Fast waves and incompressible models

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Low Mach number flows

Compressible Euler equations :

$$\partial_t \rho + \text{ div } \rho \boldsymbol{u} + = 0 \qquad \rho = \rho_* \quad \rho$$

$$\partial_t \rho \boldsymbol{u} + \text{ div } \rho \boldsymbol{u} \otimes \boldsymbol{u} + \nabla p = 0 \qquad \boldsymbol{u} = u_* \quad \boldsymbol{u}$$

$$\partial_t p + \boldsymbol{u}.\nabla p + \rho a^2 \text{ div } \boldsymbol{u} + = 0 \qquad p = \rho_* (a_*)^2 \quad p$$

$$x_i = L_* \quad x_i; \quad t = L_*/u_* \quad t \qquad \varepsilon = u_*/a_*$$

$$\partial_t \rho + \text{ div } \rho \boldsymbol{u} = 0$$

$$\partial_t \rho \boldsymbol{u} + \text{ div } \rho \boldsymbol{u} \otimes \boldsymbol{u} + \frac{1}{\varepsilon^2} \nabla p = 0$$

$$\partial_t p + \boldsymbol{u}.\nabla p + \rho a^2 \text{ div } \boldsymbol{u} = 0$$

Low Mach number flows

The incompressible limit

$$f = f_0 + M_* f_1 + M_*^2 f_2$$

- $\mathcal{O}(1/M_*^2)$: $\nabla p_0 = 0$
  - if$\partial_t p_0 = 0 \rightarrow$ div $\boldsymbol{u}_0 = 0$
  - if$D\rho_0/Dt = 0 \rightarrow \rho_0 =$ constant
- $\mathcal{O}(1/M_*)$ same analysis
- $\mathcal{O}(1)$ $\rho_0 D\boldsymbol{u}_0/Dt + \nabla p_2 = 0$

Incompressible Euler equations

$$\boxed{\begin{array}{l} \rho D\boldsymbol{u}/Dt + \nabla p = 0 \\ \text{div } \boldsymbol{u} = 0 \end{array}}$$

Low Mach number flows

The acoustic limit

Incompressible limit is **not** the unique low Mach limit of compressible eqs

- hidden assumption in incompressible asymptotic analysis
- time scale $t_* = L_*/u_*$: large time scale
- choose instead $t_* = L_*/a_*$: short time scale

$$\text{scaling becomes} \begin{cases} \frac{1}{\varepsilon}\partial_t \rho + \text{ div } \rho \boldsymbol{u} + = 0 \\ \frac{1}{\varepsilon}\partial_t \rho \boldsymbol{u} + \text{ div } \rho \boldsymbol{u} \otimes \boldsymbol{u} + \frac{1}{M_*^2}\nabla p = 0 \\ \frac{1}{\varepsilon}\partial_t p + \boldsymbol{u}.\nabla p + \rho a^2 \text{ div } \boldsymbol{u} = 0 \end{cases}$$

First example : Low Mach number flows

Superposition incompressible + acoustics

Asymptotic analysis of the acoustic limit

$$f = f_0 + M_* f_1 + M_*^2 f_2$$

- $\mathcal{O}(1/M_*^2)$: $\nabla p_0 = 0$
- $\mathcal{O}(1/M_*)$
  - $\partial_t p_0 = \partial_t p_0 = 0$
  - $\rho_0 \partial_t \boldsymbol{u}_0 + \nabla p_1 = 0$
- $\mathcal{O}(1)$: $\partial_t p_1 + \rho_0 a_0^2 \nabla . \boldsymbol{u}_0 = 0$

Linear Acoustic equations

$$\boxed{\begin{array}{l} \rho_0 \partial_t \boldsymbol{u} + \nabla p = 0 \\ \\ \partial_t p + \rho_0 a_0^2 \ \mathrm{div}\ \boldsymbol{u} = 0 \end{array}}$$

Incompressible + Acoustic superposition

Provisional conclusion (Intuition):

- General solution = Slow (incompressible) + fast (Acoustic) component
- Can we prove it ?
- Does acoustic-acoustic interactions are able to modify the dynamics of the incompressible
- Can we understand it ? component ?

