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Abstract: This work deals with stability and robust stabilisation of retarded time-delay systems by applying a new method for obtaining an envelope that bounds all the system poles. Through LMIs we are able to determine envelopes that can be applied to verify the stability of the system and can also be utilised to design robust state-feedback controllers which cope with design requirements regarding $\alpha - stability$.

Keywords: Robust Stabilisation, Stability, State-Feedback, Time-delay systems

1. INTRODUCTION

Time-delays are inherently coupled with almost every dynamical system. This can be due to the time necessary to acquire the information needed for the control, the time required to transport information, the processing time, the sampling period, the propagation time on networked systems, among many others. Although many times those delays are neglected, they may cause poor performance or, in a worst scenario, they may even cause instability. Stability for time-delay systems was discussed, among others, in Michiels and Niculescu (2014), Richard (2003) and Briat (2015).

For the stabilisation through state feedback, delay independent controllers can be devised using Riccati equations Lee et al. (1994); Shen et al. (1991), whereas the delay dependent case was designed by means of Lyapunov-Krasoviskii functionals in Fridman and Shaked (2001, 2002); Niculescu (1998a). Lyapunov-Krasoviskii functionals were also utilised for robust control of state delay systems in Xia and Jia (2003). A controller design approach through a finite LTI comparison system was developed in Cardeliquio et al. (2016) and in Cardeliquio et al. (2017). Criteria for robust stability and stabilisation was dealt in Li and Souza (1997b). Robust exponential stabilisation for systems with time-varying delays can be seen in Seuret et al. (2004). Robust stability and stabilisation for singular systems with parametric uncertainties were discussed, among others, in Xu et al. (2002) and Wang et al. (2008). Delay independent stability for uncertain systems can be seen in Luo et al. (1995) and delay dependent stability and stabilisation in Moon et al. (2001), Fridman and Shaked (2003) and Boukas (2009). The discrete counterpart was studied in Wu et al. (2009), for positive systems. Guaranteed LQR control was dealt in Kubo (2004) and robust polytopic $\mathcal{H}_\infty$ static output feedback in Eli et al. (2010). Finally, α-stability was discussed in Wang and Wang (1996) for non commensurate delays and in Niculescu (1998b) via LMIs.

The use of an envelope that ensures that all poles are contained inside it was discussed in Michiels and Niculescu (2014). Different types of envelopes were also discussed in T.Mori and Kokame (1989) and Wang (1992). In those cases, the methods utilised to establish the envelopes were not used to test stability nor to design controllers. In fact, in general, the envelope extends to the right half plane and due to that, it only provides a region where the poles are allowed to be without any guarantee about the stability of the system. The present work is based on Linear Matrix Inequalities (LMIs) instead of the singular value approach, see Michiels and Niculescu (2014), and provides a different analysis regarding the use of envelopes. A procedure to test robust stability for retarded time-delay systems is established. In addition, a robust state-feedback controller coping with project requirements regarding α-stability can be designed.

Notation. Matrices are denoted by capital letters, whilst small letters represent scalars and vectors. For real matrices or vectors the symbol ($'$) indicates transpose and for complex matrices or vectors the symbol ($*$) denotes conjugate transpose. The determinant of a matrix A is indicated by $\det(A)$. The sets of real, integer and natural numbers including zero are denoted by $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$, respectively. $\Re(.)$ is the real part of a complex number. A left eigenvector is defined as a row vector x_L satisfying $x_L A = \lambda_L x_L$, where λ_L is a left eigenvalue of the matrix A. For partitioned matrices the symbol $\bullet$ represents each one

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of its Hermitian blocks. The induced p-norm of a matrix A is given by $\|A\|_p$, $A \in \mathbb{C}^{n\times m}$. Finally, $X > 0$ ($X \geq 0$) denotes that the symmetric matrix X is positive definite (positive semi-definite).

