Some formal proofs of isomorphy and discontinuity

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In computable analysis a representation for a space X is a partial surjective mapping from the Baire space $\mathbb{N}^{\mathbb{N}}$ to X, i.e., a function $\delta\colon \subseteq \mathbb{N}^{\mathbb{N}} \to X$. A pair $\mathbf{X} = (X, \delta_{\mathbf{X}})$ of a space and its representation is called a represented space and an element φ of Baire space is called name of $x \in \mathbf{X}$ if $\delta(\varphi) = x$. Through a representation the notions of computability and continuity of operators on Baire space can be transfered to any represented space $\mathbf{X}$. Popular topics in computable analysis are proving mathematical problems computable or, if this is impossible, classifying their degree of incomputability [BG11, BDBP12, PS18]. A problem that often appears in such classifications is closed choice, where the task is "given a non-empty closed set $A \in \mathcal{A}(\mathbf{X})$ select an element $a \in A$". Here, a closed subset of a represented space is given by specifying positive information about its complement. Thus, for most choices of $\mathbf{X}$, this task is uncomputable and even discontinuous.

More formally, $\mathcal{A}(\mathbf{X})$ is defined by use of the space of open subsets $\mathcal{O}(\mathbf{X})$, which in turn (following for instance [Pau16]) can abstractly be described as the space of continuous functions from $\mathbf{X}$ to the Sierpiński space $\mathbb{S}$. Here, $\mathbb{S}$ has the two point set $\{\bot, \top\}$ as underlying set and the total function $\delta_{\mathbb{S}}$ specified by

$$\delta_{\mathbb{S}}(\varphi) = \top \quad\iff\quad \exists n \in \mathbb{N}\ \varphi(n) \neq 0$$

as representation. The function space construction from computable analysis [Wei00, Definition 3.3.13] provides a representation $[\delta_{\mathbf{X}} \to \delta_{\mathbb{S}}]$ of the continuous functions from $\mathbf{X}$ to $\mathbb{S}$ and we call the resulting represented space $\mathbb{S}^{\mathbf{X}}$. Next, identify a subset U of $\mathbf{X}$ with its characteristic function

$$\chi_U : \mathbf{X} \to \mathbb{S}, \quad \chi_U(x) := \begin{cases} \top & \text{if } x \in U, \\ \bot & \text{otherwise.} \end{cases}$$

Conveniently, χ_U is continuous if and only if U is open and therefore $\mathcal{O}(\mathbf{X})$ can be identified with $\mathbb{S}^{\mathbf{X}}$. Finally, $\mathcal{A}(\mathbf{X})$ is represented as the complements of opens, i.e., following the above

$$\delta_{\mathcal{A}(\mathbf{X})}(\varphi) = A \quad\iff\quad [\delta_{\mathbf{X}} \to \delta_{\mathbb{S}}](\varphi) = \chi_{X\setminus A}.$$

The task of closed choice on $\mathbf{X}$ is formalized as finding a realizer (in the sense of function realizabilty) of the multivalued function $C_{\mathbf{X}}\colon \subseteq \mathcal{A}(\mathbf{X}) \rightrightarrows \mathbf{X}$ defined by $C_{\mathbf{X}}(A) := A$. Or in words: a is an acceptable return value of $C_{\mathbf{X}}$ on input A if and only if a is an element of A. Note that this in particular means that the domain of $C_{\mathbf{X}}$ are the non-empty subsets of $\mathbf{X}$ and that a realizer can behave arbitrarily outside of the domain, i.e., no solution needs to be produced in this case.

The function space construction is quite complicated and for concrete spaces it is often possible to use simpler representations. For instance, one may make use of the fact that there exist infinite products and indeed $\prod_{n\in\mathbb{N}} \mathbf{X}$ is isomorphic to the function space $\mathbf{X}^{\mathbb{N}}$.

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Thus $\mathcal{O}(\mathbb{N}) \cong \prod_{i\in\mathbb{N}} \mathbb{S}$, where the right hand side uses the infinite tupling on Baire space as replacement for the more complicated function space construction. An even more concrete description of $\mathcal{O}(\mathbb{N})$ can be obtained by using the enumeration representation, where a name of an open set enumerates its elements. The representation of the corresponding concrete space $\mathbf{A}_\mathbb{N}$ of the closed subsets of the natural numbers is given by

$$\delta_{\mathbf{A}_\mathbb{N}}(\varphi) = \mathbb{N} \setminus \{n \in \mathbb{N} \mid \exists m \in \mathbb{N},\ \varphi(m) = n + 1\}.$$

That is, the information a name specifies about a closed set is an enumeration of its complement.

We formalized the proofs $\mathcal{A}(\mathbb{N}) \simeq \mathbf{A}_\mathbb{N}$ and also the isomorpy of the concrete and abstract spaces of open sets. Some of the steps in the proof are carried out in more generality than needed, for instance we prove $\mathbf{X}^\mathbb{N} \cong \prod_{i\in\mathbb{N}} \mathbf{X}$ and $\mathcal{O}(\mathbf{X}) \cong \mathcal{A}(\mathbf{X})$ (via taking the complement and realized by the identity) for an arbitrary represented spaces. The formalization is carried out in coqrep [Ste18], a coq library for computable analysis which is being developed by one of the authors. As an application we formalized a proof that closed choice on the naturals as function from $\mathbf{A}_\mathbb{N}$ to $\mathbb{N}$ is discontionuous and used the isomorphisms to conclude that the same is true for $C_\mathbb{N}$ in its original definition using $\mathcal{A}(\mathbb{N})$ as source space. Our proofs of isomorphy proceed by specifying a subset of the needed algorithms concretely, proving them correct and obtaining the remaining translations through compositionality. In particular we retain full executability even though some of our correctness proofs use classical reasoning for convenience. For obvious reasons the discontinuity statements are exempt from the executability claims.

The results themselves are well known and were picked for having fairly straight forward proofs while at the same time being well suited as examples for full formal proofs in the library that require some of the more advanced features it offers like the function space and infinite product constructions. During our work we also suplemented some previously missing parts to the library, for instance a full justification of the infinite product construction by means of a proof of the corresponding universal property. The whole project has meanwhile been made part of the coqrep library.

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