Technical report on: Multi-Robot Symmetric Formations for Gradient and Hessian Estimation with Application to Source Seeking

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This technical report provides the details and extended mathematical derivations of the results presented in the paper "Multi-Robot Symmetric Formations for Gradient and Hessian Estimation with Application to Source Seeking", published in the IEEE Transactions on Robotics, 2019.

I. Problem formulation

Let us define the signal distribution $\sigma(\mathbf{r})$ as a 3-dimensional spatial function representing the scalar field at location $\mathbf{r}$ achieving its maximum at position $\mathbf{r}^*$ where the source is located and smoothly decreasing to zero far from the source. Formally speaking, we will consider the following assumptions on this signal, where we denote its gradient and Hessian at a location $\mathbf{r}$ as $\nabla\sigma(\mathbf{r})$ and $\mathbf{H}_{\sigma(\mathbf{r})}$ respectively:

Assumption 1 *The function $\sigma : \mathbb{R}^3 \to \mathbb{R}^+$, is three times continuously differentiable, i.e. $\sigma \in \mathcal{C}^3$, and all its partial derivatives up to order three are globally bounded. Moreover, $\nabla\sigma(\boldsymbol{r}^*) = 0$, $\nabla\sigma(\boldsymbol{r}) \neq 0$, $\forall \boldsymbol{r} \neq \boldsymbol{r}^*$, and $\boldsymbol{H}_{\sigma(\boldsymbol{r}^*)} < -a\boldsymbol{I}_3$, where $a > 0$ and $\boldsymbol{I}_3 \in \mathbb{R}^{3\times 3}$ denotes the identity matrix.*

The previous assumption implies that there exist scalar L and M such that

$$|\underbrace{\sigma(\mathbf{r}) - \sigma(\mathbf{c}) - \nabla\sigma(\mathbf{c})^T(\mathbf{r}-\mathbf{c})}_{=:\varphi^{\nabla}(\mathbf{r},\mathbf{c})}| \leq L\|\mathbf{r}-\mathbf{c}\|^2$$

$$|\underbrace{\varphi^{\nabla}(\mathbf{r},\mathbf{c}) - \frac{1}{2}(\mathbf{r}-\mathbf{c})^T\mathbf{H}_{\sigma(\mathbf{c})}(\mathbf{r}-\mathbf{c})}_{=:\varphi^{H}(\mathbf{r},\mathbf{c})}| \leq M\|\mathbf{r}-\mathbf{c}\|^3$$

where $\varphi^{\nabla}(\mathbf{r},\mathbf{c})$ and $\varphi^{H}(\mathbf{r},\mathbf{c})$ correspond to the first and second order remainders of the Taylor expansion about the point $\mathbf{c}$, respectively.

II. 3-D estimation

A. Robots' formation

Consider a group of $N = 2n$, $n \in \mathbb{N}$ agents forming a symmetric configuration composed of two parallel circular formations whose centers are aligned with the $z-$axis. The center point $\mathbf{c} \in \mathbb{R}^3$ is located between the two circles at distance $D\sin\theta_F$ from each one. One half of the agents is uniformly distributed in the upper circular formation, the other half is uniformly distributed in the lower circular formation and thus, their relative vectors with respect to the center point $\mathbf{c}$ are also evenly spaced. The robots' positions in such configuration are expressed in spherical coordinates as follows:

$$\begin{aligned} &\mathbf{r}_i = \mathbf{c} + D\left[s_{\theta_i}c_{\phi_i}, s_{\theta_i}s_{\phi_i}, c_{\theta_i}\right]^T \\ &\theta_i = \begin{cases} \theta_F, & \text{if} \quad i = 2k-1 \\ \pi - \theta_F, & \text{if} \quad i = 2k, \end{cases} \qquad k = 1,\dots,n \end{aligned} \tag{1}$$

where $\mathbf{r}_i \in \mathbb{R}^3$ denotes the position of agent i, $\phi_i = 2\pi i/N$ is the azimuthal angle, c_ϕ and s_ϕ represent the cosine and sine of angle ϕ respectively, D is the radial distance to the center $\mathbf{c}$, θ_i is the polar angle and θ_F is defined such that $s_{\theta_F} = \sqrt{2/3}$.

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Note that this symmetric formation has two interesting and convenient properties. First of all, it is easy to see that all the agents are placed at the same distance from the center, i.e., $\|\mathbf{r}_i - \mathbf{c}\| = D, \forall i$. Moreover, the sum of the relative position vectors of all agents with respect to the center is equal to zero for $N = 2n$, $n \geq 2$, as proven in the sequel:

$$\sum_{i=1}^{N}(\mathbf{r}_i - \mathbf{c}) = D\sum_{i=1}^{N}\begin{bmatrix} s_{\theta_i}c_{\phi_i} \\ s_{\theta_i}s_{\phi_i} \\ c_{\theta_i}\end{bmatrix} = D\begin{bmatrix} s_{\theta_F}\sum_{i=1}^{N}c_{\phi_i} \\ s_{\theta_F}\sum_{i=1}^{N}s_{\phi_i} \\ \sum_{i=2k-1}c_{\theta_F} - \sum_{i=2k}c_{\theta_F}\end{bmatrix} = \mathbf{0}$$

since $\sum_{i=1}^{N}\cos(2\pi i/N) = \sum_{i=1}^{N}\sin(2\pi i/N) = 0$.

B. Gradient estimation

Consider the symmetric formation of agents given by (1) collecting measurements of a 3-D signal distribution $\sigma(\mathbf{r})$. The following theorem is proposed:

Theorem 1 *Assume that $\sigma(\boldsymbol{r}) : \mathbb{R}^3 \to \mathbb{R}$ satisfies Assumption 1 and the agents' formation is given by* (1). *Considering a fleet of $N = 2n$ agents with $n \geq 2$ and define*

$$\widehat{\nabla}\sigma(\boldsymbol{c}) := \frac{3}{ND^2}\sum_{i=1}^{N}\sigma(\boldsymbol{r}_i)(\boldsymbol{r}_i - \boldsymbol{c}) \tag{2}$$

then it holds

$$\|\widehat{\nabla}\sigma(\boldsymbol{c}) - \nabla\sigma(\boldsymbol{c})\| \leq 3LD. \tag{3}$$