Incompressible + Acoustic superposition

Very complex answer depending on

- Initial data
  - well-prepared initial data (initial data “close” to incompressible flow)
  - general initial data
- State Law
  - barotropic flow ($p = p(\rho)$)
  - $p = p(\rho, s)$: the sound speed (at first order) is NOT a constant
- Dissipative or not (Euler or NS)
- Boundary conditions
  - Whole space
  - Periodic BC
  - Closed vessel (slip bc or no-slip)

Incompressible + Acoustic superposition

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Slow and fast limits of hyperbolic PDEs

Let $\boldsymbol{W} \in \boldsymbol{R}^N$ solution of the hyperbolic system with a large operator

$$\begin{cases} \partial_t \boldsymbol{W} + \sum_j [A_j(\boldsymbol{W}, \varepsilon) + \frac{1}{\varepsilon} C_j] \partial_{x_j} \boldsymbol{W} = 0 \\ \boldsymbol{W}(0, \boldsymbol{x}, \varepsilon) = \boldsymbol{W}_0(\boldsymbol{x}, \varepsilon) \end{cases}$$

What is the behavior of the solutions when $\varepsilon \to 0$?

Let $\boldsymbol{n}$ be a arbitrary direction, then some eigenvalues of $\sum_j n_j(A_j + \frac{1}{\varepsilon} C_j)$ are of the form $a_k + \frac{1}{\varepsilon} c_k \to \pm\infty$ while the others (kernel of $\sum_j n_j C_j$) are simply a_k

What is the behavior of the solutions when Slow and Fast waves co-exist ?

Singular limit of hyperbolic PDEs : Slow limit

$$\partial_t \boldsymbol{W} + \sum_j A_j(\boldsymbol{W}, \varepsilon)\partial_{x_j} \boldsymbol{W} + \frac{1}{\varepsilon} \sum_j C_j \partial_{x_j} \boldsymbol{W} = 0$$

$\mathbb{L}\boldsymbol{W} = \sum_j C_j \partial_{x_j} \boldsymbol{W}$ has to be $\mathcal{O}(\varepsilon)$

Look for the solution as $\boldsymbol{W} = \boldsymbol{W}_0 + \varepsilon \boldsymbol{W}_1$ with $\mathbb{L}\boldsymbol{W}_0 = 0$, obtain :

$$\partial_t \boldsymbol{W}_0 + \sum_j A_j(\boldsymbol{W}, \varepsilon)\partial_{x_j} \boldsymbol{W}_0 + \mathbb{L}\boldsymbol{W}_1 = \mathcal{O}(\varepsilon)$$

and the solutions converge to $\boldsymbol{W}_0$ defined by :

$$\begin{cases} \mathbb{L}\boldsymbol{W}_0 = 0 \\ \partial_t \boldsymbol{W}_0 + \mathbb{P}\sum_j A_j(\boldsymbol{W}_0, 0)\partial_{x_j} \boldsymbol{W} = 0 \end{cases}$$

$\mathbb{P}$ projection on the kernel of $\mathbb{L}$

But the system has also a fast limit

$$\partial_t \boldsymbol{W} + \sum_j A_j(\boldsymbol{W}, \varepsilon)\partial_{x_j} \boldsymbol{W} + \frac{1}{\varepsilon}\sum_j C_j \partial_{x_j} \boldsymbol{W} = 0$$

Let us do the simple change of variable : $t = \varepsilon\tau$:

$$\frac{1}{\varepsilon}\partial_\tau \boldsymbol{W} + \sum_j A_j(\boldsymbol{W}, \varepsilon)\partial_{x_j} \boldsymbol{W} + \frac{1}{\varepsilon}\sum_j C_j \partial_{x_j} \boldsymbol{W} = 0$$

and when $\varepsilon \to 0$ the limiting form becomes :

$$\partial_\tau \boldsymbol{W} + \sum_j C_j \partial_{x_j} \boldsymbol{W} = 0$$

Solution are fast waves moving at velocity $\frac{1}{\varepsilon}$

Singular limit of hyperbolic PDEs

Let $\boldsymbol{W} \in \boldsymbol{R}^N$ solution of the hyperbolic system with a large operator

$$\begin{cases} \partial_t \boldsymbol{W} + \sum_j [A_j(\boldsymbol{W}, \varepsilon) + \frac{1}{\varepsilon} C_j] \partial_{x_j} \boldsymbol{W} = 0 \\ \boldsymbol{W}(0, \boldsymbol{x}, \varepsilon) = \boldsymbol{W}_0(\boldsymbol{x}, \varepsilon) \end{cases}$$

What is the behavior of the solutions when $\varepsilon \to 0$?

How Slow and Fast waves co-exist ?
Why do we think that we can split the fast and slow phenomena ?

