

On CCZ-Equivalence, Extended-Affine Equivalence and Function Twisting

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On CCZ-Equivalence, Extended-Affine Equivalence and Function Twisting

Anne Canteaut, Léo Perrin

June 18, 2018

BFA'2018



Definition (CCZ-Equivalence)

$F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$ and $G : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$ are *C(arlet)-C(harpin)-Z(inoviev)* equivalent if

$$\Gamma_G = \{(x, G(x)), \forall x \in \mathbb{F}_2^n\} = L(\{(x, F(x)), \forall x \in \mathbb{F}_2^n\}) = L(\Gamma_F),$$

where $L : \mathbb{F}_2^{n+m} \rightarrow \mathbb{F}_2^{n+m}$ is an affine permutation.

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Definition (EA-Equivalence; EA-mapping)

F and G are *E(xtended) A(ffine)* equivalent if $G(x) = (B \circ F \circ A)(x) + C(x)$, where A, B, C are affine and A, B are permutations; so that

$$\{(x, G(x)), \forall x \in \mathbb{F}_2^n\} = \begin{bmatrix} A^{-1} & 0 \\ CA^{-1} & B \end{bmatrix} (\{(x, F(x)), \forall x \in \mathbb{F}_2^n\}) .$$

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Affine permutations with such linear part are **EA-mappings**; their transposes are **TEA-mappings**

What is the relation between functions that are CCZ- but **not** EA-equivalent?

Admissible Mapping

For $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$, the affine permutation L is **admissible for F** if

$$L(\{(x, F(x)), \forall x \in \mathbb{F}_2^n\}) = \{(x, G(x)), \forall x \in \mathbb{F}_2^n\}$$

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Definition (LAT/Walsh Spectrum)

The L(inear) A(pproximation) T(able) of $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$ is

$$\mathcal{W}_F(\alpha, \beta) = \sum_{x \in \mathbb{F}_2^n} (-1)^{\alpha \cdot x + \beta \cdot F(x)} .$$

Structure of this talk

0 - CCZ-Equivalence ; Bijectivity

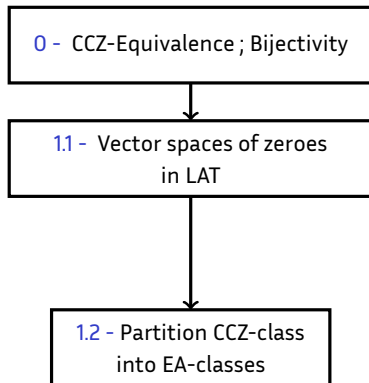
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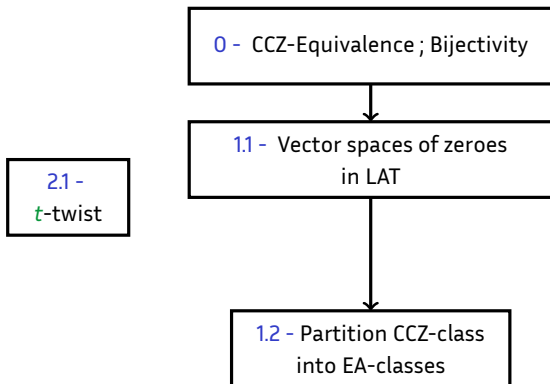