Proof: Using the first-order Taylor expansion of each measurement $\sigma(\mathbf{r}_i)$ about the point $\mathbf{c}$ and recalling that $\|\widetilde{\mathbf{r}}_i\| = D$ where $\widetilde{\mathbf{r}}_i := \mathbf{r}_i - \mathbf{c}$, then the following equation holds for all $i = 1, \dots, N$:

$$\sigma(\mathbf{r}_i) - \sigma(\mathbf{c}) = \nabla\sigma(\mathbf{c})^T\widetilde{\mathbf{r}}_i + \varphi^{\nabla}(\mathbf{r}_i, \mathbf{c}),$$

where $\varphi^{\nabla}(\mathbf{r}_i, \mathbf{c})$ denotes the remainder of the Taylor expansion. Multiplying the previous equation by $3\widetilde{\mathbf{r}}_i/(ND^2)$ and summing over $i = 1, \dots, N$, we obtain

$$\frac{3}{ND^2}\sum_{i=1}^{N}\sigma(\mathbf{r}_i)\widetilde{\mathbf{r}}_i - \frac{3\sigma(\mathbf{c})}{ND^2}\sum_{i=1}^{N}\widetilde{\mathbf{r}}_i = \\ \frac{3}{ND^2}\sum_{i=1}^{N}\nabla\sigma(\mathbf{c})^T\widetilde{\mathbf{r}}_i\widetilde{\mathbf{r}}_i + \frac{3}{ND^2}\sum_{i=1}^{N}\varphi^{\nabla}(\mathbf{r}_i, \mathbf{c})\widetilde{\mathbf{r}}_i.$$

Due to the symmetric properties of the formation, we have $\sum_{i=1}^{N}\widetilde{\mathbf{r}}_i = 0$ and thus

$$\widehat{\nabla}\sigma(\mathbf{c}) = \frac{3\nabla\sigma(\mathbf{c})}{ND^2}\sum_{i=1}^{N}\widetilde{\mathbf{r}}_i\widetilde{\mathbf{r}}_i^T + \mathbf{\Psi}^{\nabla}(D, \mathbf{c}), \tag{4}$$

where $\mathbf{\Psi}^{\nabla}(D, \mathbf{c}) = \frac{3}{ND^2}\sum_{i=1}^{N}\varphi^{\nabla}(\mathbf{r}_i, \mathbf{c})\widetilde{\mathbf{r}}_i$. We analyze the second term of the previous equation using (1) to express the position of the agents $\mathbf{r}_i$ to obtain

$$\sum_{i=1}^{N}\widetilde{\mathbf{r}}_i\widetilde{\mathbf{r}}_i^T = D^2\sum_{i=1}^{N}\begin{bmatrix} s^2_{\theta_i}c^2_{\phi_i} & \star & \star \\ s^2_{\theta_i}c_{\phi_i}s_{\phi_i} & s^2_{\theta_i}s^2_{\phi_i} & \star \\ s_{\theta_i}c_{\phi_i}c_{\theta_i} & s_{\theta_i}s_{\phi_i}c_{\theta_i} & c^2_{\theta_i}\end{bmatrix} = D^2\sum_{i=1}^{N}\begin{bmatrix} s^2_{\theta_F}c^2_{\phi_i} & \star & \star \\ s^2_{\theta_F}c_{\phi_i}s_{\phi_i} & s^2_{\theta_F}s^2_{\phi_i} & \star \\ s_{\theta_F}c_{\phi_i}c_{\theta_i} & s_{\theta_F}s_{\phi_i}c_{\theta_i} & c^2_{\theta_i}\end{bmatrix}.$$

The elements depending on the polar angle θ_i must be decomposed to consider the agents with even index $i = 2k$ and the rest with $i = 2k - 1$, $k = 1, \dots, n$, then for $n \geq 2$ we have

$$\sum_{i=1}^{N}c_{\phi_i}c_{\theta_i} = c_{\theta_F}\left(\sum_{i=2k-1}^{n}c_{\phi_i} - \sum_{i=2k}^{n}c_{\phi_i}\right) = 0$$

and the same results holds for the sine. Considering the properties of the proposed formation (1) for which $s^2_{\theta_F} = 2/3$, and applying trigonometric properties, the following equality holds:

$$\sum_{i=1}^{N}\widetilde{\mathbf{r}}_i\widetilde{\mathbf{r}}_i^T = D^2\begin{bmatrix} \frac{N}{2}s^2_{\theta_F} & \star & \star \\ 0 & \frac{N}{2}s^2_{\theta_F} & \star \\ 0 & 0 & Nc^2_{\theta_F}\end{bmatrix} = \frac{ND^2}{3}\mathbf{I}_3,$$

consequently $\widehat{\nabla}\sigma(\mathbf{c}) = \nabla\sigma(\mathbf{c}) + \mathbf{\Psi}^{\nabla}(D,\mathbf{c})$. According to Assumption 1 the term $\mathbf{\Psi}^{\nabla}(D,\mathbf{c})$ satisfies the inequality

$$\|\mathbf{\Psi}^{\nabla}(D,\mathbf{c})\| \leq \frac{3}{ND^2}\sum_{i=1}^{N}|\varphi^{\nabla}(\mathbf{r}_i,\mathbf{c})|\|\mathbf{r}_i-\mathbf{c}\| \leq 3LD.$$

which concludes the proof. ■

Corollary 1 *Let $\sigma(\boldsymbol{r}) : \mathbb{R}^3 \to \mathbb{R}$ be a quadratic function, i.e. $\sigma(\boldsymbol{r}) = \boldsymbol{r}^T\boldsymbol{S}\boldsymbol{r} + \boldsymbol{b}^T\boldsymbol{r} + d$, where $\mathbf{S}$ is negative definite, and consider the agents' formation given by* (1)*, then for a fleet of $N = 2n$ agents with $n \geq 3$ we have $\widehat{\nabla}\sigma(\boldsymbol{c}) = \nabla\sigma(\boldsymbol{c})$.*

Proof: Since $\sigma(\mathbf{r})$ is quadratic the Taylor's reminder can be computed explicitly as

$$\varphi^{\nabla}(\mathbf{r}_i,\mathbf{c}) = \widetilde{\mathbf{r}}_i^T\mathbf{S}\,\widetilde{\mathbf{r}}_i.$$

Consequently, the error term in (4) becomes

$$\mathbf{\Psi}^{\nabla}(D,\mathbf{c}) = \frac{3}{ND^2}\sum_{i=1}^{N}\varphi^{\nabla}(\mathbf{r}_i,\mathbf{c})\widetilde{\mathbf{r}}_i = \frac{3}{ND^2}\sum_{i=1}^{N}\widetilde{\mathbf{r}}_i\widetilde{\mathbf{r}}_i^T\mathbf{S}\,\widetilde{\mathbf{r}}_i$$