An Explicit linear example I

Consider the linear system

$$
\begin{aligned}
\frac{\partial r}{\partial t} \quad &+\boldsymbol{a}.\nabla r \quad +\frac{1}{\varepsilon} div\, \boldsymbol{u} = 0 \\
\frac{\partial \boldsymbol{u}}{\partial t} \quad &+\boldsymbol{a}.\nabla \boldsymbol{u} \quad +\frac{1}{\varepsilon} \nabla r = 0
\end{aligned}
$$

Warm-up : Explicit linear example II

Compact form :

$$\partial_t \boldsymbol{v} + \ \mathbb{H}\boldsymbol{v} + \frac{1}{\varepsilon} \ \mathbb{L}\boldsymbol{v} = 0$$

Notations :

$$\boldsymbol{v} = \begin{pmatrix} r \\ \boldsymbol{u} \end{pmatrix} \quad \mathbb{L}\boldsymbol{v} = \begin{pmatrix} \nabla \cdot \boldsymbol{u} \\ \nabla r \end{pmatrix}$$

$\mathbb{H}\boldsymbol{v} = \boldsymbol{a}.\nabla \boldsymbol{v}$ is a constant velocity linear advection operator

In Fourier space

$$\frac{\partial \hat{\boldsymbol{v}}(\boldsymbol{k})}{\partial t} + i[\hat{\mathbb{H}}(\boldsymbol{k}) + \frac{1}{\varepsilon}\hat{\mathbb{L}}\ (\boldsymbol{k})]\hat{\boldsymbol{v}}(\boldsymbol{k}) = 0 \quad \text{for} \quad \boldsymbol{k} \in Z^2 \tag{1}$$

where the matrix $\hat{\mathbb{H}}(\boldsymbol{k}) + 1/\varepsilon\hat{\mathbb{L}}(\boldsymbol{k})$ is equal to :

$$\begin{pmatrix} \boldsymbol{a}.\boldsymbol{k} & k_1/\varepsilon & k_2/\varepsilon \\ k_1/\varepsilon & \boldsymbol{a}.\boldsymbol{k} & 0 \\ k_2/\varepsilon & 0 & \boldsymbol{a}.\boldsymbol{k} \end{pmatrix} \tag{2}$$

This matrix is diagonalizable, its eigenvectors are :

$$s_1(\boldsymbol{k}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -k_1/\mid \boldsymbol{k} \mid \\ -k_2/\mid \boldsymbol{k} \mid \end{pmatrix}, \quad s_2(\boldsymbol{k}) = \frac{1}{\mid \boldsymbol{k} \mid} \begin{pmatrix} 0 \\ -k_2 \\ k_1 \end{pmatrix}$$

$$, \quad s_3(\boldsymbol{k}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ k_1/\mid \boldsymbol{k} \mid \\ k_2/\mid \boldsymbol{k} \mid \end{pmatrix} \tag{3}$$

with associated eigenvalues $\lambda_1 = \boldsymbol{a}.\boldsymbol{k} - \frac{\mid \boldsymbol{k} \mid}{\varepsilon}$, $\lambda_2 = \boldsymbol{a}.\boldsymbol{k}$ and $\lambda_3 = \boldsymbol{a}.\boldsymbol{k} + \frac{\mid \boldsymbol{k} \mid}{\varepsilon}$.

Note : $\hat{\mathbb{L}} s_2(\boldsymbol{k}) = 0$;in physical space $s_2(\boldsymbol{k})$ corresponds to constant density ($\nabla r = 0$) and div free vectors ($\nabla \, . \, \boldsymbol{u} = 0$)

Explicit linear example III

$$\hat{\boldsymbol{v}}(\boldsymbol{k},t) =$$

$$\begin{cases} \dfrac{1}{\sqrt{2}}(\hat{r}(\boldsymbol{k},0) - \dfrac{k_1}{|\,\boldsymbol{k}\,|}\hat{u}(\boldsymbol{k},0) - \dfrac{k_2}{|\,\boldsymbol{k}\,|}\hat{v}(\boldsymbol{k},0))e^{-i(\boldsymbol{a}.\boldsymbol{k}-|\boldsymbol{k}|/\varepsilon)t}s_1(\boldsymbol{k}) \\ +\dfrac{1}{|\,\boldsymbol{k}\,|}(-k_2\hat{u}(\boldsymbol{k},0) + k_1\hat{v}(\boldsymbol{k},0))e^{-i\boldsymbol{a}.\boldsymbol{k}t}s_2(\boldsymbol{k}) \\ +\dfrac{1}{\sqrt{2}}(\hat{r}(\boldsymbol{k},0) + \dfrac{k_1}{|\,\boldsymbol{k}\,|}\hat{u}(\boldsymbol{k},0) + \dfrac{k_2}{|\,\boldsymbol{k}\,|}\hat{v}(\boldsymbol{k},0))e^{-i(\boldsymbol{a}.\boldsymbol{k}+|\boldsymbol{k}|/\varepsilon)t}s_3(\boldsymbol{k}) \end{cases}$$

