcal{V}}^\varepsilon + \mathcal{O}(\varepsilon) \\ &= -\mathcal{L}(-t/\varepsilon) \mathbb{H}(\mathcal{L}(t/\varepsilon) \tilde{\mathcal{V}}^\varepsilon, \mathcal{L}(t/\varepsilon) \tilde{\mathcal{V}}^\varepsilon) + \mathcal{O}(\varepsilon) \end{aligned}$$

since $\mathcal{L}(t/\varepsilon)$ and \mathbb{L} commute.

Initial data : $\tilde{\mathcal{V}}^\varepsilon(t=0) = \mathcal{V}^\varepsilon(t=0)$ since $\mathcal{L}(0)$ is the identity

Limit Equation

Step 2 : Limit Equation for the filtered variable $\tilde{\mathcal{V}}^\varepsilon$

$$\frac{\partial \tilde{\mathcal{V}}^\varepsilon}{\partial t} + \mathcal{L}(-t/\varepsilon) \mathbb{H}(\mathcal{L}(t/\varepsilon) \tilde{\mathcal{V}}^\varepsilon, \mathcal{L}(t/\varepsilon) \tilde{\mathcal{V}}^\varepsilon) = \mathcal{O}(\varepsilon)$$

$$\tilde{\mathcal{V}}^0 = \lim_{\varepsilon \rightarrow 0} \tilde{\mathcal{V}}^\varepsilon$$

$$\frac{\partial \tilde{\mathcal{V}}^0}{\partial t} + \mathcal{H}(\tilde{\mathcal{V}}^0, \tilde{\mathcal{V}}^0) = 0$$

where $\mathcal{H}(\tilde{\mathcal{V}}^0, \tilde{\mathcal{V}}^0) = \lim_{\varepsilon \rightarrow 0} \mathcal{L}(-t/\varepsilon) \mathbb{H}(\mathcal{L}(t/\varepsilon) \tilde{\mathcal{V}}^0, \mathcal{L}(t/\varepsilon) \tilde{\mathcal{V}}^0)$ is a time-independent operator whose expression can be computed explicitly (see next slides)

Step 3 : Go back to the unfiltered variable \mathcal{V}^ε

$$\mathcal{V}^\varepsilon \rightarrow \mathcal{L}(t/\varepsilon) \tilde{\mathcal{V}}^0$$



Limit for the original variable

But we have

$$\mathcal{L}(t/\varepsilon)\tilde{\mathcal{V}}^0 \rightarrow P\tilde{\mathcal{V}}^0$$

since

$$\mathcal{L}(t/\varepsilon)\tilde{\mathcal{V}}^0 = \mathcal{L}(t/\varepsilon)(P\tilde{\mathcal{V}}^0 + \sum_{k,\pm} \pm a_k^\pm e^{\pm i\lambda_k t/\varepsilon} \Phi_k^\pm) \rightarrow P\tilde{\mathcal{V}}^0$$

Final result : weak limit of $\mathcal{V}^\varepsilon = P\tilde{\mathcal{V}}^0$ that satisfies

$$\frac{\partial P\tilde{\mathcal{V}}^0}{\partial t} + P\mathcal{H}(\tilde{\mathcal{V}}^0, \tilde{\mathcal{V}}^0) = 0$$



Explicit form of the limit equation for $P^{\mathcal{V}\tilde{0}}$

example : computation of the quadratic term $\mathcal{Q}(\mathbf{W}_{\text{Fast}}, \mathbf{W}_{\text{Fast}}) = (\mathbf{v} \cdot \nabla)\mathbf{v} = (\mathbf{v})_j \partial_j \mathbf{v}$

$$\begin{aligned} & (\mathcal{L}_v(t/\varepsilon)Q^{\mathcal{V}} \cdot \nabla)\mathcal{L}_v(t/\varepsilon)Q^{\mathcal{V}} = \\ & \left\{ \sum_k (a_k^+ e^{i\lambda_k t/\varepsilon} - a_k^- e^{-i\lambda_k t/\varepsilon}) \frac{\nabla \psi_k}{i\lambda_k} \right\}_j \partial_j \left\{ \sum_l (a_l^+ e^{i\lambda_l t/\varepsilon} - a_l^- e^{-i\lambda_l t/\varepsilon}) \frac{\nabla \psi_l}{i\lambda_l} \right\} = \\ & \sum_{k,l} [-a_k^+ a_l^+ e^{i(\lambda_k + \lambda_l)t/\varepsilon} - a_k^- a_l^- e^{-i(\lambda_k + \lambda_l)t/\varepsilon}] \frac{1}{\lambda_k \lambda_l} (\nabla \psi_k)_j \partial_j (\nabla \psi_l) \\ & + \sum_{k,l} [a_k^- a_l^+ e^{i(\lambda_l - \lambda_k)t/\varepsilon} + a_k^+ a_l^- e^{i(\lambda_k - \lambda_l)t/\varepsilon}] \frac{1}{\lambda_k \lambda_l} (\nabla \psi_k)_j \partial_j (\nabla \psi_l) \end{aligned}$$