According to (1), we obtain

$$\begin{aligned}
\mathbf{\Psi}^{\nabla}(D,\mathbf{c}) =& \frac{3D}{N}\sum_{i=1}^{N}\begin{bmatrix} s_{\theta_i}^2 c_{\phi_i}^2 & \star & \star \\ s_{\theta_i}^2 c_{\phi_i} s_{\phi_i} & s_{\theta_i}^2 s_{\phi_i}^2 & \star \\ s_{\theta_i} c_{\phi_i} c_{\theta_i} & s_{\theta_i} s_{\phi_i} c_{\theta_i} & c_{\theta_i}^2 \end{bmatrix}\begin{bmatrix} S_{11}s_{\theta_i}c_{\phi_i} + S_{12}s_{\theta_i}s_{\phi_i} + S_{13}c_{\theta_i} \\ S_{12}s_{\theta_i}c_{\phi_i} + S_{22}s_{\theta_i}s_{\phi_i} + S_{23}c_{\theta_i} \\ S_{13}s_{\theta_i}c_{\phi_i} + S_{23}s_{\theta_i}s_{\phi_i} + S_{33}c_{\theta_i} \end{bmatrix} \\
=& \frac{3D}{N}\sum_{i=1}^{N}\begin{bmatrix} s_{\theta_F}^2 c_{\phi_i}^2 & \star & \star \\ s_{\theta_F}^2 c_{\phi_i} s_{\phi_i} & s_{\theta_F}^2 s_{\phi_i}^2 & \star \\ s_{\theta_F} c_{\phi_i} c_{\theta_i} & s_{\theta_F} s_{\phi_i} c_{\theta_i} & c_{\theta_i}^2 \end{bmatrix}\begin{bmatrix} S_{11}s_{\theta_F}c_{\phi_i} + S_{12}s_{\theta_F}s_{\phi_i} + S_{13}c_{\theta_i} \\ S_{12}s_{\theta_F}c_{\phi_i} + S_{22}s_{\theta_F}s_{\phi_i} + S_{23}c_{\theta_i} \\ S_{13}s_{\theta_F}c_{\phi_i} + S_{23}s_{\theta_F}s_{\phi_i} + S_{33}c_{\theta_i} \end{bmatrix} \\
=& \frac{3D}{N}\sum_{i=1}^{N}\begin{bmatrix} a_x & a_y & a_z \end{bmatrix}^T
\end{aligned}$$

where $\mathbf{S} = \begin{bmatrix} S_{11} & \star & \star \\ S_{12} & S_{22} & \star \\ S_{13} & S_{23} & S_{33} \end{bmatrix}$ and

$$\begin{aligned}
a_x =& S_{11}s_{\theta_F}^3 c_{\phi_i}^3 + 2S_{12}s_{\theta_F}^3 c_{\phi_i}^2 s_{\phi_i} + 2S_{13}s_{\theta_F}^2 c_{\theta_i} c_{\phi_i}^2 \\
&+ S_{22}s_{\theta_F}^3 c_{\phi_i} s_{\phi_i}^2 + 2S_{23}s_{\theta_F}^2 c_{\theta_i} c_{\phi_i} s_{\phi_i} + S_{33}s_{\theta_F} c_{\theta_i}^2 c_{\phi_i}, \\
a_y =& S_{11}s_{\theta_F}^3 c_{\phi_i}^2 s_{\phi_i} + 2S_{12}s_{\theta_F}^3 c_{\phi_i} s_{\phi_i}^2 + 2S_{13}s_{\theta_F}^2 c_{\theta_i} c_{\phi_i} s_{\phi_i} \\
&+ S_{22}s_{\theta_F}^3 s_{\phi_i}^3 + 2S_{23}s_{\theta_F}^2 c_{\theta_i} s_{\phi_i}^2 + S_{33}s_{\theta_F} c_{\theta_i}^2 s_{\phi_i}, \\
a_z =& S_{11}s_{\theta_F}^2 c_{\theta_i} c_{\phi_i}^2 + 2S_{13}s_{\theta_F} c_{\theta_i}^2 c_{\phi_i} + 2S_{12}s_{\theta_F}^2 c_{\theta_i} c_{\phi_i} s_{\phi_i} \\
&+ S_{22}s_{\theta_F}^2 c_{\theta_i} s_{\phi_i}^2 + 2S_{23}s_{\theta_F} c_{\theta_i}^2 s_{\phi_i} + S_{33}c_{\theta_i}^3.
\end{aligned}$$

Decomposing the terms depending on c_{θ_i} in agents with even and odd index respectively we note that all terms involve a third-order polynomial in sine and cosine and applying trigonometric properties (see Appendix), we obtain that for $n \geq 3$, $a_x = a_y = a_z = 0$, and consequently $\mathbf{\Psi}^{\nabla}(D,\mathbf{c}) = 0$. ■

C. Hessian matrix estimation

Consider the symmetric formation of agents given by (1) taking measurements of a signal distribution $\sigma(\mathbf{r})$. In order to estimate the Hessian matrix of the signal $\sigma(\mathbf{r})$, three additional agents are required, two placed along the $z-$axis of the previous symmetric formation, such that the agents positions are given by (1) with $\theta_{N+1} = 0$ and $\theta_{N+2} = \pi$ and another one placed at the center $\mathbf{c}$ of the formation. The following theorem is proposed:

Theorem 2 *Assume that $\sigma(\boldsymbol{r}) : \mathbb{R}^3 \to \mathbb{R}$ satisfies Assumption 1. Let $N = 2n$ agents deployed as in* (1)*, two additional agents are defined also by* (1) *with $\theta_{N+1} = 0$ and $\theta_{N+2} = \pi$ respectively, and the agent $N+3$ is placed at the center $\mathbf{c}$ of the formation. If $n \geq 4$ then the quantity*

$$\boldsymbol{K}_{\sigma(\boldsymbol{c})} := \frac{18}{ND^4}\sum_{i=1}^{N+2}(\sigma(\boldsymbol{r}_i) - \sigma(\boldsymbol{c}))(\boldsymbol{r}_i - \boldsymbol{c})(\boldsymbol{r}_i - \boldsymbol{c})^T \tag{5}$$

satisfies

$$\Big\|\boldsymbol{K}_{\sigma(\boldsymbol{c})} - \underbrace{\begin{bmatrix} d_1 & \star & \star \\ H_{12} & d_2 & \star \\ 2H_{13} & 2H_{23} & d_3 \end{bmatrix}}_{=:\mathcal{L}(\boldsymbol{H}_{\sigma(\boldsymbol{c})})}\Big\|_F \leq \frac{18(N+2)DM}{N} \tag{6}$$

with

$$\begin{aligned} d_1 =& \frac{3}{2}H_{11} + \frac{1}{2}H_{22} + H_{33} \\ d_2 =& \frac{1}{2}H_{11} + \frac{3}{2}H_{22} + H_{33} \\ d_3 =& H_{11} + H_{22} + H_{33} + \frac{18}{N}H_{33} \end{aligned} \tag{7}$$

where the Hessian matrix at the center of the formation is denoted by $\boldsymbol{H}_{\sigma(\boldsymbol{c})} = \begin{bmatrix} H_{11} & \star & \star \\ H_{12} & H_{22} & \star \\ H_{13} & H_{23} & H_{33} \end{bmatrix}$ *and* $\|\cdot\|_F$ *indicates the Frobenius Norm.*