1.1 - Vector spaces of zeroes
in LAT

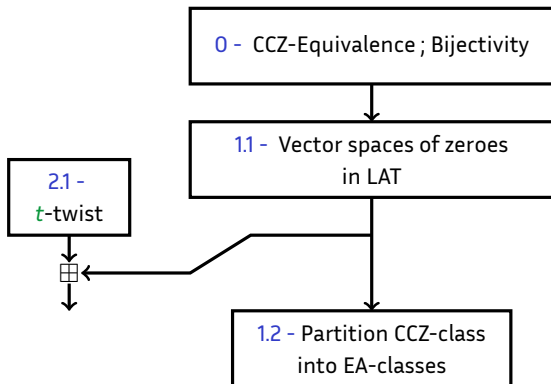
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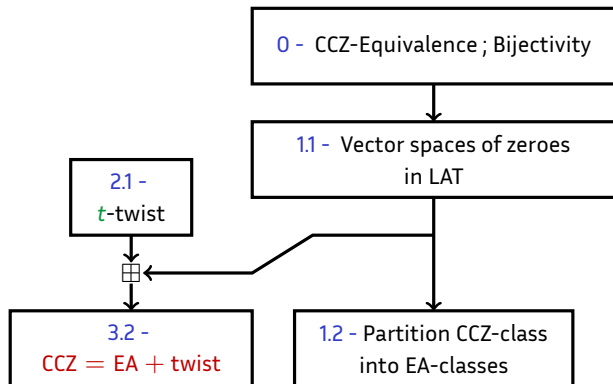
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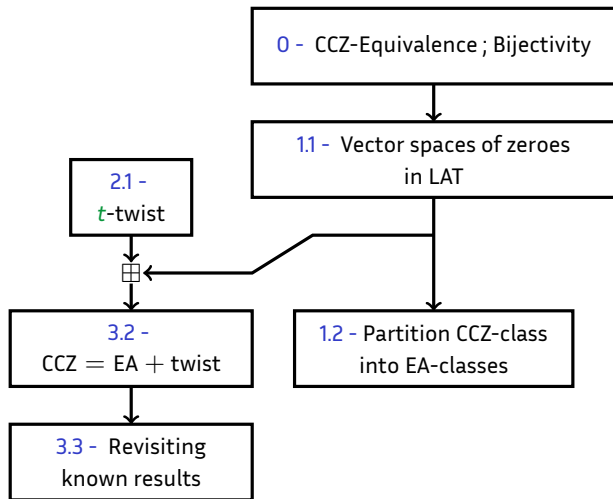
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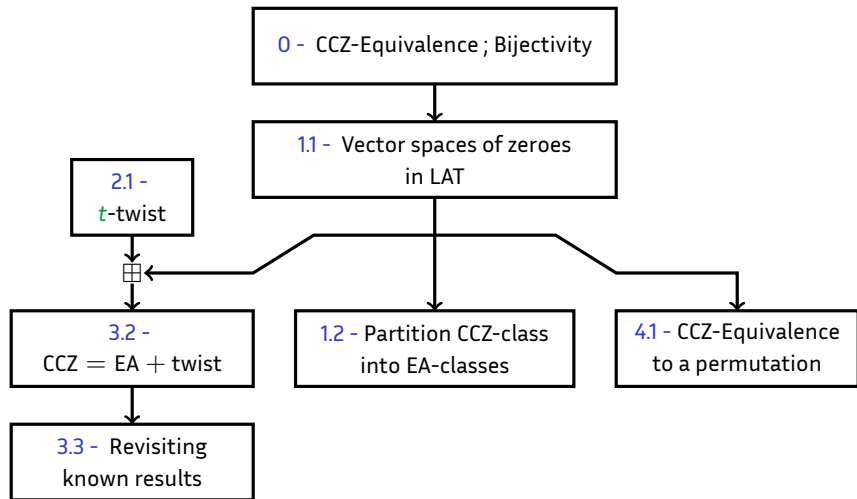
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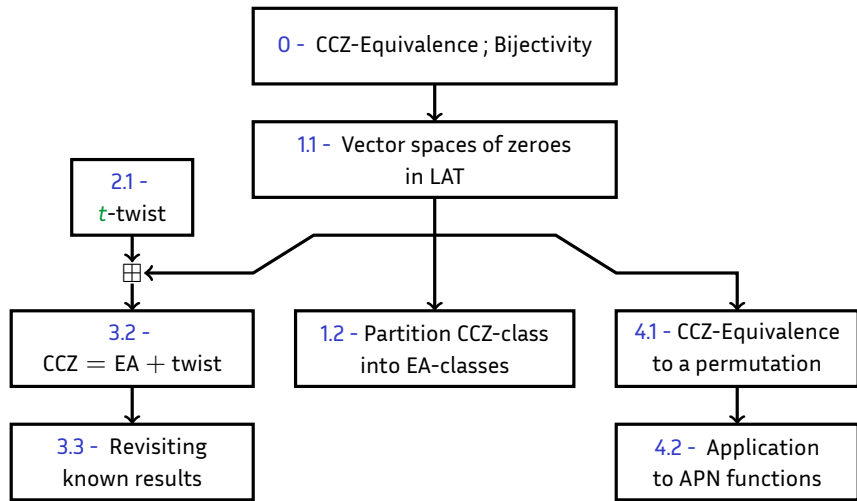
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Structure of this talk



