On nonlinear approximations and the linear hull effect

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Linear approximations

Linear approximations

$$\Pr[\alpha \cdot x + \beta \cdot F(x) = 0] \text{ far from } \frac{1}{2}$$

quantified by:

$$\mathrm{cor}_F(\alpha, \beta) = 2^{-n} \sum_{x \in \mathbb{F}_2^n} (-1)^{\alpha \cdot x + \beta \cdot F(x)}$$

since

$$\Pr[\alpha \cdot x + \beta \cdot F(x) = 0] = \frac{1}{2}(1 + \mathrm{cor}_F(\alpha, \beta))$$

Linear approximations with correlation ± 1

F has a linear approximation with correlation ± 1
iff it has a component of degree 1.

$\Rightarrow$ This never occurs for one-round SPN (except for trivial Sboxes)

An alternative formulation:

$$\mathrm{cor}_F(\alpha, \beta) = -1 + 2^{-n+2} \#\{x \in \langle \alpha \rangle^{\perp} \text{ such that } F(x) \in \langle \beta \rangle^{\perp}\}$$

$$\Rightarrow \mathrm{cor}_F(\alpha, \beta) = \pm 1 \text{ iff } F(\langle \alpha \rangle^{\perp}) = \langle \beta \rangle^{\perp} \text{ or } \mathbb{F}_2^n \setminus \langle \beta \rangle^{\perp}.$$

Linear approximations over several rounds [Daemen 95][Nyberg 01]

$$\mathrm{cor}_{G\circ F}(\alpha, \beta) = \sum_{\gamma \in \mathbb{F}_2^n} \mathrm{cor}_F(\alpha, \gamma)\, \mathrm{cor}_G(\gamma, \beta)\ .$$

If one dominant trail $(\alpha, \gamma_0, \beta)$:

$$\mathrm{cor}_{G\circ F}(\alpha, \beta) \simeq \mathrm{cor}_F(\alpha, \gamma_0)\, \mathrm{cor}_G(\gamma_0, \beta)\ .$$

Otherwise, linear hull effect.

Two-round approximations with correlation ± 1

For a two-round SPN

$$\mathbf{cor}_{L \circ S}(\alpha, \beta) = \sum_{\gamma \in \mathbb{F}_2^n} \mathbf{cor}_S(\alpha, \gamma)\, \mathbf{cor}_L(\gamma, \beta) = \mathbf{cor}_S(\alpha, L^T(\beta))\ .$$

$$\mathbf{cor}_{\mathcal{R} \circ \mathsf{Add}_k \circ \mathcal{R}}(\alpha, \beta) = \sum_{\gamma \in \mathbb{F}_2^n} (-1)^{k \cdot \gamma}\, \mathbf{cor}_S(\alpha, L^T(\gamma))\, \mathbf{cor}_S(\gamma, L^T(\beta))\ .$$

Question: can we get a correlation $\pm\mathbf{1}$ for a two-round approximation for some fixed k?

Nonlinear approximations and invariants

Nonlinear approximations

Let g and h be two balanced Boolean functions of n variables.

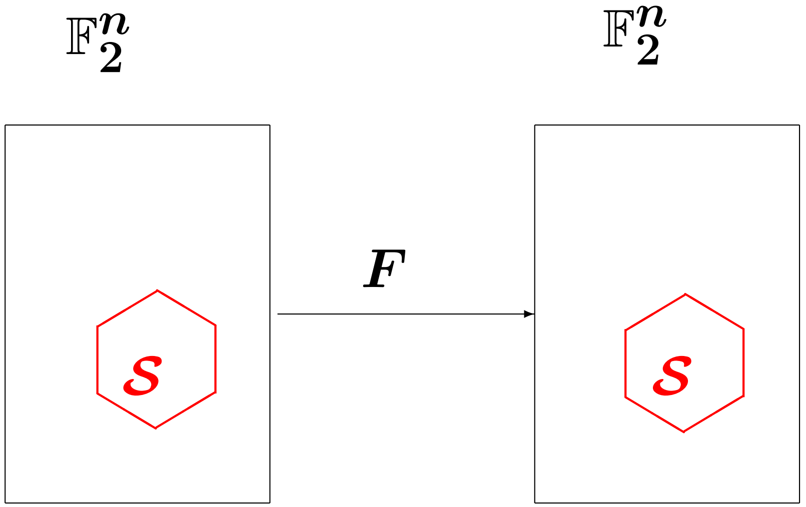
$$\Pr[g(x) + h(F(x)) = 0] \text{ far from } \frac{1}{2}.$$