Proof: Using the first-order Taylor expansion of each measurement $\sigma(\mathbf{r}_i)$ about the point $\mathbf{c}$ and recalling that $\|\widetilde{\mathbf{r}}_i\| = D$, where $\widetilde{\mathbf{r}}_i := \mathbf{r}_i - \mathbf{c}$ then the following equation holds for all $i = 1, \ldots, N+2$:

$$\sigma(\mathbf{r}_i) - \sigma(\mathbf{c}) = \nabla\sigma(\mathbf{c})^T\widetilde{\mathbf{r}}_i + \frac{1}{2}\widetilde{\mathbf{r}}_i^T\mathbf{H}_{\sigma(\mathbf{c})}\widetilde{\mathbf{r}}_i + \varphi^H(\mathbf{r}_i, \mathbf{c}),$$

where $\varphi^H(\mathbf{r}_i, \mathbf{c})$ denotes the remainder of the Taylor expansion. Pre-multiplying the previous equation by $18\widetilde{\mathbf{r}}_i/(ND^4)$ and post-multiplying by $\widetilde{\mathbf{r}}_i^T$, and then summing over $i = 1, \ldots, N+2$, we obtain

$$\begin{aligned} \mathbf{K}_{\sigma(\mathbf{c})} = & \frac{18}{ND^4}\sum_{i=1}^{N+2}\widetilde{\mathbf{r}}_i\nabla\sigma(\mathbf{c})^T\widetilde{\mathbf{r}}_i\widetilde{\mathbf{r}}_i^T + \\ & + \frac{9}{ND^4}\sum_{i=1}^{N+2}\widetilde{\mathbf{r}}_i\widetilde{\mathbf{r}}_i^T\mathbf{H}_{\sigma(\mathbf{c})}\widetilde{\mathbf{r}}_i\widetilde{\mathbf{r}}_i^T + \frac{18}{ND^4}\sum_{i=1}^{N+2}\widetilde{\mathbf{r}}_i\varphi^H(\mathbf{r}_i, \mathbf{c})\widetilde{\mathbf{r}}_i^T \end{aligned} \tag{8}$$

To analyze this equation, we decompose the summation in two parts, first, consider the N agents of the cylindrical formation and second, the two additional ones:

$$\begin{aligned} \sum_{i=1}^{N+2}\widetilde{\mathbf{r}}_i\nabla\sigma(\mathbf{c})^T\widetilde{\mathbf{r}}_i\widetilde{\mathbf{r}}_i^T =& \sum_{i=1}^{N}\widetilde{\mathbf{r}}_i\nabla\sigma(\mathbf{c})^T\widetilde{\mathbf{r}}_i\widetilde{\mathbf{r}}_i^T + \widetilde{\mathbf{r}}_{N+1}\nabla\sigma(\mathbf{c})^T\widetilde{\mathbf{r}}_{N+1}\widetilde{\mathbf{r}}_{N+1}^T + \widetilde{\mathbf{r}}_{N+2}\nabla\sigma(\mathbf{c})^T\widetilde{\mathbf{r}}_{N+2}\widetilde{\mathbf{r}}_{N+2}^T \\ =& \sum_{i=1}^{N}\widetilde{\mathbf{r}}_i\nabla\sigma(\mathbf{c})^T\widetilde{\mathbf{r}}_i\widetilde{\mathbf{r}}_i^T + D^3\begin{bmatrix} 0 & \star & \star \\ 0 & 0 & \star \\ 0 & 0 & \nabla\sigma_z(\mathbf{c}) \end{bmatrix} + D^3\begin{bmatrix} 0 & \star & \star \\ 0 & 0 & \star \\ 0 & 0 & -\nabla\sigma_z(\mathbf{c}) \end{bmatrix} \\ =& D^3\sum_{i=1}^{N}\begin{bmatrix} s_{\theta_i}c_{\phi_i}\nabla\sigma_x(\mathbf{c}) & s_{\theta_i}c_{\phi_i}\nabla\sigma_y(\mathbf{c}) & s_{\theta_i}c_{\phi_i}\nabla\sigma_z(\mathbf{c}) \\ s_{\theta_i}s_{\phi_i}\nabla\sigma_x(\mathbf{c}) & s_{\theta_i}s_{\phi_i}\nabla\sigma_y(\mathbf{c}) & s_{\theta_i}s_{\phi_i}\nabla\sigma_z(\mathbf{c}) \\ c_{\theta_i}\nabla\sigma_x(\mathbf{c}) & c_{\theta_i}\nabla\sigma_y(\mathbf{c}) & c_{\theta_i}\nabla\sigma_z(\mathbf{c}) \end{bmatrix}\begin{bmatrix} s_{\theta_i}^2c_{\phi_i}^2 & \star & \star \\ s_{\theta_i}^2c_{\phi_i}s_{\phi_i} & s_{\theta_i}^2s_{\phi_i}^2 & \star \\ s_{\theta_i}c_{\theta_i}c_{\phi_i} & s_{\theta_i}c_{\theta_i}s_{\phi_i} & c_{\theta_i}^2 \end{bmatrix} \\ =& D^3\sum_{i=1}^{N}\begin{bmatrix} s_{\theta_F}c_{\phi_i}\nabla\sigma_x(\mathbf{c}) & s_{\theta_F}c_{\phi_i}\nabla\sigma_y(\mathbf{c}) & s_{\theta_F}c_{\phi_i}\nabla\sigma_z(\mathbf{c}) \\ s_{\theta_F}s_{\phi_i}\nabla\sigma_x(\mathbf{c}) & s_{\theta_F}s_{\phi_i}\nabla\sigma_y(\mathbf{c}) & s_{\theta_F}s_{\phi_i}\nabla\sigma_z(\mathbf{c}) \\ c_{\theta_i}\nabla\sigma_x(\mathbf{c}) & c_{\theta_i}\nabla\sigma_y(\mathbf{c}) & c_{\theta_i}\nabla\sigma_z(\mathbf{c}) \end{bmatrix}\begin{bmatrix} s_{\theta_F}^2c_{\phi_i}^2 & \star & \star \\ s_{\theta_F}^2c_{\phi_i}s_{\phi_i} & s_{\theta_F}^2s_{\phi_i}^2 & \star \\ s_{\theta_F}c_{\theta_i}c_{\phi_i} & s_{\theta_F}c_{\theta_i}s_{\phi_i} & c_{\theta_i}^2 \end{bmatrix} \\ =& D^3\sum_{i=1}^{N}\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} B_1 =& \nabla\sigma_x(\mathbf{c})s_{\theta_F}^3\begin{bmatrix} c_{\phi_i}^3 & \star \\ c_{\phi_i}^2s_{\phi_i} & c_{\phi_i}s_{\phi_i}^2 \end{bmatrix} + \nabla\sigma_y(\mathbf{c})s_{\theta_F}^3\begin{bmatrix} c_{\phi_i}^2s_{\phi_i} & \star \\ c_{\phi_i}s_{\phi_i}^2 & s_{\phi_i}^3 \end{bmatrix} + \nabla\sigma_z(\mathbf{c})s_{\theta_F}^2\begin{bmatrix} c_{\phi_i}^2c_{\theta_i} & \star \\ c_{\phi_i}s_{\phi_i}c_{\theta_i} & s_{\phi_i}^2c_{\theta_i} \end{bmatrix} \\ B_2 =& \nabla\sigma_x(\mathbf{c})s_{\theta_F}^2\begin{bmatrix} c_{\phi_i}^2c_{\theta_i} \\ c_{\phi_i}^2s_{\phi_i}c_{\theta_i} \end{bmatrix} + \nabla\sigma_y(\mathbf{c})s_{\theta_F}^2\begin{bmatrix} c_{\phi_i}s_{\phi_i}c_{\theta_i} \\ s_{\phi_i}^2c_{\theta_i} \end{bmatrix} + \nabla\sigma_z(\mathbf{c})s_{\theta_F}c_{\theta_F}^2\begin{bmatrix} c_{\phi_i} \\ s_{\phi_i} \end{bmatrix} \\ B_3 =& B_2^T \\ B_4 =& \nabla\sigma_x(\mathbf{c})s_{\theta_F}c_{\theta_i}^2c_{\phi_i} + \nabla\sigma_y(\mathbf{c})s_{\theta_F}c_{\theta_i}^2s_{\phi_i} + \nabla\sigma_z(\mathbf{c})c_{\theta_F}^3. \end{aligned}$$