Explicit linear example IV

Fast oscillatory component $\hat{\boldsymbol{v}}_f(\boldsymbol{k}, t, t/\varepsilon)$

$$\frac{1}{\sqrt{2}}\left\{\begin{array}{l}(\hat{r}(\boldsymbol{k},0) - \frac{k_1}{|\boldsymbol{k}|}\hat{u}(\boldsymbol{k},0) - \frac{k_2}{|\boldsymbol{k}|}\hat{v}(\boldsymbol{k},0))e^{-i(\boldsymbol{a}.\boldsymbol{k}-\frac{|\boldsymbol{k}|}{\varepsilon})t}s_1(\boldsymbol{k}) \\ + \\ (\hat{r}(\boldsymbol{k},0) + \frac{k_1}{|\boldsymbol{k}|}\hat{u}(\boldsymbol{k},0) + \frac{k_2}{|\boldsymbol{k}|}\hat{v}(\boldsymbol{k},0))e^{-i(\boldsymbol{a}.\boldsymbol{k}+\frac{|\boldsymbol{k}|}{\varepsilon})t}s_3(\boldsymbol{k})\end{array}\right. \tag{4}$$

Explicit linear example V

Slow component belonging to the kernel of $\mathbb{L}$

$$\hat{\boldsymbol{v}}_s(\boldsymbol{k},\tau) = \frac{1}{\mid \boldsymbol{k} \mid}(-k_2\hat{u}(\boldsymbol{k},0) + k_1\hat{v}(\boldsymbol{k},0))e^{-i\boldsymbol{a}.\boldsymbol{k}t}s_2(\boldsymbol{k})$$

This component belongs to the kernel of $\mathbb{L}$ and satisfies the incompressible system

$$\begin{cases} \dfrac{\partial \boldsymbol{v}_s}{\partial t} + \mathbb{H}\boldsymbol{v}_s = 0 \\ \\ \mathbb{L}\boldsymbol{v}_s = 0 \end{cases}$$

Explicit linear example VI

What is the behavior of the solutions when $\varepsilon \to 0$?

For any ε the solution is composed of a superposition of fast and slow waves.

Does the solution converge toward something when $\varepsilon \to 0$?

- In a point-wise : NO : faster and faster oscillations
- In a weak sense (average or distribution) YES

  $e^{\pm i(\frac{|\boldsymbol{k}|}{\varepsilon})t} \to 0$

  thus the oscillatory part of the solution $\to 0$

  and the solutions converge (weakly) toward $\boldsymbol{v}_0$ that satisfies the incompressible system :

$$\begin{cases} \dfrac{\partial \boldsymbol{v}_0}{\partial t} + \mathbb{H}\boldsymbol{v}_0 = 0 \\ \mathbb{L}\boldsymbol{v}_0 = 0 \end{cases}$$

Is it true also for non-linear systems ?

Can we discard the fast component of the solution ?

- How to deal with non-linear interactions of the fast waves : non linear system contain quadratic terms e.g : $\mathcal{Q}(U, U) = (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}$

$$\boldsymbol{W} = \boldsymbol{W}_{\text{Slow}} + \boldsymbol{W}_{\text{Fast}}$$

thus

$$\begin{aligned}\mathcal{Q}(\boldsymbol{W}, \boldsymbol{W}) &= \mathcal{Q}(\boldsymbol{W}_{\text{Slow}}, \boldsymbol{W}_{\text{Slow}}) + \mathcal{Q}(\boldsymbol{W}_{\text{Slow}}, \boldsymbol{W}_{\text{Fast}}) + \mathcal{Q}(\boldsymbol{W}_{\text{Fast}}, \boldsymbol{W}_{\text{Slow}}) \\ &+ \mathcal{Q}(\boldsymbol{W}_{\text{Fast}}, \boldsymbol{W}_{\text{Fast}})\end{aligned}$$

Can we prove that non-linear interaction of fast waves : $\mathcal{Q}(\boldsymbol{W}_{\text{Fast}}, \boldsymbol{W}_{\text{Fast}})$ is not important for the slow dynamics of the system ?