quantified by:

$$\mathrm{cor}_F(g, h) = 2^{-n} \sum_{x \in \mathbb{F}_2^n} (-1)^{g(x)+h(F(x))}$$

Nonlinear invariants [Todo-Leander-Sasaki 16]

Non-trivial partition of $\mathbb{F}_2^n$ invariant under F:



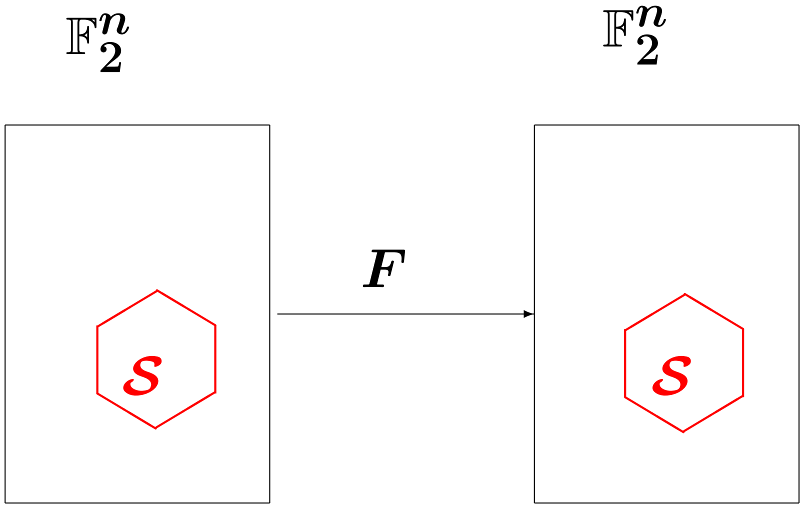
$\mathcal{S}$: any subset of $\mathbb{F}_2^n$

$$F(\mathcal{S}) = \mathcal{S}$$

or $F(\mathcal{S}) = \mathbb{F}_2^n \setminus \mathcal{S}$

The nonlinear invariant attack [Todo-Leander-Sasaki 16]

Non-trivial partition of $\mathbb{F}_2^n$ invariant under F:



$\mathcal{S}$: any subset of $\mathbb{F}_2^n$

$F(\mathcal{S}) = \mathcal{S}$

or $F(\mathcal{S}) = \mathbb{F}_2^n \setminus \mathcal{S}$

Equivalently:

Let g be the Boolean function defined by $g(x) := 1$ iff $x \in \mathcal{S}$

$$\forall x \in \mathbb{F}_2^n, g(F(x)) = g(x) \text{ or } \forall x \in \mathbb{F}_2^n, g(F(x)) = g(x) + 1$$

Such a g is called an invariant for F.

Nonlinear approximations with correlation ± 1

g is an invariant for F if and only if

$$\mathrm{cor}_F(g, g) = 2^{-n} \sum_{x \in \mathbb{F}_2^n} (-1)^{g(x)+g(F(x))} = \pm 1$$

Nonlinear approximations as a combination of linear approximations

$$\mathbf{cor}_{\boldsymbol{F}}(\boldsymbol{g},\boldsymbol{h}) = \sum_{\boldsymbol{\gamma},\boldsymbol{\gamma}' \in \mathbb{F}_2^n} \mathbf{cor}_{\boldsymbol{g}}(\boldsymbol{\gamma})\,\mathbf{cor}_{\boldsymbol{F}}(\boldsymbol{\gamma},\boldsymbol{\gamma}')\,\mathbf{cor}_{\boldsymbol{h}}(\boldsymbol{\gamma}') \;.$$

If $\boldsymbol{g} = \boldsymbol{\ell}_{\boldsymbol{\alpha}}$ and $\boldsymbol{h} = \boldsymbol{\ell}_{\boldsymbol{\beta}}$, then

$$\mathbf{cor}_{\boldsymbol{F}}(\boldsymbol{g},\boldsymbol{h}) = \mathbf{cor}_{\boldsymbol{F}}(\boldsymbol{\alpha},\boldsymbol{\beta}) \;.$$

Otherwise, we gather together several linear approximations.