2. PROBLEM STATEMENT

Consider the uncertain retarded linear time-delay system with N delays, whose minimal realisation is given by

$$\dot{x}(t) = \sum_{i=0}^{N} A_i x\left(t - \tau_i\right), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state variable, $0 = \tau_0 < \tau_1 < \cdots < \tau_N$ are the delays, $A_i \in \mathbb{R}^{n\times n}$ for all $i \in \{0, \ldots, N\}$ and the system matrices belong to a convex polytope

$$\mathcal{P} := co\left\{\left[A_0^\ell, \cdots, A_N^\ell\right], \ell \in 1, \ldots, N_v\right\}, \tag{2}$$

defined by the convex combination of N_v vertices. Each matrix can be individually defined as (Bernussou et al., 1989)

$$A_i := \left\{\sum_{\ell=1}^{N_i} \xi_i^\ell A_i^\ell, \ \sum_{\ell=1}^{N_i} \xi_i^\ell = 1, \ \xi_i^\ell \geq 0\right\}. \tag{3}$$

Hence, $N_v = \Pi_{i=0}^{N} N_i$.

Example 1. As an example consider the matrix

$$A_1 = \begin{bmatrix} -2 & [0 \ \ 1] \\ [-1 \ \ 3] & 2 \end{bmatrix} \tag{4}$$

where [0 1] and [-1 3] represents the parametric uncertainty on A_1. This matrix can be rewritten as

$$A_1 = \xi_1^1 A_1^1 + \xi_1^2 A_1^2 + \xi_1^3 A_1^3 + \xi_1^4 A_1^4 \tag{5}$$

where $\xi_1^1 + \xi_1^2 + \xi_1^3 + \xi_1^4 = 1$, $\xi_1^1,\ \xi_1^2,\ \xi_1^3,\ \xi_1^4 \geq 0$.

It is easy to see that the convex combination that creates A_1 for this particular transition matrix is

$$\begin{aligned} A_1 = \xi_1^1 \begin{bmatrix} -2 & 0 \\ -1 & 2 \end{bmatrix} + \xi_1^2 \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix} \\ + \xi_1^3 \begin{bmatrix} -2 & 0 \\ 3 & 2 \end{bmatrix} + \xi_1^4 \begin{bmatrix} -2 & 1 \\ -1 & 2 \end{bmatrix}. \end{aligned} \tag{6}$$

System (1) is exponentially stable if and only if all zeros of its characteristic equation

$$\det\left(sI - \sum_{i=0}^{N} A_i e^{-s\tau_i}\right) = 0 \tag{7}$$

are in the open left half-plane (Bellman and Cooke, 1963).

The following Proposition introduces an envelope that completely surround all of its poles.

Proposition 1. Let λ be any real number. If there exist a characteristic root s_0 of equation (7), such that $s_0 = \lambda + \jmath\omega$ and if there exist matrices $T = T' > 0$, $Q_i = Q_i' > 0$, for all $i \in \{0, \ldots, N\}$ and a scalar μ that satisfy

$$\begin{bmatrix} T & T & \ldots & T \\ & Q_0 & & \\ & & \ddots & \\ & & & Q_N \end{bmatrix} \geq 0, \tag{8}$$

and

$$\begin{bmatrix} \mu T & A_0^\ell Q_0 e^{-\lambda\tau_0} & \ldots & A_N^\ell Q_N e^{-\lambda\tau_N} \\ \bullet & Q_0 & 0 & 0 \\ \bullet & 0 & \ddots & 0 \\ \bullet & 0 & 0 & Q_N \end{bmatrix} \geq 0, \tag{9}$$

for all $\ell \in \{1, \ldots, N_v\}$, then

$$|s_0| \leq \sqrt{\mu}. \tag{10}$$

Proof: First, let us consider the nominal system without any uncertainties, i.e., $N_v = 1$. The following inequality is always true, which is easily verifiable applying Schur's complement

$$\begin{bmatrix} A_i Q_i A_i' e^{-2\lambda\tau_i} & \bullet \\ A_i' e^{-(\lambda - \jmath\omega)\tau_i} & Q_i^{-1} \end{bmatrix} \geq 0. \tag{11}$$

