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ON THE RIGID-LID APPROXIMATION OF SHALLOW WATER BINGHAM MODEL

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ABSTRACT. This paper discusses the well posedness of an initial value problem describing the motion of a Bingham fluid in a basin with a degenerate bottom topography. A physical interpretation of such motion is discussed. The system governing such motion is obtained from the Shallow Water-Bingham models in the regime where the Froude number degenerates, i.e taking the limit of such equations as the Froude number tends to zero. Since we are considering equations with degenerate coefficients, then we shall work with weighted Sobolev spaces in order to establish the existence of a weak solution. In order to overcome the difficulty of the discontinuity in Bingham's constitutive law, we follow a similar approach to that introduced in [G. DUVAUT and J.-L. LIONS, Springer-Verlag, 1976]. We study also the behavior of this solution when the yield limit vanishes. Finally, a numerical scheme for the system in 1D is furnished.

Keywords. Bingham fluid, Muckenhoupt weights, Variational inequality, Low Froude number.
AMS subject classification: 35Q30, 76N10, 35B65, 35D35.

1. INTRODUCTION

Bingham fluids constitute a crucial topic of study on which many applicable researches are conducted. As an interesting example we suggest the snow that appears in some important natural phenomena such as avalanches. Technically speaking, the avalanche dynamics can be described through different perspectives, such as the center of mass consideration [27] and the density one. In addition, many approaches consider the avalanche to be a deformable body whereas others describe it as a granular material. Not only the dynamics of snow, but also its constitutive behavior can be described according to different points of views such as Newtonian fluid and Bingham fluid. In many literatures, snow is considered to be a non-Newtonian fluid. One of the reasons is that Newtonian fluids adapt rapidly to deform themselves where they reach a negligible depth in an unbounded space, on the other hand, snow will pile in such a case and thus will rest with a finite depth. This can be explained by the fact that upon being in rest state, snow achieves a yield value with a non zero shear stress, and thus a threshold stress value should be exceeded to start deformation. Another reason is the experimental results of the analysis of the avalanche's velocity profile along its depth, which reveals the viscosity dependence of the shear [24] [26]. These two reasons triggered many authors in the literature to treat snow as a Bingham fluid which is characterized mainly by a non constant stress that depends on viscosity and that may differ in the same body of mass. Hence, one would notice some portions flowing while others moving in a bulk motion as solids. More precisely, at low shear stress, Bingham fluids possess high viscosity, and thus they behave as a shear thinning fluid at low or zero speed. As the shear stress reaches a certain limit-denoted yield stress- one notices a sudden drop in the viscosity, while above the yield stress, snow behaves like a low viscosity liquid.

The need to use reduced models arises from the difficulty that the three dimensional Cauchy momentum equations induce in the analytical and numerical studies, especially if we consider a fee boundary case. In this sequel, the reduced model adopted is the Shallow water-Bingham model that is derived using depth integration of the Navier-Stokes-Bingham system with free surface, taking into account that the horizontal length scale is much greater than the vertical one. Since there is no well posedness result on the initial system and the approximated one, then any mathematical justification of this procedure is still as far as we know an interesting open problem due to the wide range of applicability especially in numerical applications. For instance, in [9], [16], the authors proposed several models of a Shallow water type system for Bingham fluid. The resulting system is given by

$$\begin{cases} \partial_t h + \operatorname{div}(hv) = 0 \\ \partial_t(hv) + \operatorname{div}(hv \otimes v) - \operatorname{div}(\sigma) + \frac{1}{2\,\mathrm{Fr}^2}\nabla h^2 = hf, \end{cases} \tag{1.1}$$

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where $h(t,x)$ is the water height, $v(t,x)$ is the horizontal velocity. We denote by $hv\otimes v$ the matrix with component hv_iv_j, Fr is the Froude number and σ is the stress tensor given by

$$\sigma = \begin{cases} 2\mu h\,\mathrm{D}(v) + \lambda h \operatorname{div} v\,\mathbb{I} + gh\dfrac{\mathrm{D}(v)}{|\,\mathrm{D}(v)|} & \text{if } \mathrm{D}(v)\neq 0, \\ |\sigma| < gb & \text{if } \mathrm{D}(v) = 0, \end{cases} \tag{1.2}$$

where $\mathrm{D}(v)$ represents the symmetric part of the velocity gradient given by