Is it true also for non-linear systems ?

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- How to deal with non-linear interactions of the fast waves : non linear system contain quadratic terms e.g : $\mathcal{Q}(U,U) = (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$

$$\boldsymbol{W} = \boldsymbol{W}_{\text{Slow}} + \boldsymbol{W}_{\text{Fast}}$$

thus

$$\begin{aligned}\mathcal{Q}(\boldsymbol{W},\boldsymbol{W}) \; &= \mathcal{Q}(\boldsymbol{W}_{\text{Slow}},\boldsymbol{W}_{\text{Slow}}) + \mathcal{Q}(\boldsymbol{W}_{\text{Slow}},\boldsymbol{W}_{\text{Fast}}) + \mathcal{Q}(\boldsymbol{W}_{\text{Fast}},\boldsymbol{W}_{\text{Slow}}) \\ &+\mathcal{Q}(\boldsymbol{W}_{\text{Fast}},\boldsymbol{W}_{\text{Fast}})\end{aligned}$$

Can we prove that non-linear interaction of fast waves : $\mathcal{Q}(\boldsymbol{W}_{\text{Fast}},\boldsymbol{W}_{\text{Fast}})$ is not important for the slow dynamics of the system ?

Counter-example : Turbulence and Reynolds stresses !

Some notations

$$\text{The variables} \quad : \quad \mathscr{V}^\varepsilon = (p^\varepsilon, \boldsymbol{v})^t$$

$$\text{The equations} \quad : \quad \partial_t \mathscr{V}^\varepsilon + \mathbb{H}(\mathscr{V}^\varepsilon, \mathscr{V}^\varepsilon) + \frac{1}{\varepsilon} \mathbb{L}\mathscr{V}^\varepsilon = \mathcal{O}(\varepsilon)$$

$\mathbb{H}(\mathscr{V}, \mathscr{V})$ is a non-linear operator (at most quadratic)

$$\mathbb{H}(\mathscr{V}, \mathscr{V}) = \begin{pmatrix} (\boldsymbol{v} \cdot \nabla) p \\ (\boldsymbol{v} \cdot \nabla) \boldsymbol{v} \end{pmatrix}$$

$\mathbb{L}\mathscr{V}$ is the constant coefficient linear operator

$$\mathbb{L}\mathscr{V} = \begin{pmatrix} \nabla \cdot \boldsymbol{v} \\ \nabla p \end{pmatrix}$$

The proof strategy

S. Schochet, E. Grenier, P.L.Lions-N.Masmoudi, B. Desjardins...

1. Introduce a filtered variable $\tilde{\mathscr{V}}^{\varepsilon} = \mathcal{F}\mathscr{V}^{\varepsilon}$ to remove the oscillations
2. Prove that the filtered variable $\tilde{\mathscr{V}}^{\varepsilon} \to \tilde{\mathscr{V}}^{0}$ satisfying some equation $\partial_t \mathscr{V}^0 + \mathcal{H}(\mathscr{V}^0, \mathscr{V}^0) = 0$ with $\mathcal{H}$ time-independent.
3. Prove that the original variable $\mathscr{V}^{\varepsilon} \to \mathcal{F}^{-1}\tilde{\mathscr{V}}^{0}$
4. Since $\mathcal{F}^{-1}\tilde{\mathscr{V}}^{0} \to P\tilde{\mathscr{V}}^{0}$ where P is the L^2 projection on the kernel of $\mathbb{L}$

Result

$\mathscr{V}^{\varepsilon} \to \overline{\mathscr{V}} = P\tilde{\mathscr{V}}^{0}$ and $\overline{\mathscr{V}}$ satisfies :

$$\partial_t \overline{\mathscr{V}} + P\mathcal{H}(\mathscr{V}^0, \mathscr{V}^0) = 0$$

The wave operator $\mathbb{L}$

$$\mathbb{L}\mathscr{V} = \begin{pmatrix} \nabla \cdot \boldsymbol{v} \\ \nabla p \end{pmatrix}$$