Outline

- 1 **CCZ-Equivalence and Vector Spaces of 0**
- 2 Function Twisting
- 3 Necessary and Efficient Conditions for CCZ-Equivalence to a Permutation
- 4 Conclusion

Plan of this Section

- 1 **CCZ-Equivalence and Vector Spaces of 0**
 - Vector Spaces of Zeroes
 - Partitioning a CCZ-Class into EA-Classes
- 2 Function Twisting
- 3 Necessary and Efficient Conditions for CCZ-Equivalence to a Permutation
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Walsh Zeroes

For all $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$, we have

$$\mathcal{W}_F(\alpha, 0) = \sum_{x \in \mathbb{F}_2^n} (-1)^{\alpha \cdot x + 0 \cdot F(x)} = 0.$$

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Definition (Walsh Zeroes)

The *Walsh zeroes* of $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$ is the set

$$\mathcal{Z}_F = \{u \in \mathbb{F}_2^n \times \mathbb{F}_2^m, \mathcal{W}_F(u) = 0\} \cup \{0\}.$$

With $\mathcal{V} = \{(x, 0), \forall x \in \mathbb{F}_2^n\} \subset \mathbb{F}_2^{n+m}$, we have $\mathcal{V} \subset \mathcal{Z}_F$.

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Note that if $\Gamma_G = L(\Gamma_F)$, then $\mathcal{Z}_G = (L^T)^{-1}(\mathcal{Z}_F)$.

Admissibility for F

Lemma

Let $L : \mathbb{F}_2^{n+m} \rightarrow \mathbb{F}_2^{n+m}$ be a linear permutation. It is admissible for $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$ if and only if

$$L^T(\mathcal{V}) \subseteq \mathcal{Z}_F$$

Admissibility of EA-mappings

EA-mappings are admissible for all $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$:

$$\begin{bmatrix} A & 0 \\ C & B \end{bmatrix}^T (\mathcal{V}) = \begin{bmatrix} A^T & C^T \\ 0 & B^T \end{bmatrix} \left(\left\{ \begin{bmatrix} x \\ 0 \end{bmatrix}, \forall x \in \mathbb{F}_2^n \right\} \right) = \mathcal{V}.$$

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Theorem (Budaghyan, Carlet (2011))

The CCZ-class of a bent function contains only its EA-class.

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Proof.

A function is bent

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Proof.

A function is bent

- ⇒ no zeroes outside of \mathcal{V}
- ⇒ no vector spaces of zeroes other than \mathcal{V}
- ⇒ only 1 EA-class



Permutations

We define

$$\mathcal{V}^\perp = \{(0, y), \forall y \in \mathbb{F}_2^m\} \subset \mathbb{F}_2^{n+m}.$$

Lemma

$F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$ is a permutation if and only if

$$\mathcal{V}^\perp \subset \mathcal{Z}_F.$$

EA-classes imply vector spaces

Lemma

let F , G and G' be such that $\Gamma_G = L(\Gamma_F)$ and $\Gamma_{G'} = L'(\Gamma_F)$.

If $L(\mathcal{V}) = L'(\mathcal{V})$, then G and G' are EA-equivalent.

EA-classes imply vector spaces

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The Lemma gives us hope!

1 EA-class \implies 1 vector space of zeroes of dimension n in \mathcal{Z}_n

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Reality takes it back...

The converse of the lemma is wrong.

Counter-example

Let $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n$ be a permutation and let

$$M_n = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}.$$

It holds that

$$\begin{aligned} \Gamma_{F^{-1}} &= \{ (x, F(x)), \forall x \in \mathbb{F}_2^n \} \\ &= \{ (F^{-1}(y), (F \circ F^{-1})(y)), \forall y \in \mathbb{F}_2^n \} \\ &= \{ (F^{-1}(y), y), \forall y \in \mathbb{F}_2^n \} \\ &= M_n(\Gamma_F). \end{aligned}$$

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The contradiction

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$$\Rightarrow M_n(\mathcal{V}) = \mathcal{V}^\perp \neq I_n(\mathcal{V})$$

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$$\implies M_n(\mathcal{V}) = \mathcal{V}^\perp \neq I_n(\mathcal{V})$$

... but M_n and I_n send Γ_F in the same EA-class

(namely that of F).