$$\mathrm{D}(v) = \frac{1}{2}(\nabla v + \nabla^t v).$$

As mentioned above, the Shallow water Bingham model is still for today an open problem. We remark that, in general, the usual strategy of the existence theory is based on two steps: the first one consists of finding u_n, a solution of a suitably chosen approximating problem (a Galerkin approximation for example). The second one amounts to establish uniform estimates in proper spaces and one can then obtains u, a solution to System (1.3)-(1.4), as a "weak" limit of a suitable chosen subsequence of u_n. The central difficulty is the passage to the limit in the nonlinear terms and in the discontinuous constitutive law. Actually, the ideas developed in studying the well posedness issue of Shallow water equations (or compressible Navier-Stokes equations) for Newtonian fluids (see [22], [8], [30], [15]) don't work for non Newtonian ones. As a forward step in this approach, we study here the rigid lid approximation of such system, that we shall call Bingham lake equations. Bingham lake equations characterizing lake equations decode many natural as well as industrial phenomena. The most interesting example is that of mixed flows of Bingham type fluids (such as petroleum) in closed supply pipes. Two types of flows are exhibited inhere characterizing a transition phase from free surface (i.e when only part of the section of pipe is filled and the pressure is being known: atmospheric pressure), to pressurized flow (i.e when the section of the pipe is full, and the pressure is an unknown). The authors in [5] have studied such kind of flows occurring for Newtonian fluids thoroughly. They presented a new model called PFS model, based on coupling the free surface part equations derived from the incompressible Navier Stokes or Euler systems, and the pressurized part equations derived from the Compressible Euler equations. A finite volume discretization have been studied in [5], and a kinetic formulation of such models is presented in [6].

Mathematically, this system can be obtained from (1.1)-(1.2) system by passing to the limit (Fr $\to 0$), where the initial height converges to a non constant function $b(x)$ depending on the space variables only. The obtained model is given by:

$$\begin{cases} \partial_t(bu) + \operatorname{div}(bu\otimes u) - \operatorname{div}\sigma + b\nabla p &= bf, \\ \operatorname{div}(bu) &= 0. \end{cases} \tag{1.3}$$

The shear stress σ satisfies the special constitutive law of a Bingham fluid (see [3]) :

$$\sigma := \begin{cases} 2\mu b\,\mathrm{D}(u) + \lambda b \operatorname{div} u\,\mathbb{I} + gb\dfrac{\mathrm{D}(u)}{|\,\mathrm{D}(u)|} & \text{if } \mathrm{D}(u)\neq 0, \\ |\sigma| < g & \text{if } \mathrm{D}(u) = 0. \end{cases} \tag{1.4}$$

Here, $u(t,x)$ denotes the velocity vector, p the pressure, $f(t,x)$ the known external force, μ and λ the Lamé viscosity coefficients, g the yield limit. We couple the system (1.3) with the so called Lions boundary conditions given by

$$bu\cdot n = 0 \qquad (\sigma\cdot n)\cdot\tau + \kappa(x)bu\cdot\tau = 0 \quad (t,x)\in(0,T)\times\partial\Omega, \tag{1.5}$$

and with the initial data (defined in a weak sence, see Theorem 4.2)

$$u(t,x)|_{t=0} = u_0(x), \quad x\in\Omega. \tag{1.6}$$

In (1.5), n and τ denote respectively the unit normal and tangential vectors to the boundary, whereas $\kappa(x)$ is the curvature of $\partial\Omega$. Lions boundary conditions, the particular case of Navier boundary condition that were first used by Navier in 1872, regard that there is a stagnant layer of fluid close to the wall allowing a fluid to slip, and the slip velocity is proportional to the shear stress.

Since the singular term $\mathrm{D}(u)/|\,\mathrm{D}(u)|$ is not always defined, numerical and mathematical obstacles appear, which forced many authors to develop new formulations of the problem in order to tackle the difficulty. One of the approaches used was introduced in [10] for an unsteady flow of an incompressible fluid having its Cauchy stress given implicitly and relating both the symmetric part of the velocity gradient $\mathrm{D}(u)$ and the deviatoric part of the Cauchy stress. The authors regard the stress tensor σ as a new variable (along with the density and the velocity vector), and this results in a dissipation rate in the form of a Young function depending on $\mathrm{D}(u)$ and σ^t. The second approach, that we will adopt in the sequel, was initiated by G. Duvaut and J. L. Lions [12] who replaced the original system by a variational inequality to get rid of the singularity.