Nonlinear approximations

and the linear hull effect

Transforming nonlinear invariants into linear approximations

Let $\boldsymbol{g}$ be a balanced nonlinear invariant for $\boldsymbol{F}$.

We can always define a permutation $\mathcal{G}$ such that $\boldsymbol{\alpha} \cdot \mathcal{G}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x})$.

Then,

$$\begin{aligned} \boldsymbol{g}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{F}(\boldsymbol{x})) &= \boldsymbol{\alpha} \cdot \mathcal{G}(\boldsymbol{x}) + \boldsymbol{\alpha} \cdot (\mathcal{G} \circ \boldsymbol{F})(\boldsymbol{x}) \\ &= \boldsymbol{\alpha} \cdot \boldsymbol{y} + \boldsymbol{\alpha} \cdot (\mathcal{G} \circ \boldsymbol{F} \circ \mathcal{G}^{-1})(\boldsymbol{y}) \end{aligned}$$

The nonlinear approximation of $\boldsymbol{F}$ defined by $(\boldsymbol{g}, \boldsymbol{g})$ corresponds to the linear approximation $(\boldsymbol{\alpha}, \boldsymbol{\alpha})$ of $\boldsymbol{F}^{\mathcal{G}, \mathcal{G}^{-1}} = \mathcal{G} \circ \boldsymbol{F} \circ \mathcal{G}^{-1}$.

$$\mathbf{cor}_{\boldsymbol{F}^{\mathcal{G}, \mathcal{G}^{-1}}}(\boldsymbol{\alpha}, \boldsymbol{\alpha}) = \sum_{\gamma_1, \gamma_2 \in \mathbb{F}_2^n} \mathbf{cor}_{\mathcal{G}_\alpha}(\gamma_1)\, \mathbf{cor}_{\boldsymbol{F}}(\gamma_1, \gamma_2)\, \mathbf{cor}_{\mathcal{G}_\alpha}(\gamma_2)$$

The other components of $\mathcal{G}$ do not matter!

$\mathcal{G}$-shifted trails

$$
\begin{aligned}
E^{\mathcal{G},\mathcal{G}^{-1}}_{(k_0,\ldots,k_t)} &= \mathcal{G} \circ \mathcal{R}_{k_t} \circ \mathcal{R}_{k_{t-1}} \circ \cdots \circ \mathcal{R}_{k_0} \circ \mathcal{G}^{-1} \\
&= \mathcal{R}^{\mathcal{G},\mathcal{G}^{-1}}_{k_t} \circ \mathcal{R}^{\mathcal{G},\mathcal{G}^{-1}}_{k_{t-1}} \circ \cdots \circ \mathcal{R}^{\mathcal{G},\mathcal{G}^{-1}}_{k_0} .
\end{aligned}
$$

$$
\mathrm{cor}_{E^{\mathcal{G},\mathcal{G}^{-1}}_{(k_0,\ldots,k_t)}}(\alpha, \beta) = \sum_{\gamma_1,\ldots,\gamma_{t-1} \in \mathbb{F}_2^n} \prod_{i=0}^{t-1} \mathrm{cor}_{\mathcal{R}^{\mathcal{G},\mathcal{G}^{-1}}_{k_i}}(\gamma_i, \gamma_{i+1}) \, .
$$