- $L^2(\Omega) \times (L^2(\Omega))^2 = \mathrm{Ker}\mathbb{L} \oplus \mathrm{Im}\mathbb{L}$
  $\mathrm{Ker}\mathbb{L} = \{(p, \boldsymbol{v}); p = cte, \nabla \cdot \boldsymbol{v} = 0\}$
  $\mathrm{Im}\mathbb{L} = \{(p, \boldsymbol{v}); \int p = 0, \exists \Phi \boldsymbol{v} = \nabla \Phi\}$
- Spectrum of $\mathbb{L}$ on $\mathrm{Im}\mathbb{L}$
  Let $\{\psi_k, k \geq 1\}$ the eigenvectors of the Laplace operator

$$-\Delta \psi_k = \lambda_k^2 \psi_k \qquad \lambda_k > 0$$

then the eigenvectors of $\mathbb{L}$ are :

$$\Phi_k^{\pm} = \begin{bmatrix} \psi_k \\ \pm \dfrac{\nabla \psi_k}{i\lambda_k} \end{bmatrix} \quad \text{with} \quad \mathbb{L}\Phi_k^{\pm} = \pm i\lambda_k \Phi_k^{\pm}$$

The solution operator $\mathcal{L}$ of the wave equation

Let $\mathcal{L}(t)$ be the group $(\mathcal{L}(t), t \in \boldsymbol{R})$ defined by

$$\mathcal{L}(t) = \exp(-\mathbb{L}t) \tag{5}$$

In other words

$$\mathscr{V}(t,x) = \mathcal{L}(t)\mathscr{V}_0(x) \quad \text{means that} \quad \frac{\partial \mathscr{V}}{\partial t} + \mathbb{L}\mathscr{V} = 0 \quad \text{with} \quad \mathscr{V}(t=0,x) = \mathscr{V}_0(x)$$

Using the expression of the spectrum of $\mathbb{L}$ we can have an explicit representation of the solution operator $\mathcal{L}(t)$: Let P be the L^2 projection on $\mathrm{Ker}\mathbb{L}$

Expression of the solution at time t

$$\text{if} \quad \mathscr{V}(0) = P\mathscr{V}(0) + \sum_{k,\pm} a_k^{\pm}\Phi_k^{\pm} \quad \text{then} \quad \mathscr{V}(t) = P\mathscr{V}(0) + \sum_{k,\pm} \pm a_k^{\pm} e^{\pm i\lambda_k t}\Phi_k^{\pm}$$

$a_k^- = (a_k^+)^*$ conjugate (real functions)

Step 1 : Filtered variable $\tilde{\mathscr{V}}^\varepsilon$

$$\partial_t \mathscr{V}^\varepsilon + \mathbb{H}(\mathscr{V}^\varepsilon, \mathscr{V}^\varepsilon) + \frac{1}{\varepsilon} \mathbb{L}\mathscr{V}^\varepsilon = \mathcal{O}(\varepsilon)$$

$$\text{introduce the filtered variable } \tilde{\mathscr{V}}^\varepsilon = \mathcal{L}(-t/\varepsilon)\mathscr{V}^\varepsilon$$

with

$$\mathcal{L}(t) = \exp(-\mathbb{L}t)$$

From the definition of $\mathcal{L}$, we deduce that

$$\begin{aligned}
\frac{\partial \tilde{\mathscr{V}}^\varepsilon}{\partial t} &= \frac{\mathbb{L}}{\varepsilon}\tilde{\mathscr{V}}^\varepsilon + \mathcal{L}(-t/\varepsilon)\frac{\partial \mathscr{V}^\varepsilon}{\partial t} \\
&= \frac{\mathbb{L}}{\varepsilon}\tilde{\mathscr{V}}^\varepsilon - \mathcal{L}(-t/\varepsilon)\mathbb{H}(\mathcal{L}(t/\varepsilon)\tilde{\mathscr{V}}^\varepsilon, \mathcal{L}(t/\varepsilon)\tilde{\mathscr{V}}^\varepsilon) - \mathcal{L}(-t/\varepsilon)\frac{\mathbb{L}}{\varepsilon}\mathcal{L}(t/\varepsilon)\tilde{\mathscr{V}}^\varepsilon + \mathcal{O}(\varepsilon) \\
&= -\mathcal{L}(-t/\varepsilon)\mathbb{H}(\mathcal{L}(t/\varepsilon)\tilde{\mathscr{V}}^\varepsilon, \mathcal{L}(t/\varepsilon)\tilde{\mathscr{V}}^\varepsilon) + \mathcal{O}(\varepsilon)
\end{aligned}$$

since $\mathcal{L}(t/\varepsilon)$ and $\mathbb{L}$ commute.