Following the same steps that in previous cases, applying trigonometric properties and considering robots with even and odd index we obtain that for $n \geq 4$:

$$\sum_{i=1}^{N} B_1 = \mathbf{0}, \quad \sum_{i=1}^{N} B_2 = \mathbf{0}, \quad \sum_{i=1}^{N} B_3 = \mathbf{0}, \quad \sum_{i=1}^{N} B_4 = \mathbf{0},$$

and consequently for $n \geq 4$ the following equation holds:

$$\sum_{i=1}^{N+2} \widetilde{\mathbf{r}}_i \nabla\sigma(\mathbf{c})^T \widetilde{\mathbf{r}}_i \widetilde{\mathbf{r}}_i^T = \mathbf{0}.$$

In order to analyze the second term of the right-side of equation (8) we decompose again the summation in two parts, considering first the N robots of the cylindrical formation and second, the two additional ones:

$$\begin{aligned}
\sum_{i=1}^{N+2} \widetilde{\mathbf{r}}_i \widetilde{\mathbf{r}}_i^T \mathbf{H}_{\sigma(\mathbf{c})} \widetilde{\mathbf{r}}_i \widetilde{\mathbf{r}}_i^T =& \sum_{i=1}^{N} \widetilde{\mathbf{r}}_i \widetilde{\mathbf{r}}_i^T \mathbf{H}_{\sigma(\mathbf{c})} \widetilde{\mathbf{r}}_i \widetilde{\mathbf{r}}_i^T + \widetilde{\mathbf{r}}_{N+1} \widetilde{\mathbf{r}}_{N+1}^T \mathbf{H}_{\sigma(\mathbf{c})} \widetilde{\mathbf{r}}_{N+1} \widetilde{\mathbf{r}}_{N+1}^T + \widetilde{\mathbf{r}}_{N+2} \widetilde{\mathbf{r}}_{N+2}^T \mathbf{H}_{\sigma(\mathbf{c})} \widetilde{\mathbf{r}}_{N+2} \widetilde{\mathbf{r}}_{N+2}^T \\
=& \sum_{i=1}^{N} \widetilde{\mathbf{r}}_i \widetilde{\mathbf{r}}_i^T \mathbf{H}_{\sigma(\mathbf{c})} \widetilde{\mathbf{r}}_i \widetilde{\mathbf{r}}_i^T + D^4 \begin{bmatrix} 0 & \star & \star \\ 0 & 0 & \star \\ 0 & 0 & H_{33} \end{bmatrix} + D^4 \begin{bmatrix} 0 & \star & \star \\ 0 & 0 & \star \\ 0 & 0 & H_{33} \end{bmatrix} \\
=& D^4 \sum_{i=1}^{N} \begin{bmatrix} s_{\theta_i}^2 c_{\phi_i}^2 & \star & \star \\ s_{\theta_i}^2 c_{\phi_i} s_{\phi_i} & s_{\theta_i}^2 s_{\phi_i}^2 & \star \\ s_{\theta_i} c_{\theta_i} c_{\phi_i} & s_{\theta_i} c_{\theta_i} s_{\phi_i} & c_{\theta_i}^2 \end{bmatrix} \begin{bmatrix} H_{11} & \star & \star \\ H_{12} & H_{22} & \star \\ H_{13} & H_{23} & H_{33} \end{bmatrix} \begin{bmatrix} s_{\theta_i}^2 c_{\phi_i}^2 & \star & \star \\ s_{\theta_i}^2 c_{\phi_i} s_{\phi_i} & s_{\theta_i}^2 s_{\phi_i}^2 & \star \\ s_{\theta_i} c_{\theta_i} c_{\phi_i} & s_{\theta_i} c_{\theta_i} s_{\phi_i} & c_{\theta_i}^2 \end{bmatrix} \\
&+ D^4 \begin{bmatrix} 0 & \star & \star \\ 0 & 0 & \star \\ 0 & 0 & 2H_{33} \end{bmatrix} \\
=& D^4 \sum_{i=1}^{N} \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} + D^4 \begin{bmatrix} 0 & \star & \star \\ 0 & 0 & \star \\ 0 & 0 & 2H_{33} \end{bmatrix},
\end{aligned}$$