Adding them for all $i \in \{0, \ldots, N\}$ leads to

$$\begin{bmatrix} \sum_{i=0}^{N} A_i Q_i A_i' e^{-2\lambda\tau_i} & \bullet \\ \sum_{i=0}^{N} A_i' e^{-(\lambda - \jmath\omega)\tau_i} & \sum_{i=0}^{N} Q_i^{-1} \end{bmatrix} \geq 0, \tag{12}$$

where we can apply Schur's complement and utilise (9) to get

$$\mu T \geq \Sigma \left(\sum_{i=0}^{N} Q_i^{-1}\right)^{-1} \Sigma^*, \tag{13}$$

where $\Sigma \triangleq \sum_{i=0}^{N} A_i e^{-(\lambda + \jmath\omega)\tau_i}$.

Notice that from (8)

$$T \geq \sum_{i=0}^{N} T Q_i^{-1} T, \tag{14}$$

which multiplying through the left and through the right by T^{-1} and calculating its inverse leads to

$$T \leq \left(\sum_{i=0}^{N} Q_i^{-1}\right)^{-1}. \tag{15}$$

Therefore, using this result in (13), implies that

$$\mu T \geq \Sigma T \Sigma^*. \tag{16}$$

Finally, let $s_0 = \lambda + j\omega$ be an eigenvalue of Σ associated with a right-eigenvector v. It is well known, see Strang (2016) and Bulirsch and Stoer (1993), that left and right eigenvalues are equal. Hence, s_0 is also an eigenvalue of Σ associated with a left-eigenvector x_L, with dimension $1 \times n$. In this case, we can multiply inequality (16) to the left by x_L and to the right by its conjugated transpose, x_L^*, obtaining

$$\mu x_L T x_L^* \geq x_L \Sigma T \Sigma^* x_L^* \tag{17}$$

and since $x_L \neq 0$ and $T > 0$,

$$\mu \geq (\lambda + \jmath\omega)(\lambda - \jmath\omega), \tag{18}$$

leading to

$$|s_0| \leq \sqrt{\mu}. \tag{19}$$

Finally, for the uncertain system, it follows immediately from the linear dependence of the LMIs that the solution of the problem in question is obtained calculating it in each one of the vertices, $\ell \in \{1, \ldots, N_v\}$, of the polytope of uncertainties $\mathcal{P}$, given by (2), which concludes the proof. $\square$

3. STABILITY

We shall now see how one can use the envelope to analyse the stability of a retarded time-delay system.

Proposition 2. Let $\lambda = \lambda_0 \in \mathbb{R}$, $\mu = \lambda_0^2 - \varepsilon$, for some $\varepsilon > 0$. If there exist $T, Q_i > 0$, for all $i \in \{0, \ldots, N\}$ such that (8) and (9) are both satisfied, then the envelope lies entirely on the left side of the vertical axis crossing λ_0.

Proof: From (19) we have that if $\lambda + \jmath\omega$ is a root of the system, then

$$|\lambda + \jmath\omega| \leq \sqrt{\lambda_0^2 - \varepsilon}, \tag{20}$$

which can be rewritten as

$$\lambda^2 + \omega^2 \leq \lambda_0^2 - \varepsilon. \tag{21}$$

This expression, clearly, is never going to be satisfied with $\lambda \geq \lambda_0$, which implies that it cannot exist parts of the envelope to the right side of the vertical axis passing through λ_0. $\square$

With this result, we propose a change of coordinates through the new variable $s = z - d$, with $d > 0$. Calculating the envelope for z, (7) becomes

$$\det\left(zI - (A_0 + dI) - \sum_{i=1}^{N} A_i e^{-z\tau_i} e^{d\tau_i}\right) = 0, \tag{22}$$

allowing us to work with an equivalent problem on the new parameters

$$\begin{aligned} \tilde{A}_0 &= A_0 + dI, \\ \tilde{A}_i &= A_i e^{d\tau_i}, \text{ for all } i \in \{1, \ldots, N\}. \end{aligned} \tag{23}$$

From equation (19) we have that $\lambda^2 + \omega^2 \leq \mu$ directly implies $\omega = \pm\sqrt{\mu - \lambda^2}$, for $\mu \geq \lambda^2$. The envelope is then defined on the complex plane by the set of points where $\omega = \pm\sqrt{\mu - \lambda^2}$, when $\mu \geq \lambda^2$. For $\mu < \lambda^2$ we define the envelope as "closed".

