On CCZ-Equivalence, Extended-Affine Equivalence and Function Twisting

Anne Canteaut, Léo Perrin

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Journées C2

informatics mathematics

Cryptographic Properties

$F : \mathbb{F}_2^n \to \mathbb{F}_2^m$ and $G : \mathbb{F}_2^n \to \mathbb{F}_2^m$ are functions (e.g. S-Boxes).

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Definition (DDT/LAT)

The D(ifference) D(istribution) T(able) of $F : \mathbb{F}_2^n \to \mathbb{F}_2^m$ is

$$\mathcal{D}_F(\alpha, \beta) = \# \{x, F(x \oplus \alpha) \oplus F(x) = \beta\}$$

The L(inear) A(pproximation) T(able) of $F : \mathbb{F}_2^n \to \mathbb{F}_2^m$ is

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Big APN Problem

Is there an APN permutation on $2t$ bits such that $\max(\text{DDT}) = 2$?

Equivalence Relations (1/2)

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Definition (EA-Equivalence; EA-mapping)

F and G are *E(xtented) A(ffine) equivalent* if, up to translations, $G(x) = (B \circ F \circ A)(x) + C(x)$, where A, B, C are linear and A, B are permutations; so that

$$\{(x, G(x)), \forall x \in \mathbb{F}_2^n\} = \begin{bmatrix} A^{-1} & 0 \\ CA^{-1} & B \end{bmatrix} \left(\{(x, F(x)), \forall x \in \mathbb{F}_2^n\}\right) .$$

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Affine permutations with such linear part are **EA-mappings**.

Equivalence Relations (2/2)

Definition (CCZ-Equivalence)

$F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$ and $G : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$ are *C(arlet)-C(harpin)-Z(inoviev) equivalent* if, up to translations,

$$\Gamma_G = \left\{(x, G(x)), \forall x \in \mathbb{F}_2^n\right\} \;=\; L\left(\left\{(x, F(x)), \forall x \in \mathbb{F}_2^n\right\}\right) = L(\Gamma_F)\,,$$

where $L : \mathbb{F}_2^{n+m} \rightarrow \mathbb{F}_2^{n+m}$ is a linear permutation.

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- CCZ-equivalence preserves the distribution of the coefficients in the DDT and the LAT.
- It does not preserve bijectivity.
- It does not preserve the algebraic degree.

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- CCZ-equivalence preserves the distribution of the coefficients in the DDT and the LAT.
- It does not preserve bijectivity.
- It does not preserve the algebraic degree.
- It plays a crucial role in the investigation of the big APN problem.

The Problem with CCZ-Equivalence

Admissible Mapping

For $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$, the affine permutation L is **admissible for F** if

$$L\left(\{(x, F(x)), \forall x \in \mathbb{F}_2^n\} \right) = \{(x, G(x)), \forall x \in \mathbb{F}_2^n\}$$

for a well defined function $G : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$.

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for a well defined function $G : \mathbb{F}_2^n \to \mathbb{F}_2^m$.

1. How do we find admissible mapping?
2. Is there a simpler way of seeing CCZ-equivalence?
3. How do we know if a function is CCZ-equivalent to a permutation?

Structure of this talk

Outline

Plan of this Section

Walsh Zeroes

Definition (Walsh Zeroes)

The *Walsh zeroes* of $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$ is the set

$$\mathcal{Z}_F = \{u \in \mathbb{F}_2^n \times \mathbb{F}_2^m, \mathcal{W}_F(u) = 0\} \cup \{0\} .$$

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For all $F : \mathbb{F}_2^n \to \mathbb{F}_2^m$, we have

$$\mathcal{W}_F(\alpha, 0) = \sum_{x \in \mathbb{F}_2^n} (-1)^{\alpha \cdot x + 0 \cdot F(x)} = 0$$

so that

$$\mathcal{V} = \{(x, 0^m), \forall x \in \mathbb{F}_2^n\} \subset \mathcal{Z}_F \subset \mathbb{F}_2^{n+m}$$

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so that

$$\mathcal{V} = \{(x, 0^m), \forall x \in \mathbb{F}_2^n\} \subset \mathcal{Z}_F \subset \mathbb{F}_2^{n+m}$$

Note that if $\Gamma_G = L(\Gamma_F)$, then $\mathcal{Z}_G = (L^T)^{-1}(\mathcal{Z}_F)$.