Making the converse work (1/2)

Definition (CCZ-invariants)

The **CCZ-invariants** of $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n$ are the affine permutations L of \mathbb{F}_2^{n+n} such that

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Examples

- For an involution, M_n is a CCZ-invariant.
- For a quadratic function q , there are CCZ-invariants with the following linear parts:

$$\begin{bmatrix} I_n & 0 \\ \Delta_{\alpha} q & I_n \end{bmatrix}.$$

Making the converse work (2/2)

Theorem (Number of EA-classes)

For $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$, let:

- s_F be the number of vector spaces of dimension n in \mathcal{Z}_F
- c_F be the number of CCZ-invariants of F
- e_F be the number of EA-classes in the CCZ-class of F .

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Corollary

If $c_F = 1$, then we do have a bijection between EA-classes and vector spaces of 0 of dimension n in \mathcal{Z}_F .

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- 2 Function Twisting**
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- 4 Conclusion

Plan of this Section

- 1 CCZ-Equivalence and Vector Spaces of 0
- 2 **Function Twisting**
 - The Twist
 - CCZ = EA + Twist
 - Revisiting some Results
- 3 Necessary and Efficient Conditions for CCZ-Equivalence to a Permutation
- 4 Conclusion

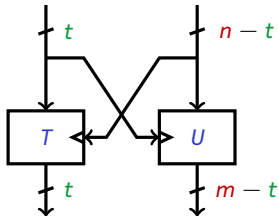
EA-equivalence is a simple sub-case of CCZ-Equivalence...

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What must we add to EA-equivalence to fully describe CCZ-Equivalence?

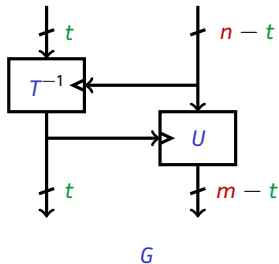
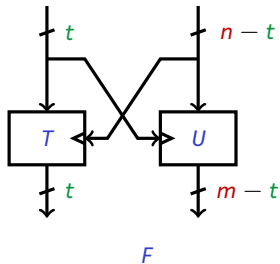
Definition of the Twist

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If T is a permutation for all secondary inputs, then we define the t -twist equivalent of F as G , where

$$G(x, y) = (T_y^{-1}(x), U_{T_y^{-1}(x)}(y))$$

for all $(x, y) \in \mathbb{F}_2^t \times \mathbb{F}_2^{n-t}$.

Examples of Twisting

- Inversion is an n -twist.

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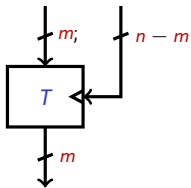
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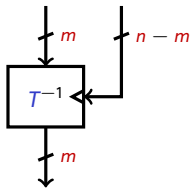
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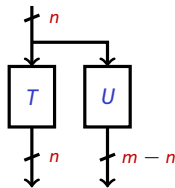
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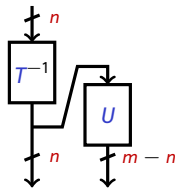
$t = m$ (start)



$t = m$ (end)



$t = n$ (start)



$t = n$ (end)

Swap Matrices

The **swap matrix** permuting \mathbb{F}_2^{n+m} is defined for $t \leq \min(n, m)$ as

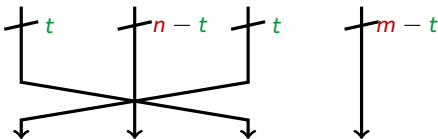
$$M_t = \begin{bmatrix} 0 & 0 & I_t & 0 \\ 0 & I_{n-t} & 0 & 0 \\ I_t & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{m-t} \end{bmatrix}.$$

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It has a simple interpretation:

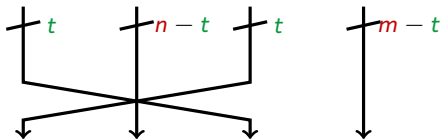


Swap Matrices

The **swap matrix** permuting \mathbb{F}_2^{n+m} is defined for $t \leq \min(n, m)$ as

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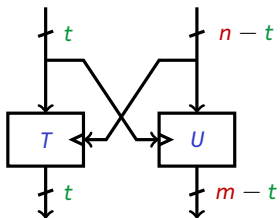
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For all $t \leq \min(n, m)$, M_t is an **orthogonal** and **symmetric involution**.