Now, if the envelope is closed on the z-plan before $z = d$, it will be closed before the origin on the s-plan, guaranteeing stability for the original system. Hence, the existence of a solution for (9) and (8), for the modified system (23), with $\mu = d^2 - \varepsilon$ and $\lambda = d$, for some $d > 0$ and $\varepsilon > 0$, it implies that the original system (1) is stable.

4. STATE-FEEDBACK

We now address the stabilisation problem. Consider the system

$$\dot{x}(t) = \sum_{i=0}^{N} A_i x(t - \tau_i) + Bu(t), \tag{24}$$

with A_i given by (3), B defined as

$$B := \left\{ \sum_{\ell=1}^{M} \eta^\ell B^\ell, \ \sum_{\ell=1}^{M} \eta^\ell = 1, \ \eta^\ell \geq 0 \right\} \tag{25}$$

and the polytope redefined as

$$\mathcal{P} := co\left\{ \left[A_0^\ell, \cdots, A_N^\ell, B^\ell\right], \ell \in 1, \ldots, N_v \right\}, \tag{26}$$

with $N_v = M\Pi_{i=0}^{N} N_i$.

We want to control this system by means of a state feedback control law $u(t) = \sum_{i=0}^{N} K_i x(t - \tau_i) \in \mathbb{R}^m$ to be designed through LMIs. This controller copes with project requirements and adds a certain degree of robustness to the closed-loop system. As will be shown the controller can be memoryless, i.e., $K_i \leftarrow 0$, for $i \geq 1$ or can even use some of the delayed states.

Theorem 1. There is a state-feedback control that stabilises the system (1) if there exist matrices $T = T' > 0$, $Q_i = Q_i' > 0$, Y_i $\forall i \in \{0, \ldots, N\}$ and positive scalars d, ε, with $\mu = d^2 - \varepsilon$, $\lambda = d$, such that

$$\begin{bmatrix} \mu T & \left(\tilde{A}_0^\ell Q_0 + \tilde{B}_0^\ell Y_0\right) e^{-\lambda\tau_0} & \ldots & \left(\tilde{A}_N^\ell Q_N + \tilde{B}_N^\ell Y_N\right) e^{-\lambda\tau_N} \\ \bullet & Q_0 & 0 & 0 \\ \bullet & 0 & \ddots & 0 \\ \bullet & 0 & 0 & Q_N \end{bmatrix} \geq 0 \tag{27}$$

and (8) are all satisfied, where $\tilde{A}_i$ is given by (23) and $\tilde{B}_i = Be^{d\tau_i}$ for all $i \in \{0, \ldots, N\}$. In this case, the controller matrices are given by $K_i = Y_i Q_i^{-1}$.

Proof: Applying Schur's complement in (27) we get exactly (9) with $A_i \leftarrow \tilde{A}_i + \tilde{B}_i K_i$, which completes the proof. $\square$

Is it possible to go one step further and design a controller that guarantees α-stability. Making the change of variables $z = s + d$, with $d = d^* + \alpha$, $d^* > 0$, $\alpha > 0$, it implies that if an envelope lies completely before d^* on the z-plane, then it will lies completely on the left side of the vertical line $\Re(s) = -\alpha$ on the s-plan.

5. NUMERICAL EXAMPLES

Let us illustrate the results of this work with some examples. For the first example we consider a system without uncertainties. All other examples take into account parametric uncertainties on the system matrices.