Admissibility for F

Lemma

Let $L : \mathbb{F}_2^{n+m} \rightarrow \mathbb{F}_2^{n+m}$ be a linear permutation. It is admissible for $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$ if and only if

$$L^T(\mathcal{V}) \subseteq \mathcal{Z}_F$$

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Example

EA-mappings are admissible for all $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$:

$$\begin{bmatrix} A & 0 \\ C & B \end{bmatrix}^T (\mathcal{V}) = \begin{bmatrix} A^T & C^T \\ 0 & B^T \end{bmatrix} \left(\left\{ \begin{bmatrix} x \\ 0 \end{bmatrix}, \forall x \in \mathbb{F}_2^n \right\} \right) = \mathcal{V} .$$

Permutations

We define

$$\mathcal{V}^{\perp} = \{(0^n, y), \forall y \in \mathbb{F}_2^m\} \subset \mathbb{F}_2^{n+m}.$$

Lemma

$F : \mathbb{F}_2^n \to \mathbb{F}_2^m$ is a permutation if and only if

$$\mathcal{V}^{\perp} \subset \mathcal{Z}_F .$$

EA-classes imply vector spaces

Lemma

*let F, G and G' be such that $\Gamma_G = L(\Gamma_F)$ and $\Gamma_{G'} = L'(\Gamma_F)$.
If $L(\mathcal{V}) = L'(\mathcal{V})$, then G and G' are EA-equivalent.*

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1 EA-class $\implies$ 1 vector space of zeroes of dimension n in $\mathcal{Z}_n$

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Reality takes it back...

The converse of the lemma is wrong.

Outline

Plan of this Section

EA-equivalence is a simple sub-case of CCZ-Equivalence...

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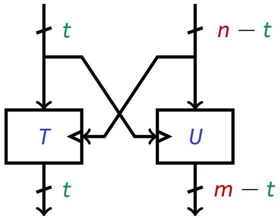
What must we add to EA-equivalence to fully describe CCZ-Equivalence?

Definition of the Twist

Any function $F : \mathbb{F}_2^n \to \mathbb{F}_2^m$ can be projected on $\mathbb{F}_2^t \times \mathbb{F}_2^{m-t}$.



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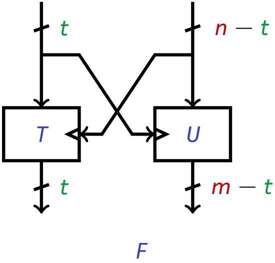

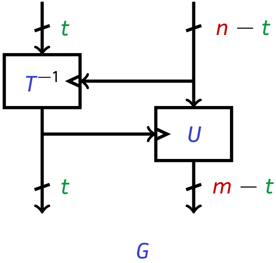

If T is a permutation for all secondary inputs, then we define the t-twist equivalent of F as G, where

$$G(x, y) = \left(T_y^{-1}(x), U_{T_y^{-1}(x)}(y)\right)$$

for all $(x, y) \in \mathbb{F}_2^t \times \mathbb{F}_2^{n-t}$.

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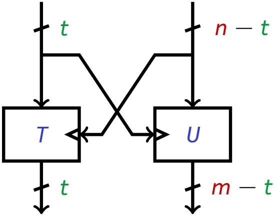
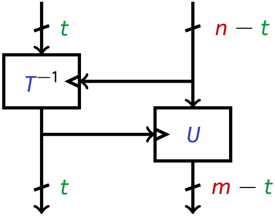


F



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for all $(x, y) \in \mathbb{F}_2^t \times \mathbb{F}_2^{n-t}$.