Initial data : $\tilde{\mathscr{V}}^\varepsilon(t=0) = \mathscr{V}^\varepsilon(t=0)$ since $\mathcal{L}(0)$ is the identity

Limit Equation

Step 2 : Limit Equation for the filtered variable $\tilde{\mathscr{V}}^{\varepsilon}$

$$\frac{\partial \tilde{\mathscr{V}}^{\varepsilon}}{\partial t} + \mathcal{L}(-t/\varepsilon)\mathbb{H}(\mathcal{L}(t/\varepsilon)\tilde{\mathscr{V}}^{\varepsilon}, \mathcal{L}(t/\varepsilon)\tilde{\mathscr{V}}^{\varepsilon}) = \mathcal{O}(\varepsilon)$$

$\tilde{\mathscr{V}}^{0} = \lim_{\varepsilon \to 0} \tilde{\mathscr{V}}^{\varepsilon}$

$$\frac{\partial \tilde{\mathscr{V}}^{0}}{\partial t} + \mathcal{H}(\tilde{\mathscr{V}}^{0}, \tilde{\mathscr{V}}^{0}) = 0$$

where $\mathcal{H}(\tilde{\mathscr{V}}^{0}, \tilde{\mathscr{V}}^{0}) = \lim_{\varepsilon \to 0} \mathcal{L}(-t/\varepsilon)\mathbb{H}(\mathcal{L}(t/\varepsilon)\tilde{\mathscr{V}}^{0}, \mathcal{L}(t/\varepsilon)\tilde{\mathscr{V}}^{0})$ is a time-independent operator whose expression can be computed explicitly (see next slides)

Step 3 : Go back to the unfiltered variable $\mathscr{V}^{\varepsilon}$

$$\mathscr{V}^{\varepsilon} \to \mathcal{L}(t/\varepsilon)\tilde{\mathscr{V}}^{0}$$

Limit for the original variable

But we have

$$\mathcal{L}(t/\varepsilon)\tilde{\mathscr{V}}^0 \rightarrow P\tilde{\mathscr{V}}^0$$

since

$$\mathcal{L}(t/\varepsilon)\tilde{\mathscr{V}}^0 = \mathcal{L}(t/\varepsilon)(P\tilde{\mathscr{V}}^0 + \sum_{k,\pm} \pm a_k^\pm e^{\pm i\lambda_k t/\varepsilon}\Phi_k^\pm) \rightharpoonup P\tilde{\mathscr{V}}^0$$

Final result : weak limit of $\mathscr{V}^\varepsilon = P\tilde{\mathscr{V}}^0$ that satisfies

$$\frac{\partial P\tilde{\mathscr{V}}^0}{\partial t} + P\mathcal{H}(\tilde{\mathscr{V}}^0, \tilde{\mathscr{V}}^0) = 0$$

Explicit form of the limit equation for $P\tilde{\mathscr{V}}^0$

example : computation of the quadratic term $\mathcal{Q}(\boldsymbol{W}_{\mathrm{Fast}}, \boldsymbol{W}_{\mathrm{Fast}}) = (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = (\boldsymbol{v})_j\partial_j\boldsymbol{v}$

$$
\begin{aligned}
&(\mathcal{L}_v(t/\varepsilon)Q\mathscr{V}\cdot\nabla)\mathcal{L}_v(t/\varepsilon)Q\mathscr{V} = \\
&\{\sum_k (a_k^+ e^{i\lambda_k t/\varepsilon} - a_k^- e^{-i\lambda_k t/\varepsilon})\frac{\nabla\psi_k}{i\lambda_k}\}_j\partial_j\{\sum_l (a_l^+ e^{i\lambda_l t/\varepsilon} - a_l^- e^{-i\lambda_l t/\varepsilon})\frac{\nabla\psi_l}{i\lambda_l}\} = \\
&\sum_{k,l}[-a_k^+ a_l^+ e^{i(\lambda_k+\lambda_l)t/\varepsilon} - a_k^- a_l^- e^{-i(\lambda_k+\lambda_l)t/\varepsilon}]\frac{1}{\lambda_k\lambda_l}(\nabla\psi_k)_j\partial_j(\nabla\psi_l) \\
&+\sum_{k,l}[a_k^- a_l^+ e^{i(\lambda_l-\lambda_k)t/\varepsilon} + a_k^+ a_l^- e^{i(\lambda_k-\lambda_l)t/\varepsilon}]\frac{1}{\lambda_k\lambda_l}(\nabla\psi_k)_j\partial_j(\nabla\psi_l)
\end{aligned}
$$

$\lim_{\varepsilon\to 0}$ (distribution) of all the terms is 0 except when $k = l$ and we get :

$$
(\mathcal{L}_v(t/\varepsilon)Q\mathscr{V}\cdot\nabla)\mathcal{L}_v(t/\varepsilon)Q\mathscr{V} \to \sum_k [a_k^- a_k^+ + a_k^+ a_k^-]\frac{1}{\lambda_k^2}(\nabla\psi_k)_j\partial_j(\nabla\psi_k) = \sum_k \frac{|a_k^+|^2}{\lambda_k^2}\nabla(|\nabla\psi_k|^2/2)
$$

On the average (weak limit) fast k-waves interact with l-waves only if $k = l$ and the result is a gradient

the result of the interaction between fast waves and slow dynamics is a gradient !