Example 2. Consider the following system matrices

$$\left[\begin{array}{c|c} A_0 & A_1 \end{array}\right] = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0.5413 \\ -2 & -3 & -1.0827 & -1.6240 \end{array}\right],$$

with $\tau = 0.4$ and $B = [0 \quad 1]'$, we have an unstable system with poles at 1.3194 and 2.4125. Choosing $\alpha = 1$, $d = \mu(A_0) + \|A_1\|$, where $\mu(.)$ is a matrix measure (T.Mori and Kokame, 1989), and applying Theorem 1 we achieve α-stability as can be seen in Figure 1.

The gains for the controller are

$$[K_0 | K_1] = [\,35.2114 \ -15.2267 | -1.0869 \ -4.5434\,].$$

One remark that is interesting to highlight, it is that in computational terms, for systems without uncertainties is possible to minimise μ through a standard generalised eigenvalue problem approach. However, for the uncertain case this is no longer possible. The problem is then solved by a linear search on μ.

For the next examples, let us describe the uncertainty as $A = A^1 + \Delta A^{2^*}$, with $0 \leq \Delta \leq \Delta_{\max}$. Hence, the vertices of the polytope are A^1 and $A^2 := A^1 + \Delta_{\max} A^{2^*}$. Now, for comparison purposes, we will design two controllers. One for the nominal plant neglecting the uncertainty and, therefore, fixing $\Delta = 0$, and the second one taking the uncertainty into account. For both controllers designed we

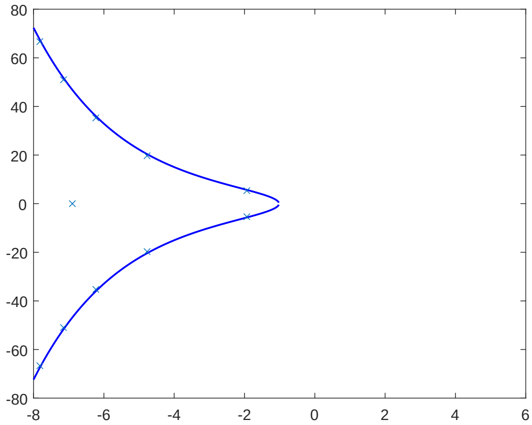

Fig. 1. α-stability, $\alpha = 1$

execute the following procedure. Starting with $\Delta = 0$ we calculate all roots of the closed-loop system characteristic equation using QPmR (Vyhldal (2013)). Incrementing $\Delta = \Delta + p$, where p is small step, e.g., $p = 0.05$, we recalculate the poles and we repeat this procedure until the systems reach instability. With this procedure we can verify that the robustness is not only valid for $0 \le \Delta \le \Delta_{\max}$ but for a higher interval. Two numerical examples are shown below.

Example 3. Consider the following matrices for the system (24)

$$A_0 = \begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix} + \Delta \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix}$$
$$A_1 = \begin{bmatrix} 0 & -0.5413 \\ 1.0827 & 1.6240 \end{bmatrix}$$
$$B = \begin{bmatrix} 0 & 1 \end{bmatrix}',$$

where $0 \le \Delta \le 2.5$ represents a parametric uncertainty and $\tau = 0.4$. Imposing $\Delta = 0$ and applying Theorem 1 to the system obtained, we have $K_1 = [7.1766 \quad -15.0622]$ and $K_2 = [-1.0764 \quad -1.9799]$. For the designed K we get that the system is stable for $0 \le \Delta \le 1.1$.

Now, lets apply Theorem 1 considering all vertices.

$$\left[\begin{array}{c|c} A_0^1 & A_0^2 \end{array}\right] = \left[\begin{array}{cc|cc} 0 & -1 & 2.5 & -1 \\ 2 & 3 & 7 & -4.5 \end{array}\right].$$

The new controller gains obtained are $K_1 = [103.7664 \quad -19.2698]$ and $K_2 = [-1.0823 \quad -6.1536]$. Applying once again the procedure described above, we have that the system is stable for $0 \le \Delta \le 4.9$. Furthermore, for $0 \le \Delta \le 2.5$, not only the poles are on the left half-plane, but our procedure ensures that they are all inside the envelope. A plot with a variety of linear combinations of (3), i.e., different values of Δ, is ploted altogether with the envelope on Figure 2.