In this paper, we will first adopt the methodology of variational inequality used in [12] to prove the existence of weak solution of the incompressible Bingham fluid confined to a shallow basin with a varying bottom topography but with some changes in the nature of spaces used. Again recalling that we are dealing with a degenerate bathymetry $b(x)$, meaning that $b(x)$ may vanish on the boundary, we will prove that the solution exists in some weighted Sobolev space where the weight is assumed to be a Muckenhoupt type. For more details about these spaces, we refer the reader to [23]. Then, we will discuss the behavior of solution when when g tends to zero. It is good to mention that the vanishing viscosity limit (μ tends to zero) of Bingham fluids is studied by J.-L. Lions [21] in a bounded open set of $\mathbb{R}^2$ without bottom topography. However, an improvement of weak regularity on the solution is required in order to achieve the convergence on the nonlinear term. Yet, as long as we can't improve the regularity of our solution due to the degeneracy of our equations, then we are not able to study such limit in our case.

The paper consists of 6 parts. In the next part (Section 2), we will introduce the spaces that we prove the solution in and some preliminary results related to the functional analysis of the problem. In addition, we will accomplish the equivalence between the initial system and the variational inequality introduced in Section 3. Then comes Section 4 where we exhibit the existence of solution via a Galerkin method. In Sections 5, we study the behavior of the solution when g tends to zero. Finally, in Section 6, a numerical scheme is implemented in 1D verifying our theoretical results.

2. Functional spaces

Before we start the analysis, and since we are going to work with weighted Sobolev spaces, let us first give a brief definition of some of the spaces and preliminaries that we shall need in the sequel, especially the constraints on the weight b and the domain Ω.

Domain: (**I**) For an integer m, $1 \leq m \leq 2$, we set $Q_m = (0,1)^m$. We assume that there exists a bi-Lipschitz

mapping

$$B : Q_2 \to \Omega, \quad \text{such that} \quad B(\overline{Q_1}) = \partial\Omega.$$

Weight: (**II**) We define a space function $b(x)$, locally integrable and belonging to the *Muckenhoupt* class $\mathcal{A}_q$. Generally speaking, for a weight of *Muckenhoupt* type, the definition of trace operator is well defined. More precisely, one can check that if $b \in \mathcal{A}_q$, we have $u \in W_b^{1,q}(\Omega) \hookrightarrow W_{loc}^{1,q}(\overline{\Omega})$ and hence there is a linear trace operator $\gamma_{0,b} : W_{loc}^{1,q}(\overline{\Omega}) \to L_{loc}^1(\overline{\Omega})$. Though it is well defined, yet we lack characterization of such trace regarding a general Muckenhoupt weight. That's why we restrict ourselves in what follows to a more specific weight that provides a characterization of the boundary terms. Its expression is given in a neighborhood of the boundary $V(\partial\Omega)$ by

$$b = \rho^\alpha(x),\ 0 < \alpha < 1/2\ \ \rho(x) = \mathrm{dist}(x, \partial\Omega) \text{ for } x \in V(\partial\Omega). \tag{2.1}$$

In this situation, the definition of the trace is more accurate. For more details about examples of weights satisfying Muckenhoupt condition, we refer the reader to [14], [18], [29].

Weighted Sobolev spaces: As mentioned in the introduction, we will prove our solution in Weighted Sobolev space V_b. We introduce the following weighted spaces:

$$D_b(\Omega) = \big\{\phi \in C_0^\infty(\Omega); \mathrm{div}(b\phi) = 0 \text{ in } \Omega,\ \ b\phi\cdot n = 0 \ \text{ on } \ \partial\Omega\big\},$$

$$L_b^q(\Omega) = \big\{\phi\,;\int_\Omega |\phi|^q\, b\,dx < \infty\big\},$$

$$(L_b^q(\Omega))' = L_{b'}^{q'}(\Omega) \quad \text{with} \quad \frac{1}{q} + \frac{1}{q'} = 1 \text{ and } b' = b^{-\frac{1}{q-1}},$$

$$H_b(\Omega) = \big\{\phi;\phi \in L_b^2(\Omega), \mathrm{div}(b\phi) = 0,\ b\phi\cdot n = 0 \text{ on } \ \partial\Omega\big\},$$

$$V_b(\Omega) = \big\{\phi;\phi \in H_b^1(\Omega), \mathrm{div}(b\phi) = 0,\ b\phi\cdot n = 0 \text{ on } \ \partial\Omega\big\}.$$

For more details about *Muckenhoupt* classes and Weighted Sobolev spaces we refer the reader to [23] and [2].