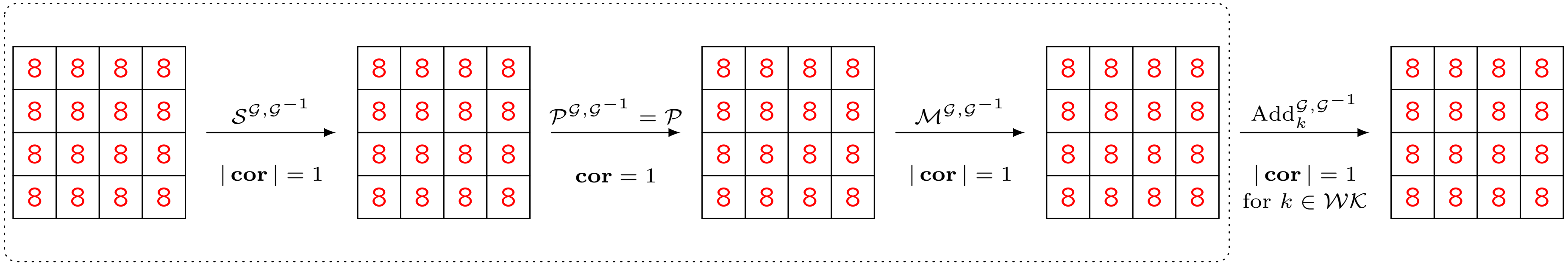
A one-round $\mathcal{G}$-shifted trail on Midori-64

$\mathcal{G} = (G, \ldots, G)$ where G is a bijection on 4 bits such that $\langle 8, G(x) \rangle = g(x)$ with $g(x) = x_3x_2 + x_2 + x_1 + x_0$ invariant for the Sbox, i.e.

$$|\,\mathbf{cor}_{S^{G,G^{-1}}}(8,8)| = 1\ .$$

$$|\,\mathbf{cor}_{\mathcal{M}^{\mathcal{G},\mathcal{G}^{-1}}}((8,\ldots,8),(8,\ldots,8))\,| = 1\ .$$

$\Rightarrow$ Iterative one-round trail with correlation ± 1:



A two-round shifted trail on Midori-64 [Beyne 18]

For $g(x) = x_0x_2 + x_0 + x_1 + x_3$ and $\alpha =$ 0x5, the Sbox satisfies

$$g(S(x)) + \alpha \cdot x = 1 \; .$$

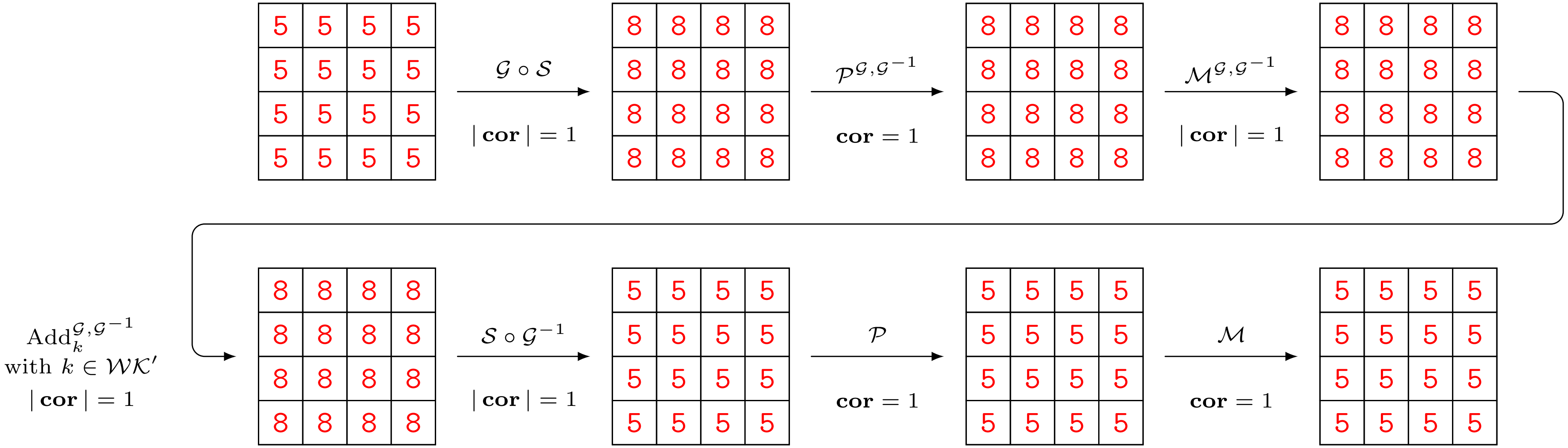
We choose a 4-bit bijection G such that $\langle 8, G(x)\rangle = g(x)$.