Example 4. Let us consider A_0, A_1 and B from the previous example, with $\tau = 0.2$, $0 \le \Delta \le 1.5$ and $\alpha = 1$. Designing a controller for the nominal system we get $K_1 = [43.8852 \quad -18.9928]$, $K_2 = [-1.0825 \quad -3.6587]$. Calculating the poles for each increment of Δ we verify stability for $0 \le \Delta \le 2.9$. Designing the robust controller through Theorem 1, with $\Delta = 1.5$, we get

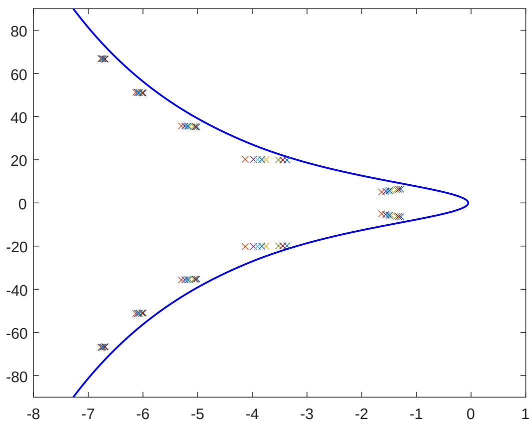

Fig. 2. Envelope for an Uncertain Retarded Time-Delay System

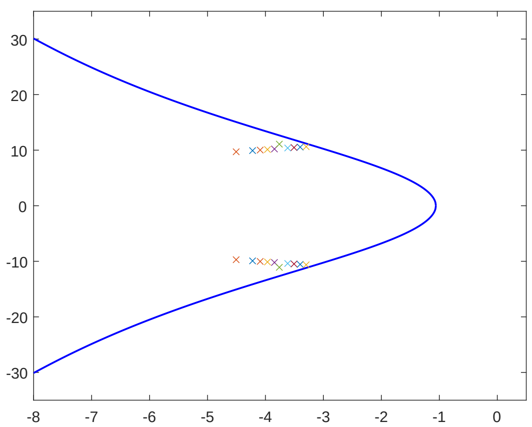

Fig. 3. α-stability Envelope for an Uncertain Retarded Time-Delay System

$K_1 = [126.8156 \quad -23.0812]$, $K_2 = [-1.0826 \quad -6.9346]$ and stability for $0 \le \Delta \le 5.2$. The envelope and the poles of the characteristic equation of (1) for different values of ξ_0^ℓ, respecting (3), can be seem on Figure 3

It is then clear the superiority of the robust controller in comparison to the controller designed for the nominal system when there are parametric uncertainties present.

Example 5. Let us consider the matrices (Li and Souza, 1997a), (Moon et al., 1998) and (Moon et al., 2001)

$$A_0 = \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix} + \Delta \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}$$
$$A_1 = \begin{bmatrix} -1 & 0 \\ -0.8 & -0.1 \end{bmatrix}$$
$$B = \begin{bmatrix} 0 & 1 \end{bmatrix}', \|\Delta\| \le 1.$$

In Li and Souza (1997a) the largest upper bound for the delay is $\tau_{max} = 0.5557$. In Moon et al. (1998) they showed that, in fact, the system is stabilisable for all delays. Applying Theorem 1, choosing properly the vertices that bounds Δ, we confirm that the system is stable for all delays and we get $K_1 = [-18.1576 \quad -1.8716]$, $K_2 =$

$[1.0399 \quad 0.1020]$. We went one step further and achieve independent stabilisation for $\|\Delta\| \leq 12$.

6. CONCLUSION

Through an LMI approach, it was possible to use envelopes not only to study stability but to design robust feedback controllers for retarded time-delay systems. The controller designed is robust to parametric uncertainties and can guarantee delay independent stability ($\alpha = 0$) or delay dependent $\alpha - stability$, for every $\tau \leq \tau^*$. Ongoing work establishes a similar approach for neutral time-delay systems.

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