Trace: As mentioned above in (2.1), we will adopt a special choice of Muckenhopt type weights that provide a good characterization of the trace. In fact, this result is proved by A. Nekvinda via the following theorem.

Theorem 2.1. *([Theorem 2.8, [25]]). Suppose that Hypothesis (**I**) holds. Then for $b = \rho^\alpha(x)$, $-1 < \alpha < q-1$, there exists a unique bounded linear operator*

$$T_b^{1,q}(\,\partial\Omega) : W_b^{1,q}(\Omega) \to W^{1-\frac{1+\alpha}{q},q}(\,\partial\Omega),$$

such that

$$T_b^{1,q}(\,\partial\Omega)(u) = u_{|\,\partial\Omega}.$$

As a consequence of this theorem, we remark that if $f \in H^1_{\rho^\alpha}(\Omega)$, $0 < \alpha < 1$ the trace of f is well defined and belong to $L^2(\,\partial\Omega) \hookrightarrow L_b^2(\,\partial\Omega)$. This helps us define the boundary integrals coming from the Navier boundary conditions.

Throughout this paper, we assume that the domain Ω satisfies Hypothesis (**I**), and b satisfies Hypothesis (**II**), and any other restrictions would be specified .

Notation Remark: Let us fix some notations which will be used throughout the sequel:

- $\mathrm{D}(u) : \mathrm{D}(v) = \sum_{i,j=1}^{2} \mathrm{D}_{i,j}(u)\,\mathrm{D}_{i,j}(v)$.
- $\nabla b \otimes u := (\,\partial_j b\, u_i)_{1\le i,j\le 2}$.
- We say that u is a b-divergence free or u satisfies the b-incompressibility condition if $\mathrm{div}(bu) = 0$.

Remark 2.1. *We should remark that in the sequel of the study, we will need to have $q = \frac{3}{2}$, i.e the weight b should belong to $A_{\frac{3}{2}}$. In fact, this choice is for the sake of bounding the nonlinear term $(u\cdot\nabla)u$ in lemma 4.1. Notice that in studying the Stokes problem, we just need to take $q = 2$ which is necessary to have a weighted version of Poincaré (or Korn's inequality). The reader can refer to lemma 4.1 in [2] for a detailed explanation of such choice.*

3. Variational Inequality

Following Duvaut and Lions in [12], we will derive in this section a variational inequality and prove (at least formally) that the resolution of this variational inequality is equivalent to solve the problem (1.3)-(1.4) in a weak sense (solution of variational problem). Indeed, let us establish first the variational formulation of system (1.3)-(1.4). For this sake, we consider a set V of functions $v : \Omega \to \mathbb{R}^2$ of enough regularity and such that $\mathrm{div}(bv) = 0$ in Ω and $bv\cdot n = 0$ in $\partial\Omega$. Then, we multiply (1.3) by $(v-u)$ and integrate in space. We get

$$\int_\Omega \partial_t u\cdot(v-u)\,b\,dx + \int_\Omega \underbrace{\mathrm{div}(bu\otimes u)}_{\mathrm{div}(bu)\cdot u + bu\cdot\nabla u}\cdot(v-u)\,dx - \int_\Omega \mathrm{div}\,\sigma\cdot(v-u)\,dx$$

$$+\int_\Omega \nabla p\cdot(v-u)\,b\,dx = \int_\Omega f\cdot(v-u)\,b\,dx.$$

Accounting for the Navier boundary conditions and the relation $\operatorname{div}(bv) = 0$, we can write

$$\int_\Omega \partial_t u \cdot (v-u)\, b\, dx + \int_\Omega (u\cdot\nabla)u\cdot(v-u)\, b\, dx + \int_\Omega \sigma : \mathrm{D}(v-u)\, dx - \int_{\partial\Omega} \sigma\cdot n\cdot(v-u)\, ds$$
$$- \int_\Omega p \underbrace{\operatorname{div}(bv-bu)}_{=0}\, dx + \int_{\partial\Omega} p \underbrace{(bv-bu)\cdot n}_{=0}\, ds = \int_\Omega f\cdot(v-u)\, b\, dx.$$

Thus

$$\begin{aligned}\int_\Omega \partial_t u \cdot (v-u)\, b\, dx + \int_\Omega (u\cdot\nabla)u\cdot(v-u)\, b\, dx + 2\mu\int_\Omega \mathrm{D}(u):\mathrm{D}(v-u)\, b\, dx \\ + \lambda \int_\Omega \operatorname{div} u \operatorname{div}(v-u)\, b\, dx + \int_\Omega g\frac{\mathrm{D}(u)}{|\mathrm{D}(u)|} : \mathrm{D}(v-u)\, b\, dx + \int_{\partial\Omega} \kappa u\cdot(v-u)\, b\, ds = \int_\Omega f\cdot(v-u)\, b\, dx.\end{aligned} \tag{3.1}$$