Equivalently,

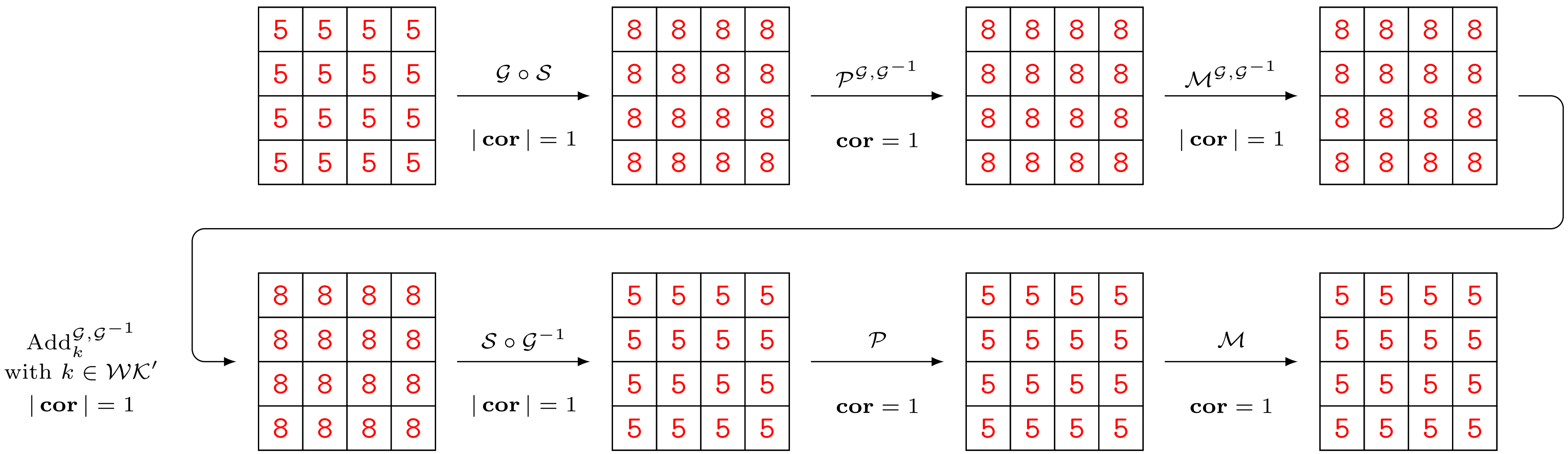
$$\mathrm{cor}_S(\ell_\alpha, g) = \mathrm{cor}_{G \circ S}(\alpha, 8) = -1 \; .$$

$$|\, \mathrm{cor}_{\mathcal{M}^{\mathcal{G},\mathcal{G}^{-1}}}((8, \ldots, 8), (8, \ldots, 8))\,| = 1 \; .$$

A two-round shifted trail on Midori-64 [Beyne 18]



A two-round shifted trail on Midori-64 [Beyne 18]



This is a two-round linear approximation with correlation $\pm\mathbf{1}$!

A 4-round $\mathcal{G}$-shifted trail on Midori-64

G is a bijection on 4 bits such that $\langle 8, G(x)\rangle = g(x)$
with $g(x) = x_3x_2x_1 + x_3x_1 + x_3 + x_2 + x_1 + x_0$ invariant for the Sbox:

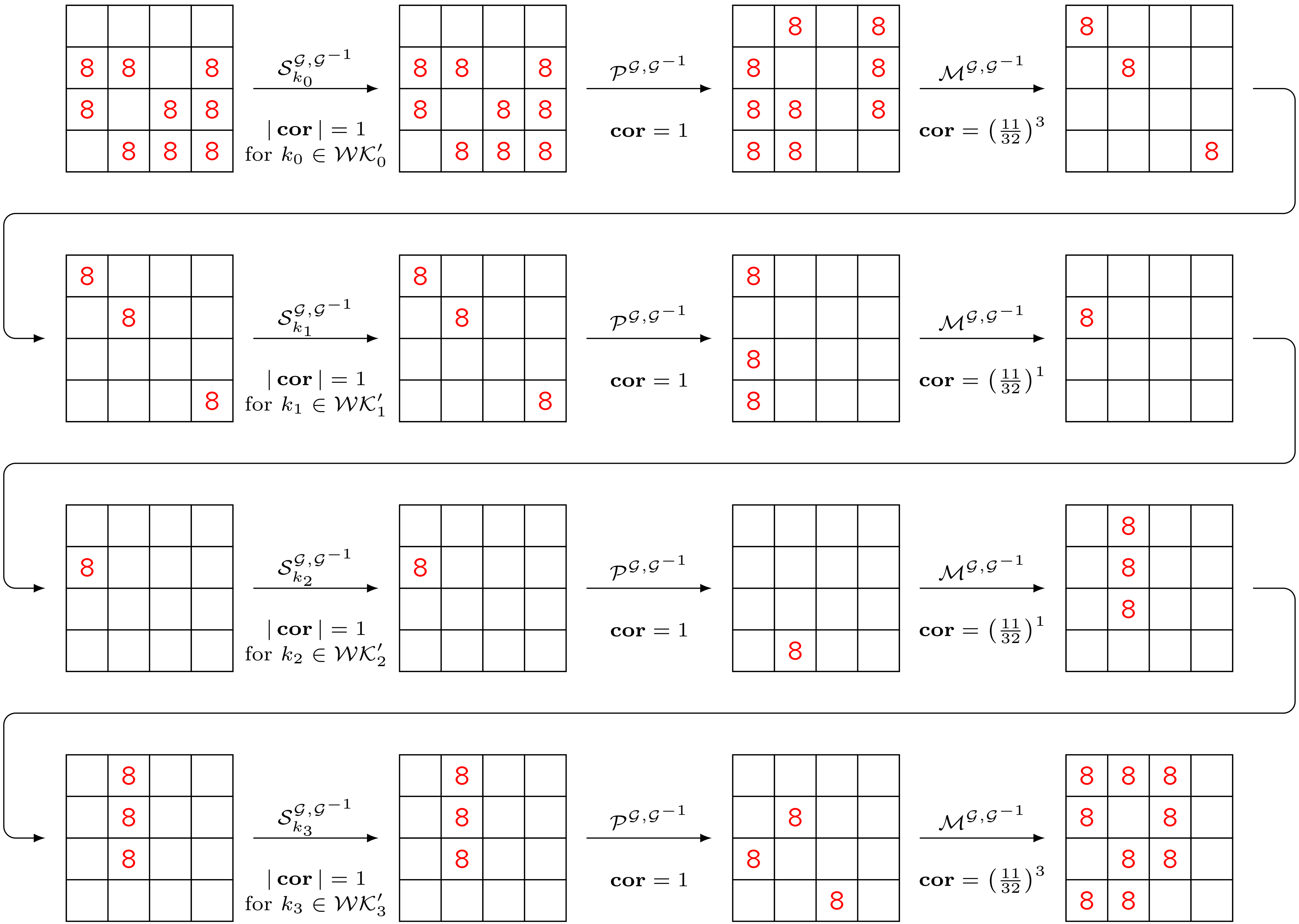
$$|\,\mathbf{cor}_{S^{G,G^{-1}}}(8,8)| = 1\ .$$

But,

$$|\,\mathbf{cor}_{\mathrm{M}^{\mathrm{G},\mathrm{G}^{-1}}}(\alpha, \mathrm{M}\alpha)\,| = \frac{11}{32}$$

if $\alpha \neq (0,0,0,0)$ and all $\alpha_i \in \{0, 8\}$.

A 4-round $\mathcal{G}$-shifted trail on Midori-64



A 4-round $\mathcal{G}$-shifted trail on Midori-64

The weak keys are those equal to $\mathbf{0}$ or $\mathbf{1}$ in all active cells.