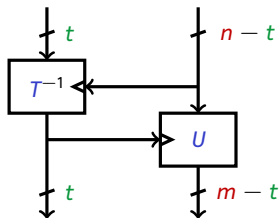
Swap Matrices and Twisting

$$F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$$



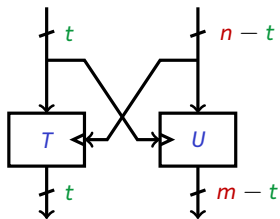
t -twist

$$G : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$$



Swap Matrices and Twisting

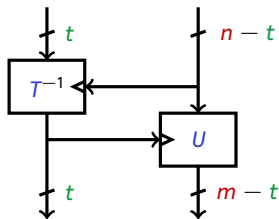
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$$\Gamma_F = \{ (x, F(x)), \forall x \in \mathbb{F}_2^n \}$$

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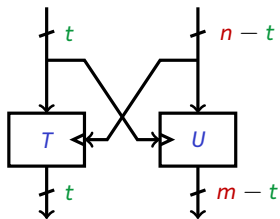


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M_t

Swap Matrices and Twisting

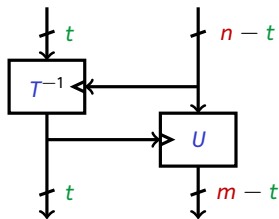
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\longleftrightarrow M_t \longleftrightarrow

$$\mathcal{W}_F(u) = \mathcal{W}_G(M_t(u))$$

Twisting and CCZ-Class

Lemma

Twisting preserves the CCZ-equivalence class.

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Main Result

Theorem

If $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$ and $G : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$ are CCZ-equivalent, then

$$\Gamma_G = (B \times M_t \times A)(\Gamma_F),$$

where A and B are EA-mappings and where

$$t = \dim(\text{proj}_{\mathcal{V}^\perp}((A^T \times M_t \times B^T)(\mathcal{V}))) .$$

In other words, EA-equivalence and twists are sufficient to fully describe CCZ-equivalence!

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Corollary

If a function is CCZ-equivalent but not EA-equivalent to another function, then they have to be EA-equivalent to functions for which a t -twist is possible.

Proof sketch

1. As F is CCZ-equivalent to G , there is a linear permutation $L : \mathbb{F}_2^{n+m} \rightarrow \mathbb{F}_2^{n+m}$ such that

$$\Gamma_G = L(\Gamma_F) \text{ and } L^T(\mathcal{V}) \subset \mathcal{Z}_F.$$

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1+2+lem. As $L^T(\mathcal{V}) = (A^T \times M_t)(\mathcal{V})$, the functions G and G' such that $\Gamma_G = L(\Gamma_F)$ and $\Gamma_{G'} = (A^T \times M_t)(\Gamma_F)$ are EA-equivalent.

We conclude that

$$\Gamma_G = (B \times M_t \times A)(\Gamma_F).$$

Usage?

What can we do with this knowledge?

Boolean Functions

Theorem (Budaghyan, Carlet (2011))

The CCZ-class of $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2$ is limited to its EA-class.

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$\implies F(x||y) = x \oplus f(y), \forall (x, y) \in \mathbb{F}_2 \times \mathbb{F}_2^{n-1}$,

\implies 1-twisting F does not change the EA-class

\implies it is impossible to leave the EA-class of F



Modular Addition (1/2)

Theorem (Schulte-Geers'13)

Addition modulo 2^m is CCZ-equivalent to

$$q(x, y) = (0, x_0y_0, x_0y_0 + x_1y_1, \dots, x_0y_0 + \dots + x_{n-2}y_{n-2}),$$

where $\Gamma_{\boxplus} = L(\Gamma_q)$ with

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It holds that

$$L^{-1} = \underbrace{\begin{bmatrix} I_m & 0 & 0 \\ I_m & I_m & 0 \\ I_m & 0 & I_m \end{bmatrix}}_{A_1} \times \underbrace{\begin{bmatrix} 0 & 0 & I_m \\ 0 & I_m & 0 \\ I_m & 0 & 0 \end{bmatrix}}_{M_m} \times \underbrace{\begin{bmatrix} I_m & 0 & 0 \\ I_m & I_m & 0 \\ 0 & I_m & I_m \end{bmatrix}}_{A_2}.$$

Modular Addition (2/2)