Summary for single phase flows

When it goes well :
Weak limit of the solutions of **compressible** systems :

$$\begin{cases} \partial_t \boldsymbol{W} + \sum_j A_j(\boldsymbol{W},\varepsilon)\partial_{x_j}\boldsymbol{W} + \frac{1}{\varepsilon}\mathbb{L}\boldsymbol{W} = 0 \\ \boldsymbol{W}(0,\boldsymbol{x},\varepsilon) = \boldsymbol{W}_0(\boldsymbol{x},\varepsilon) \end{cases}$$

are the solutions of the **incompressible** system

$$\begin{cases} \partial_t \boldsymbol{W} + \mathbb{P}\sum_j A_j(\boldsymbol{W},0)\partial_{x_j}\boldsymbol{W} = 0 \\ \mathbb{L}\boldsymbol{W} = 0 \\ \boldsymbol{W}(0,\boldsymbol{x}) = \mathbb{P}\boldsymbol{W}_0(\boldsymbol{x}) \end{cases}$$

where $\mathbb{P}$ is the projection on $\ker(\mathbb{L})$.

In general for these systems :

decoupling between fast waves and slow dynamics

Some Comments on numerical approximation by upwind schemes

Continous Acoustic operator :

$$\mathbb{L}\mathscr{V} = \begin{pmatrix} \nabla \cdot \boldsymbol{v} \\ \nabla p \end{pmatrix}$$

$\mathscr{V} \in Ker\mathbb{L} : p \equiv Constant \quad and \quad \nabla \cdot \boldsymbol{v} = 0$

Discrete Acoustic operator :

$$\mathbb{L}_h\mathscr{V}_h = \frac{1}{|C_i|}\sum_l h \begin{pmatrix} \boldsymbol{u}_l.\boldsymbol{n}_{il} + \Delta_{il} p \\ p_l \boldsymbol{n}_{il} + \Delta_{il} U \boldsymbol{n}_{il} \end{pmatrix}$$

Some Comments on numerical approximation by upwind schemes

Discrete Acoustic operator :

$$\mathbb{L}_h \mathscr{V}_h = \frac{1}{|C_i|} \sum_l h \begin{pmatrix} \boldsymbol{u}_l . \boldsymbol{n}_{il} + \Delta_{il} p \\ p_l + \Delta_{il} U \boldsymbol{n}_{il} \end{pmatrix}$$

can be rewritten :

$$\mathbb{L}_h \mathscr{V}_h = \frac{1}{|C_i|} \sum_l h \begin{pmatrix} (\Delta_{il} p - \Delta_{il} U) \\ (\Delta_{il} p - \Delta_{il} U) \boldsymbol{n}_{il} \end{pmatrix}$$

$\mathbb{L}_h \mathscr{V}_h = 0$ can be considered as a system of 3 $\times$ number of cells equations for the number of edges /2 variables $\Delta_{il} p - \Delta_{il} U$

There are Non zero solutions of this system

p is not a constant

Need to modify the artificial dissipation of upwind schemes.

Many different approaches : choose the one you like.

Some Comments and open questions on 2-phase flows

Well prepared initial data :

- Single Mach number : one velocity - one pressure model
  - 4 equations (homogeneous model)
  - 5 equations
- Formal asymptotic expansion : Existence of some Low Mach number model
- Two Mach numbers :
  - 6 equations : two velocities - one pressure model
  - 7 equations : two velocities - two pressures model
- Formal asymptotic expansion : much more complex : may depend on assumed relationship between the two Mach numbers

Some Comments and open questions on 2-phase flows

General initial data :

- Single phase flows : possible to separate acoustic from incompressible
- Two-phase flows : Is it possible (even for 4 equations model) ?
- Large operator is not a constant coefficient operator
- Speed of sound is not a constant
- Need to understand what is acoustic in a 2-phase system