Correlation of the trail:

$$\left(\frac{\mathbf{11}}{\mathbf{32}}\right)^{\mathbf{8}} = \mathbf{2^{-12.325}}$$

Correlation of the approximation:

$$\mathbf{cor}_{(\mathcal{R}_{k_3} \circ \ldots \mathcal{R}_{k_0})^{\mathcal{G},\mathcal{G}^{-1}}(\alpha,\alpha)} \simeq \mathbf{2^{-12.16}}$$

What's about the other trails? For the first 2 rounds:

- For $G_1 = [\mathtt{0}, \mathtt{8}, \mathtt{c}, \mathtt{4}, \mathtt{a}, \mathtt{2}, \mathtt{6}, \mathtt{e}, \mathtt{9}, \mathtt{1}, \mathtt{d}, \mathtt{5}, \mathtt{3}, \mathtt{b}, \mathtt{f}, \mathtt{7}]$,
  35,937 $\mathcal{G}_1$-shifted linear trails having a nonzero correlation
- For $G_2 = [\mathtt{0}, \mathtt{9}, \mathtt{a}, \mathtt{1}, \mathtt{8}, \mathtt{2}, \mathtt{3}, \mathtt{f}, \mathtt{c}, \mathtt{4}, \mathtt{d}, \mathtt{5}, \mathtt{6}, \mathtt{e}, \mathtt{b}, \mathtt{7}]$,
  282,184 $\mathcal{G}_2$-shifted linear trails having a nonzero correlation

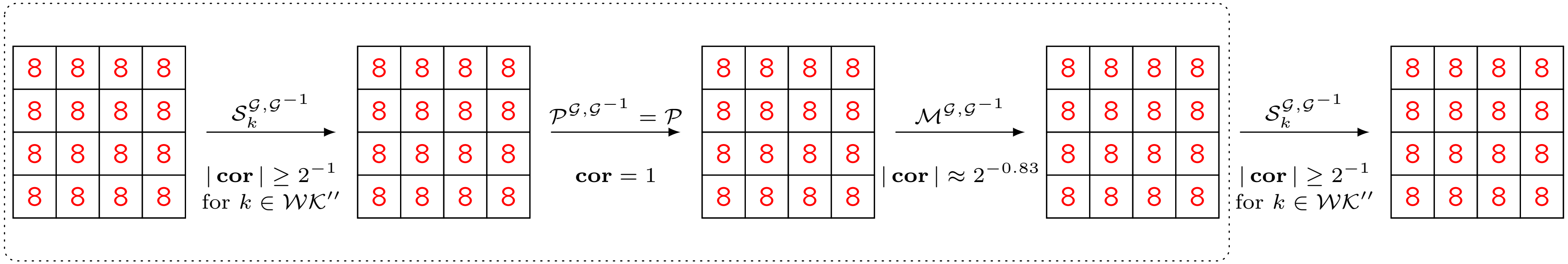
Another 1-round $\mathcal{G}$-shifted trail on Midori-64

$\mathcal{G} = (G', G, \ldots, G)$ where G is a bijection on 4 bits such that $\langle 8, G(x)\rangle = g(x)$ with $g(x) = x_3x_2 + x_2 + x_1 + x_0$ invariant for S, $\langle 8, G'(x)\rangle = g'(x)$ with $g'(x) = x_3x_2x_1 + x_3x_1 + x_3 + x_2 + x_1 + x_0$.

$$|\,\mathrm{cor}_{S_k^{G', G'^{-1}}}(8, 8)| = \begin{cases} 1 & \text{if } k \in \{0,1\} \\ 2^{-1} & \text{if } k \notin \{0,1\} \end{cases} \quad .$$

$$|\,\mathrm{cor}_{\mathcal{M}^{\mathcal{G},\mathcal{G}^{-1}}}((8, \ldots, 8), (8, \ldots, 8))\,| \simeq 2^{-0.83}$$