Lemma

Let $T_z^{\boxplus} : \mathbb{F}_2^m \rightarrow \mathbb{F}_2^m$ be defined by

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Let $v = T_z^{\boxplus}(x)$. Then:

$$\begin{cases} v_0 & = x_0 \\ v_{i+1} & = x_i + x_{i+1} + v_i z_i \end{cases} \quad \text{and, convertly,} \quad \begin{cases} x_0 & = v_0 \\ x_{i+1} & = x_i + v_{i+1} + v_i z_i. \end{cases}$$

Outline

- 1 CCZ-Equivalence and Vector Spaces of 0
- 2 Function Twisting
- 3 Necessary and Efficient Conditions for CCZ-Equivalence to a Permutation**
- 4 Conclusion

Plan of this Section

- 1 CCZ-Equivalence and Vector Spaces of 0
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- 3 **Necessary and Efficient Conditions for CCZ-Equivalence to a Permutation**
 - Efficient Criteria
 - Applications to APN Functions
- 4 Conclusion

Another Problem

How do we know if a function is CCZ-equivalent to a permutation?

Remainder

Recall that F is a permutation if and only if $\mathcal{V} \subset \mathcal{Z}_F$ and $\mathcal{V}^\perp \subset \mathcal{Z}_F$.

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Recall that F is a permutation if and only if $\mathcal{V} \subset \mathcal{Z}_F$ and $\mathcal{V}^\perp \subset \mathcal{Z}_F$.

Lemma

G is CCZ-equivalent to a permutation if and only if

$$V = L(\mathcal{V}) \subset \mathcal{Z}_G \text{ and } V' = L(\mathcal{V}^\perp) \subset \mathcal{Z}_G$$

for some linear permutation L . Note that

$$\text{span}(V \cup V') = \mathbb{F}_2^n \times \mathbb{F}_2^m.$$

3-Spaces Criteria

3-space criteria

Let $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$, not be a permutation. If it is CCZ-equivalent to a permutation then \mathcal{Z}_F must contain at least 3 vector spaces of zeroes of dimension n .

Projected Spaces Criteria

Key observation

The projections

$$p : (x, y) \mapsto x \text{ and } p' : (x, y) \mapsto y$$

mapping $\mathbb{F}_2^n \times \mathbb{F}_2^m$ to \mathbb{F}_2^n and \mathbb{F}_2^m respectively are **linear**.

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Thus, if G is CCZ-equivalent to a permutation then $p(V)$ and $p(V')$ are subspaces of \mathbb{F}_2^n whose span is \mathbb{F}_2^n .

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Projected Spaces Criteria

If $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$ is CCZ-equivalent to a permutation, then there are at least two subspaces of dimension $n/2$ in $p(\mathcal{Z}_F)$ and in $p'(\mathcal{Z}_F)$.

QAM

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None of them are CCZ-equivalent to a permutation

Göloğlu's Candidates (1/2)

Göloğlu's introduced APN functions

$$f_k : x \mapsto x^{2^k+1} + (x + x^{2^{n/2}})^{2^k+1}$$

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*Unfortunately, f_k are not equivalent to permutations on $n = 4, 8$ and does not seem to be equivalent to one on $n = 12$ (we say "it does not seem to be equivalent to a permutation" since checking the existence of CCZ-equivalent permutations **requires huge amount of computing and is infeasible on $n = 12$; our program was still running at the time of writing**).*

Göloğlu's Candidates (2/2)

n	cardinal proj.	time proj. (s)	time BasesExtraction (s)
12	1365	0.066	0.0012
16	21845	16.79	0.084
20	349525	10096.00	37.48

Time needed to show that f_k is **not** CCZ-equivalent to a permutation.

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 - Summary
 - Open Problems

Conclusion

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- $CCZ = EA + \text{Twist}$, both of which have a simple interpretation.
- Efficient criteria to know if a function is CCZ-equivalent to a permutation...
- ... implemented using a very efficient vector space extraction algorithm (not presented)

The Fourier transform solves everything!

Open Problems

EA-equivalence

How can we efficiently check the EA-equivalence of two functions?

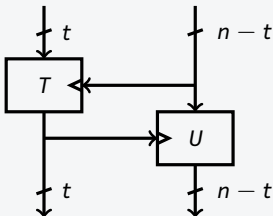
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Conjecture

If the CCZ-class of a permutation P is not reduced to the EA-classes of P and P^{-1} , then P has the following decomposition



where **both** T and U are keyed permutations.