The identiy is a 0-twist, functional inversion is an n-twist.

Swap Matrices

The swap matrix permuting $\mathbb{F}_2^{n+m}$ is defined for $t \leq \min(n, m)$ as

$$M_t = \begin{bmatrix} 0 & 0 & I_t & 0 \\ 0 & I_{n-t} & 0 & 0 \\ I_t & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{m-t} \end{bmatrix} .$$

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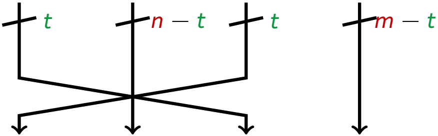

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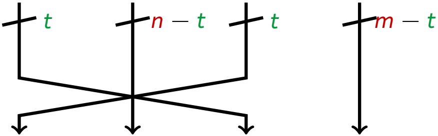
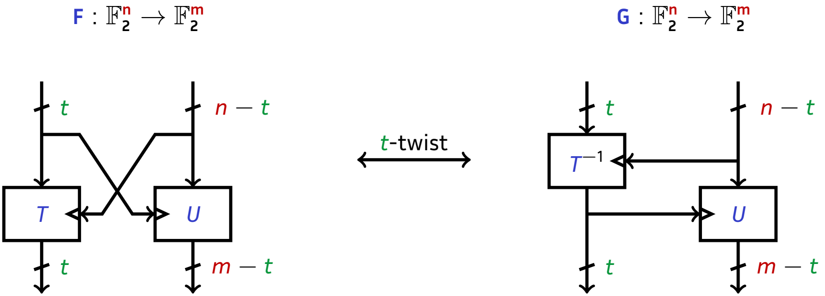


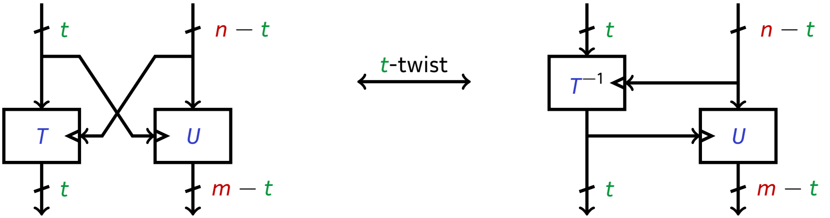
For all $t \leq \min(n, m)$, M_t is an **orthogonal** and **symmetric involution**.

Swap Matrices and Twisting



Swap Matrices and Twisting

$\mathbf{F} : \mathbb{F}_2^{\mathbf{n}} \to \mathbb{F}_2^{\mathbf{m}}$ $\mathbf{G} : \mathbb{F}_2^{\mathbf{n}} \to \mathbb{F}_2^{\mathbf{m}}$



$$\Gamma_F = \{ (x, F(x)), \forall x \in \mathbb{F}_2^n \} \quad \xleftrightarrow{M_t} \quad \Gamma_G = \{ (x, G(x)), \forall x \in \mathbb{F}_2^n \}$$

Swap Matrices and Twisting

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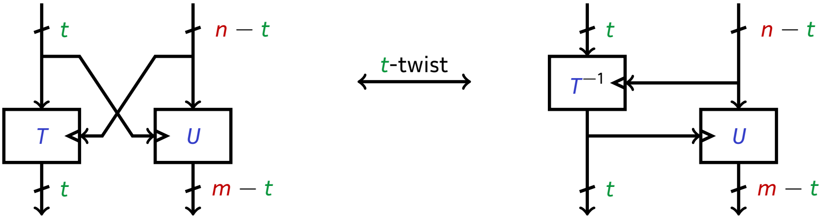

$\Gamma_F = \{ (x, F(x)), \forall x \in \mathbb{F}_2^n \}$ $\overset{M_t}{\longleftrightarrow}$ $\Gamma_G = \{ (x, G(x)), \forall x \in \mathbb{F}_2^n \}$

$$\mathcal{W}_F(u) = \mathcal{W}_G(M_t(u))$$

Twisting and CCZ-Class

Lemma

Twisting preserves the CCZ-equivalence class.