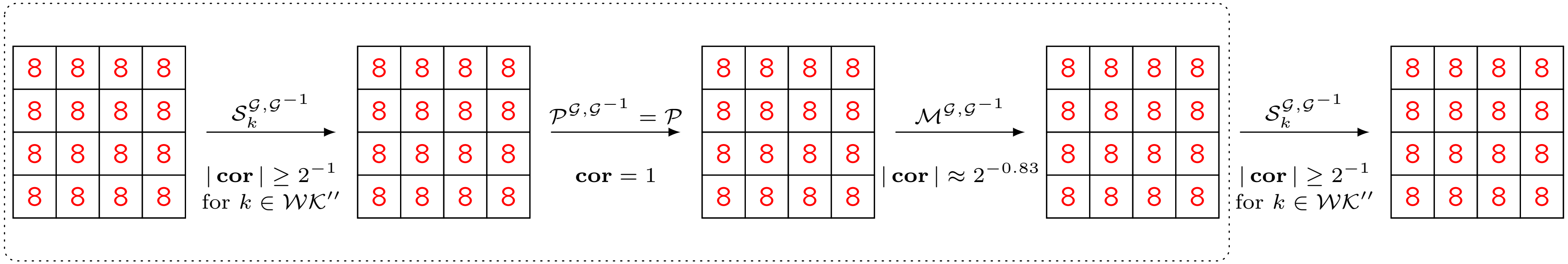
Another 1-round $\mathcal{G}$-shifted trail on Midori-64



Correlation of the 16-round trail:

$$\geq \left(\mathbf{2}^{-\mathbf{1.83}}\right)^{\mathbf{16}} = \mathbf{2}^{-\mathbf{29.28}}$$

Another 1-round $\mathcal{G}$-shifted trail on Midori-64



Correlation of the 16-round trail:

$$\geq \left(\mathbf{2}^{-\mathbf{1.83}}\right)^{\mathbf{16}} = \mathbf{2}^{-\mathbf{29.28}}$$

Correlation over 16 rounds:

different from the correlation of the trail.

Focus on a single column

$\mathcal{G} = (G', G, G, G)$ with

$$|\,\mathbf{cor}_{S_k^{G',G'^{-1}}}(8,8)| = \begin{cases} 1 & \text{if } k \in \{0,1\} \\ 2^{-1} & \text{if } k \notin \{0,1\} \end{cases} .$$

$$|\,\mathbf{cor}_{\mathbf{M}^{\mathcal{G},\mathcal{G}^{-1}}}((8,8,8,8),(8,8,8,8))\,| = 2^{-0.83}$$

$$\begin{bmatrix}8\\8\\8\\8\end{bmatrix} \xrightarrow[|\,\mathbf{cor}\,| \geq 2^{-1}]{\mathcal{S}_{k_0}^{\mathcal{G},\mathcal{G}^{-1}}} \begin{bmatrix}8\\8\\8\\8\end{bmatrix} \xrightarrow[|\,\mathbf{cor}\,| \approx 2^{-0.83}]{\mathbf{M}^{\mathcal{G},\mathcal{G}^{-1}}} \begin{bmatrix}8\\8\\8\\8\end{bmatrix} \xrightarrow[|\,\mathbf{cor}\,| \geq 2^{-1}]{\mathcal{S}_{k_1}^{\mathcal{G},\mathcal{G}^{-1}}} \begin{bmatrix}8\\8\\8\\8\end{bmatrix} \xrightarrow[|\,\mathbf{cor}\,| \approx 2^{-0.83}]{\mathbf{M}^{\mathcal{G},\mathcal{G}^{-1}}} \begin{bmatrix}8\\8\\8\\8\end{bmatrix}$$

If $k_1 \in \left(\mathbb{F}_2^4 \setminus \{(0,0,*,*)\}\right) \times \{(0,0,*,*)\}^3$,

$$\mathbf{cor}_{\mathcal{R}_{k_1} \circ \mathcal{R}_{k_0}}((8,8,8,8),(8,8,8,8)) = 0$$

Open problems

- When can we approximate the correlation with a single trail?
- Nonlinear approximations as a method for clustering linear approximations to capture the linear hull effect?
  - How general is this?
  - How can we find the appropriate approximation?