$\lim_{\varepsilon \rightarrow 0}$ (distribution) of all the terms is 0 except when $k = l$ and we get :

$$(\mathcal{L}_v(t/\varepsilon)Q^{\mathcal{V}} \cdot \nabla)\mathcal{L}_v(t/\varepsilon)Q^{\mathcal{V}} \rightarrow \sum_k [a_k^- a_k^+ + a_k^+ a_k^-] \frac{1}{\lambda_k^2} (\nabla \psi_k)_j \partial_j (\nabla \psi_k) = \sum_k \frac{|a_k^+|^2}{\lambda_k^2} \nabla(|\nabla \psi_k|^2/2)$$

On the average (weak limit) fast k-waves interact with l-waves only if $k = l$ and the result is a gradient

the result of the interaction between fast waves and slow dynamics is a gradient !



Summary for single phase flows

When it goes well :

Weak limit of the solutions of **compressible** systems :

$$\begin{cases} \partial_t \mathbf{W} + \sum_j A_j(\mathbf{W}, \varepsilon) \partial_{x_j} \mathbf{W} + \frac{1}{\varepsilon} \mathbb{L} \mathbf{W} = 0 \\ \mathbf{W}(0, \mathbf{x}, \varepsilon) = \mathbf{W}_0(\mathbf{x}, \varepsilon) \end{cases}$$

are the solutions of the **incompressible** system

$$\begin{cases} \partial_t \mathbf{W} + \mathbb{P} \sum_j A_j(\mathbf{W}, 0) \partial_{x_j} \mathbf{W} = 0 \\ \mathbb{L} \mathbf{W} = 0 \\ \mathbf{W}(0, \mathbf{x}) = \mathbb{P} \mathbf{W}_0(\mathbf{x}) \end{cases}$$

where \mathbb{P} is the projection on $\ker(\mathbb{L})$.

In general for these systems :

decoupling between fast waves and slow dynamics

Some Comments on numerical approximation by upwind schemes

Continuous Acoustic operator :

$$\mathbb{L}\mathcal{V} = \begin{pmatrix} \nabla \cdot \mathbf{v} \\ \nabla p \end{pmatrix}$$

$\mathcal{V} \in \text{Ker}\mathbb{L} : p \equiv \text{Constant}$ and $\nabla \cdot \mathbf{v} = 0$

Discrete Acoustic operator :

$$\mathbb{L}_h \mathcal{V}_h = \frac{1}{|C_i|} \sum_l h \begin{pmatrix} \mathbf{u}_l \cdot \mathbf{n}_{il} + \Delta_{il} p \\ p_l \mathbf{n}_{il} + \Delta_{il} U \mathbf{n}_{il} \end{pmatrix}$$

Some Comments on numerical approximation by upwind schemes

Discrete Acoustic operator :

$$\mathbb{L}_h \mathcal{V}_h = \frac{1}{|C_i|} \sum_l h \begin{pmatrix} \mathbf{u}_l \cdot \mathbf{n}_{il} + \Delta_{il} p \\ \rho_l + \Delta_{il} U \mathbf{n}_{il} \end{pmatrix}$$

can be rewritten :

$$\mathbb{L}_h \mathcal{V}_h = \frac{1}{|C_i|} \sum_l h \begin{pmatrix} (\Delta_{il} p - \Delta_{il} U) \\ (\Delta_{il} p - \Delta_{il} U) \mathbf{n}_{il} \end{pmatrix}$$

$\mathbb{L}_h \mathcal{V}_h = 0$ can be considered as a system of $3 \times$ number of cells equations for the number of edges $/2$ variables $\Delta_{il} p - \Delta_{il} U$

There are Non zero solutions of this system

p is not a constant

Need to modify the artificial dissipation of upwind schemes.

Many different approaches : choose the one you like.

Some Comments and open questions on 2-phase flows

Well prepared initial data :

- Single Mach number : one velocity - one pressure model
 - 4 equations (homogeneous model)
 - 5 equations
- Formal asymptotic expansion : Existence of some Low Mach number model
- Two Mach numbers :
 - 6 equations : two velocities - one pressure model
 - 7 equations : two velocities - two pressures model
- Formal asymptotic expansion : much more complex : may depend on assumed relationship between the two Mach numbers



Some Comments and open questions on 2-phase flows

General initial data :

- Single phase flows : possible to separate acoustic from incompressible
- Two-phase flows : Is it possible (even for 4 equations model) ?
- Large operator is not a constant coefficient operator
- Speed of sound is not a constant
- Need to understand what is acoustic in a 2-phase system