where

$$\begin{aligned}
C_1 =& s_{\theta_i}^4 \begin{bmatrix} H_{11} c_{\phi_i}^4 + 2H_{12} c_{\phi_i}^3 s_{\phi_i} + H_{22} c_{\phi_i}^2 s_{\phi_i}^2 & \star \\ H_{11} c_{\phi_i}^3 s_{\phi_i} + 2H_{12} c_{\phi_i}^2 s_{\phi_i}^2 + H_{22} c_{\phi_i} s_{\phi_i}^3 & H_{11} c_{\phi_i}^2 s_{\phi_i}^2 + 2H_{12} c_{\phi_i} s_{\phi_i}^3 + H_{22} s_{\phi_i}^4 \end{bmatrix} + 2H_{13} s_{\theta_i}^3 \begin{bmatrix} c_{\phi_i}^3 c_{\theta_i} & \star \\ c_{\phi_i}^2 s_{\phi_i} c_{\theta_i} & c_{\phi_i} s_{\phi_i}^2 \end{bmatrix} \\
&+ 2H_{23} s_{\theta_i}^3 \begin{bmatrix} c_{\phi_i}^2 s_{\phi_i} c_{\theta_i} & \star \\ c_{\phi_i} s_{\phi_i}^2 c_{\theta_i} & s_{\phi_i}^3 c_{\theta_i} \end{bmatrix} + H_{33} s_{\theta_i}^2 c_{\theta_i}^2 \begin{bmatrix} c_{\phi_i}^2 & \star \\ c_{\phi_i} s_{\phi_i} & s_{\phi_i}^2 \end{bmatrix} \\
C_2 =& s_{\theta_i}^3 c_{\theta_i} \begin{bmatrix} H_{11} c_{\phi_i}^3 + 2H_{12} c_{\phi_i}^2 s_{\phi_i} + H_{22} c_{\phi_i} s_{\phi_i}^2 \\ H_{11} c_{\phi_i}^2 s_{\phi_i} + 2H_{12} c_{\phi_i} s_{\phi_i}^2 + H_{22} s_{\phi_i}^3 \end{bmatrix} + 2 s_{\theta_i}^2 c_{\theta_i}^2 \begin{bmatrix} H_{13} c_{\phi_i}^2 + H_{23} c_{\phi_i} s_{\phi_i} \\ H_{13} c_{\phi_i} s_{\phi_i} + H_{23} s_{\phi_i}^2 \end{bmatrix} + H_{33} s_{\theta_i} c_{\theta_i}^3 \begin{bmatrix} c_{\phi_i} \\ s_{\phi_i} \end{bmatrix} \\
C_3 =& C_2^T \\
C_4 =& s_{\theta_i}^2 c_{\theta_i}^2 (H_{11} c_{\phi_i}^2 + 2H_{12} c_{\phi_i} s_{\phi_i} + H_{22} s_{\phi_i}^2) + s_{\theta_i} c_{\theta_i}^3 (H_{13} c_{\phi_i} + H_{23} s_{\phi_i}) + H_{33} c_{\theta_i}^4.
\end{aligned}$$

Applying trigonometric properties, the fact that $s_{\theta_F} = \sqrt{2/3}$ and considering robots with even and odd index we obtain that for $n \geq 4$:

$$\frac{9}{ND^4} \sum_{i=1}^{N+2} \widetilde{\mathbf{r}}_i \widetilde{\mathbf{r}}_i^T \mathbf{H}_{\sigma(\mathbf{c})} \widetilde{\mathbf{r}}_i \widetilde{\mathbf{r}}_i^T = \begin{bmatrix} \frac{3}{2} H_{11} + \frac{1}{2} H_{22} + H_{33} & \star & \star \\ H_{12} & \frac{1}{2} H_{11} + \frac{3}{2} H_{22} + H_{33} & \star \\ 2H_{13} & 2H_{23} & H_{11} + H_{22} + H_{33} + \frac{18}{N} H_{33} \end{bmatrix}.$$

The error term defined by

$$\mathbf{\Psi}^H(D, \mathbf{c}) = \frac{18}{ND^4} \sum_{i=1}^{N+2} \widetilde{\mathbf{r}}_i \varphi^H(\mathbf{r}_i, \mathbf{c}) \widetilde{\mathbf{r}}_i^T$$

satisfies

$$\|\mathbf{\Psi}^H(D, \mathbf{c})\|_F \leq \frac{18}{ND^4} \sum_{i=1}^{N+2} |\varphi^H(\mathbf{r}_i, \mathbf{c})| \|\widetilde{\mathbf{r}}_i\|^2 \leq \frac{18(N+2)DM}{N}$$

which concludes the proof. ■

Note that once $\mathbf{K}_{\sigma(\mathbf{c})}$ is computed, an estimate $\widehat{\mathbf{H}}_{\sigma(\mathbf{c})}$ for the true Hessian $\mathbf{H}_{\sigma(\mathbf{c})}$ is given by the unique solution of the linear system

$$\mathcal{L}(\widehat{\mathbf{H}}_{\sigma(\mathbf{c})}) = \mathbf{K}_{\sigma(\mathbf{c})} \tag{9}$$

which can be obtained by inverting the operator $\mathcal{L}$.

Corollary 2 *Let $\sigma(\boldsymbol{r}) : \mathbb{R}^3 \to \mathbb{R}$ be a quadratic function, i.e. $\sigma(\boldsymbol{r}) = \boldsymbol{r}^T\boldsymbol{S}\boldsymbol{r} + \boldsymbol{b}^T\boldsymbol{r} + a$, where $\boldsymbol{S}$ is negative definite. Considering $N = 2n$ robots deployed as in (1), two additional robots are defined also by (1) with $\theta_{N+1} = 0$ and $\theta_{N+2} = \pi$ respectively, and the robot $N+3$ is placed at the center $\boldsymbol{c}$ of the formation. Then for $n \geq 4$ the matrix $\boldsymbol{K}_{\sigma(\boldsymbol{c})}$ defined in (5) satisfies $\boldsymbol{K}_{\sigma(\boldsymbol{c})} = \mathcal{L}(\boldsymbol{H}_{\sigma(\boldsymbol{c})})$ and therefore $\widehat{\boldsymbol{H}}_{\sigma(\boldsymbol{c})} = \boldsymbol{H}_{\sigma(\boldsymbol{c})}$.*