Main Result

Theorem

If $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$ and $G : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$ are CCZ-equivalent, then

$$\Gamma_G = (B \times M_t \times A)(\Gamma_F) ,$$

where A and B are EA-mappings and where

$$t = \dim \left(proj_{\mathcal{V}^\perp} \left((A^T \times M_t \times B^T)(\mathcal{V}) \right) \right) .$$

In other words, EA-equivalence and twists are sufficient to fully describe CCZ-equivalence!

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In other words, EA-equivalence and twists are sufficient to fully describe CCZ-equivalence!

Corollary

If a function is CCZ-equivalent but not EA-equivalent to another function, then they have to be EA-equivalent to functions for which a t-twist is possible.

Outline

Plan of this Section

Another Problem

How do we know if a function is CCZ-equivalent to a permutation?

Remainder

Recall that F is a permutation if and only if $\mathcal{V} \subset \mathcal{Z}_F$ and $\mathcal{V}^{\perp} \subset \mathcal{Z}_F$.

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Lemma

G is CCZ-equivalent to a permutation if and only if

$$V = L(\mathcal{V}) \subset \mathcal{Z}_G \text{ and } V' = L(\mathcal{V}^{\perp}) \subset \mathcal{Z}_G$$

for some linear permutation L. Note that

$$span\left(V \cup V'\right) = \mathbb{F}_2^n \times \mathbb{F}_2^m .$$

Projected Spaces Criterion

Key observation

The projections

$$p : (x, y) \mapsto x \text{ and } p' : (x, y) \mapsto y$$

mapping $\mathbb{F}_2^n \times \mathbb{F}_2^m$ to $\mathbb{F}_2^n$ and $\mathbb{F}_2^m$ respectively are **linear**.

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Thus, If G is CCZ-equivalent to a permutation then $p(V)$ and $p(V')$ are subspaces of $\mathbb{F}_2^n$ whose span is $\mathbb{F}_2^n$.

We deduce that $\dim(p(V)) + \dim(p(V')) \leq n$

Projected Spaces Criterion

If $F : \mathbb{F}_2^n \to \mathbb{F}_2^m$ is CCZ-equivalent to a permutation, then there are at least two subspaces of dimension $n/2$ in $p(\mathcal{Z}_F)$ and in $p'(\mathcal{Z}_F)$.

QAM

Yu et al. (DCC'14) generated 8180 8-APN quadratic functions from "*QAM*" (matrices).

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None of them are CCZ-equivalent to a permutation

Göloğlu's Candidates (1/2)

Göloğlu's introduced APN functions

$$f_k : x \mapsto x^{2^k+1} + (x + x^{2^{n/2}})^{2^k+1}$$

for $n = 4t$. They have the *subspace property* of the Kim mapping.

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> *Unfortunately, f_k are not equivalent to permutations on $n = 4, 8$ and does not **seem** to be equivalent to one on $n = 12$ (we say "it does not seem to be equivalent to a permutation" since checking the existence of CCZ-equivalent permutations **requires huge amount of computing** and is infeasible on $n = 12$; our program was still running at the time of writing).*

Göloğlu's Candidates (2/2)

n	cardinal proj.	time proj. (s)	time `BasesExtraction` (s)
12	1365	0.066	0.0012
16	21845	16.79	0.084
20	349525	10096.00	37.48

Time needed to show that f_k is **not** CCZ-equivalent to a permutation.

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Conclusion

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- CCZ $=$ EA $+$ Twist, both of which have a simple interpretation.
- Efficient criteria to know if a function is CCZ-equivalent to a permutation...
- ... implemented using a very efficient vector space extraction algorithm (not presented)

Open problem

How can we efficiently check the EA-equivalence of two functions?

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https://eprint.iacr.org/2018/713

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Thank you!