Proof: The results directly follow from Theorem 2, since in the case of quadratic functions the Taylor's remainders are zero, i.e. $\varphi^H(\mathbf{r}_i, \mathbf{c}) = 0, \forall i = 1, \dots, N$, therefore $\boldsymbol{\Psi}^H(D, \mathbf{c}) = 0$. ■

III. Noise and approximation error analysis

In this section we address the problem of noisy measurements and their effect on the gradient and Hessian estimation, and we compare it with the approximation errors of equations (3) and (6). We limit the analysis to the 3-D scenario. We assume that the measurements are corrupted by Gaussian zero-mean white noise which models possible small scale spatial variations due to turbulence or local perturbation:

$$y(\mathbf{r}_i) = \sigma(\mathbf{r}_i) + \upsilon_i(\mathbf{r}_i), \quad \upsilon_i(\mathbf{r}_i) \sim \mathcal{N}(0, \upsilon^2)$$

where υ^2 represents the variance of the noise. We further assume that the noise is independent in each robot measurement, i.e. $\mathbb{E}[\upsilon_i(\mathbf{r}_i)\upsilon_j(\mathbf{r}_j)] = 0, i \neq j$ which is a realistic assumption if $\|\mathbf{r}_i - \mathbf{r}_j\| > D_{min}$ where D_{min} represents the spatial correlation distance of the disturbance. We start by considering the effect on the gradient estimation. Since the noise is additive in the measurement, the gradient estimate becomes

$$\widehat{\nabla}\sigma(\mathbf{c}) := \frac{3}{ND^2}\sum_{i=1}^{N}\sigma(\mathbf{r}_i)(\mathbf{r}_i - \mathbf{c}) + \underbrace{\frac{3}{ND^2}\sum_{i=1}^{N}\upsilon_i(\mathbf{r}_i - \mathbf{c})}_{\boldsymbol{\Psi}_\upsilon^\nabla(D,\mathbf{c})}$$

The term $\boldsymbol{\Psi}_\upsilon^\nabla(D, \mathbf{c})$ represents the effect of noise. It has clearly zero mean, i.e. $\mathbb{E}[\boldsymbol{\Psi}_\upsilon^\nabla(D, \mathbf{c})] = 0$ and its expected standard deviation is given by:

$$\mathsf{s}_\nabla(D) := \sqrt{\mathbb{E}[\|\boldsymbol{\Psi}_\upsilon^\nabla(D, \mathbf{c})\|^2]} = \frac{3}{ND^2}(\mathbb{E}[\left(\sum_{i=1}^{N}\upsilon_i\widetilde{\mathbf{r}}_i\right)^T\left(\sum_{i=1}^{N}\upsilon_i\widetilde{\mathbf{r}}_i\right)])^{\frac{1}{2}} = \frac{3}{ND^2}(\mathbb{E}[\sum_{i=1}^{N}\sum_{j=1}^{N}\upsilon_i\upsilon_j\widetilde{\mathbf{r}}_i^T\widetilde{\mathbf{r}}_j])^{\frac{1}{2}}$$

$$= \frac{3\upsilon}{ND^2}\left(\sum_{i=1}^{N}\widetilde{\mathbf{r}}_i^T\widetilde{\mathbf{r}}_i\right)^{\frac{1}{2}} = \frac{3\upsilon}{N^{\frac{1}{2}}D},$$

where we used the results from the proof of Theorem 1. Note that this term is a monotonically decreasing function of the formation radius and the number of robots. This term has to be compared with the approximation error provided in (3) which is monotonically increasing with the formation radius. This observation shows a clear trade-off when deciding the formation radius in order to balance the effect of approximation error and the noise error. As a rule of thumb we can try to design the formation radius D_∇^* to exactly balance these two errors:

$$\mathsf{s}_\nabla(D_\nabla^*) = 3LD_\nabla^* \Longrightarrow D_\nabla^* = \frac{\upsilon^{\frac{1}{2}}}{L^{\frac{1}{2}}N^{\frac{1}{4}}}, \quad \mathsf{s}_\nabla(D_\nabla^*) = \frac{3(\upsilon L)^{\frac{1}{2}}}{N^{\frac{1}{4}}}$$

which provide a rule-of-thumb to chose the formation radius if L, υ and N are known. The previous expression shows that both the minimal estimation error and the optimal formation radius decrease as the number of robots increases, however it is important to keep in mind that we must have $\|\mathbf{r}_i - \mathbf{r}_j\| > D_{min}$ for the previous equation to be realistic. Considering the cylindrical formation (1) we have that the minimal distance between two robots in the formation is $\min\|\mathbf{r}_i - \mathbf{r}_j\| = 2\sqrt{2/3}D_\nabla^*\sin(\pi/n)$ and then $D_\nabla^* > D_{min}/(2\sqrt{2/3}\sin(\pi/n))$ for the previous equation to be realistic.

We now consider the effect on the Hessian estimation. Since even in this case the noise is additive in the estimate of the Hessian, it is straightforward to see that to the estimate $\widehat{\mathbf{H}}_{\sigma(\mathbf{c})}$ there is the additional term:

$$\boldsymbol{\Psi}_\upsilon^H(D, \mathbf{c}) = \frac{18}{ND^4}\sum_{i=1}^{N+2}(\upsilon_i - \upsilon_c)(\mathbf{r}_i - \mathbf{c})(\mathbf{r}_i - \mathbf{c})^T$$

Such matrix has clearly zero mean, while the expected the expected squared Frobenius norm is given by:

$$
\begin{aligned}
\mathbb{E}[\|\mathbf{\Psi}_v^H(D,\mathbf{c})\|_F^2] &= \mathbb{E}\left[\text{trace}(\mathbf{\Psi}_v^H(D,\mathbf{c})(\mathbf{\Psi}_v^H(D,\mathbf{c}))^T)\right] \\
&= \frac{18^2}{N^2D^8}\mathbb{E}\left[\text{trace}\left(\left(\sum_{i=1}^{N+2}(v_i-v_c)\widetilde{\mathbf{r}}_i\widetilde{\mathbf{r}}_i^T\right)\left(\sum_{i=1}^{N+2}(v_i-v_c)\widetilde{\mathbf{r}}_i\widetilde{\mathbf{r}}_i^T\right)^T\right)\right] \\
&= \frac{18^2}{N^2D^8}\mathbb{E}\left[\text{trace}\left(\sum_{i=1}^{N+2}\sum_{j=1}^{N+2}(v_i-v_c)(v_j-v_c)\widetilde{\mathbf{r}}_i\widetilde{\mathbf{r}}_i^T\widetilde{\mathbf{r}}_j\widetilde{\mathbf{r}}_j^T\right)\right] \\
&= \frac{18^2}{N^2D^8}\mathbb{E}\left[\sum_{i=1}^{N+2}\sum_{j=1}^{N+2}(v_i-v_c)(v_j-v_c)\text{trace}\left(\widetilde{\mathbf{r}}_i^T\widetilde{\mathbf{r}}_j\widetilde{\mathbf{r}}_j^T\widetilde{\mathbf{r}}_i\right)\right] \\
&= \frac{18^2}{N^2D^8}\mathbb{E}\left[\sum_{i=1}^{N+2}\sum_{j=1}^{N+2}(v_i-v_c)(v_j-v_c)(\widetilde{\mathbf{r}}_i^T\widetilde{\mathbf{r}}_j)^2\right] \\
&= \frac{18^2}{N^2D^8}\left(\sum_{i=1}^{N+2}\mathbb{E}[v_i^2]\|\widetilde{\mathbf{r}}_i\|^4+\sum_{i=1}^{N+2}\sum_{j=1}^{N+2}\mathbb{E}[v_c^2](\widetilde{\mathbf{r}}_i^T\widetilde{\mathbf{r}}_j)^2\right) = \frac{18^2v^2}{N^2D^8}\left(D^4(N+2)+\sum_{i=1}^{N+2}\sum_{j=1}^{N+2}(\widetilde{\mathbf{r}}_i^T\widetilde{\mathbf{r}}_j)^2\right) \\
&= \frac{18^2v^2}{N^2D^8}\left(D^4(N+2)+D^4\sum_{i=1}^{N+2}\sum_{j=1}^{N+2}(c_{\phi_i}c_{\phi_j}s_{\theta_i}s_{\theta_j}+s_{\phi_i}s_{\phi_j}s_{\theta_i}s_{\theta_j}+c_{\theta_i}c_{\theta_j})^2\right) \\
&= \frac{18^2v^2}{N^2D^4}\left((N+2)+4+4Nc_{\theta_F}^2+\sum_{i=1}^{N}\sum_{j=1}^{N}(c_{(\phi_i-\phi_j)}s_{\theta_i}s_{\theta_j}+c_{\theta_i}c_{\theta_j})^2\right) \\
&= \frac{18^2v^2}{N^2D^4}\left((N+2)+4+4Nc_{\theta_F}^2+\sum_{i=1}^{N}\sum_{j=1}^{N}(c^2_{(\phi_i-\phi_j)}s^4_{\theta_F}+c^4_{\theta_F})^2\right) \\
&= \frac{18^2v^2}{N^2D^4}\big(N+2+(\frac{N}{3}+2)^2\big)
\end{aligned}
$$

Thus, the expected standard deviation is can be expressed as follows:

$$
\begin{aligned}
\mathsf{s}_H(D) &:= (\mathbb{E}[\|\mathbf{\Psi}_v^H(D,\mathbf{c})\|_F^2])^{\frac{1}{2}} \\
&= \frac{18v}{ND^2}\big(N+2+(\frac{N}{3}+2)^2\big)^{\frac{1}{2}} \approx \frac{6v}{D^2}
\end{aligned}
$$

where the last approximation is obtained for $N \gg 1$. As compared to the effect of noise in the gradient estimation $\mathsf{s}_\nabla(D)$, the standard deviation $\mathsf{s}_H(D)$ does not decrease to zero as the number of robots increases, and it is also more sensitive to the formation radius since it is inversely proportional to the square of the radius. This last fact is to be expected since we are estimating second order derivatives via finite differences and therefore measurement noise is amplified more than when computing first derivatives.

Similar consideration can be derived for the Hessian in terms of finding the optimal radius D_H^* to balance the approximation error and the noise error in the scenario $N \gg 1$:

$$
\mathsf{s}_H(D_\nabla^*)=18MD_H^* \Longrightarrow D_H^*=\Big(\frac{v}{3M}\Big)^{\frac{1}{3}},\ \ \mathsf{s}_H(D_H^*)=\frac{18(vM)^{\frac{1}{3}}}{3^{\frac{1}{3}}}
$$

From a practical perspective, the choice of the optimal radius formation should be chosen to balance all approximation errors and noise effects in both the gradient and the Hessian. Since the noise error increases rapidly for decreasing formation radius a sensible choice for the radius is the largest between D_∇^* and D_H^*.

Appendix

This appendix presents several relations and trigonometric properties used to prove the lemmas and corollaries proposed in this paper. An important property of the uniformly distributed circular formation exploited in this paper is that $\sum_{i=1}^N \cos(\frac{m2\pi}{N}) = \sum_{i=1}^N \sin(\frac{m2\pi}{N}) = 0$, for $N > m$. To prove this result, consider the well known geometric series property $\sum_{i=1}^N r^i = \frac{r(1-r^N)}{1-r}$, $r \neq 1$. If we take $r = e^{\frac{m2\pi}{N}j}$ where $j^2 = -1$ and $N > m$ to avoid the situation in which $r = 1$, then

$$
\sum_{i=1}^N e^{\frac{m2\pi}{N}ij} = \frac{e^{\frac{m2\pi}{N}j}(1-e^{m2\pi j})}{1-e^{\frac{m2\pi}{N}j}} = 0
$$

since $e^{m2\pi j} = \cos(m2\pi) + j\sin(m2\pi) = 1$, $\forall m$. Combining this result with trigonometric properties we obtain the following useful equations:

$$\begin{aligned}
&c_\phi^2 = (1 + c_{(2\phi)})/2, \quad c_\phi^2 s_\phi = s_\phi - s_\phi^3 \quad c_\phi s_\phi^2 = c_\phi - c_\phi^3,\\
&c_\phi^3 = (3c_\phi + c_{(3\phi)})/4, \quad c_\phi^3 s_\phi = 3s_{(2\phi)}/8 + (s_{(4\phi)} - s_{(2\phi)})/8,\\
&s_\phi^3 = (3s_\phi - s_{(3\phi)})/4, \quad c_\phi s_\phi^3 = 3s_{(2\phi)/}8 - (s_{(4\phi)} + s_{(2\phi)})/8,\\
&c_\phi^4 = 3/8 + c_{(2\phi)}/2 + c_{(4\phi)}/8, \quad c_\phi s_\phi = s_\phi/2,\\
&s_\phi^4 = c_\phi^4 - c_{(2\phi)}, \quad c_\phi^2 s_\phi^2 = 1/2 + c_{(2\phi)}/2 - c_\phi^4.
\end{aligned}$$