In equation (3.1), the ratio $\frac{\mathrm{D}(u)}{|\mathrm{D}(u)|}$ is not always defined. The physical understanding of this term indicates that it should be interpreted when $\mathrm{D}(u) = 0$ as "any trace-free symmetric matrix with norm less or equal to one". In the mathematical language such quantity is called "multivalued". This fundamental difficulty makes the approximation of the problem (3.1) already a complex challenge , and has motivated a large literature, see [12], [10] and references therein. Below, we follow the procedure in [12] to derive a "formal" equivalent form of the variational formulation (3.1). Indeed, regarding the Cauchy-Schwartz inequality

$$\mathrm{D}(f):\mathrm{D}(g) = \sum_{i,j} \mathrm{D}_{i,j}(f)\, \mathrm{D}_{i,j}(g) \le |\mathrm{D}(f)||\mathrm{D}(g)|,$$

as $\mathrm{D}(u) \neq 0$, we have

$$\begin{aligned}\int_\Omega \frac{\mathrm{D}(u):\mathrm{D}(v-u)}{|\mathrm{D}(u)|}\, b\, dx &= \int_\Omega \left[\frac{1}{|\mathrm{D}(u)|}\mathrm{D}(u):\mathrm{D}(v) - \frac{1}{|\mathrm{D}(u)|}\mathrm{D}(u):\mathrm{D}(u)\right] b\, dx \\ &\le \int_\Omega \left[\frac{1}{|\mathrm{D}(u)|}|\mathrm{D}(u)||\mathrm{D}(v)| - \frac{1}{|\mathrm{D}(u)|}|\mathrm{D}(u)|^2\right] b\, dx \\ &\le \int_\Omega \left[|\mathrm{D}(v)| - |\mathrm{D}(u)|\right] b\, dx.\end{aligned}$$

Hence we get

$$\begin{aligned}\int_\Omega f\cdot(v-u)\, b\, dx \le \int_\Omega \partial_t u\cdot(v-u)\, b\, dx + \int_\Omega u\cdot\nabla u\cdot(v-u)\, b\, dx + 2\mu\int_\Omega \mathrm{D}(u):\mathrm{D}(v-u)\, b\, dx \\ + \lambda\int_\Omega \operatorname{div} u \operatorname{div}(v-u)\, b\, dx + g\int_\Omega (|\mathrm{D}(v)| - |\mathrm{D}(u)|)\, b\, dx + \int_{\partial\Omega} \kappa u\cdot(v-u)\, b\, ds \qquad \text{for all } v\in V.\end{aligned} \tag{3.2}$$

By now we have proved that if (u,p) is a formal solution of (1.3), then it would satisfy (3.2). Reciprocally, assuming that u is a solution of the variational inequality (3.2) such that $\mathrm{D}(u) \neq 0$ a.e. in Ω. Letting $\theta w = v-u$ in the previous inequality, one gets

$$\begin{aligned}\int_\Omega f\cdot(\theta w)\, b\, dx \le \int_\Omega \partial_t u\cdot(\theta w)\, b\, dx + \int_\Omega (u\cdot\nabla)u\cdot\theta w\, b\, dx + 2\mu\int_\Omega \mathrm{D}(u):\mathrm{D}(\theta w)\, b\, dx \\ + \lambda\int_\Omega \operatorname{div} u \operatorname{div}(\theta w)\, b\, dx + g\int_\Omega (|\mathrm{D}(u+\theta w)| - |\mathrm{D}(u)|)\, b\, dx + \int_{\partial\Omega} \kappa u\cdot(\theta w)\, b\, ds \qquad \text{for all } w\in V.\end{aligned}$$

Now, as we divide by θ we get:

$$\begin{aligned}\int_\Omega f\cdot w\, b\, dx \le \int_\Omega \partial_t u\cdot w\, b\, dx + \int_\Omega u\cdot\nabla u\cdot w\, b\, dx + 2\mu\int_\Omega \mathrm{D}(u):\mathrm{D}(w)\, b\, dx \\ + \lambda\int_\Omega \operatorname{div} u \operatorname{div} w\, b\, dx + \underbrace{g\int_\Omega \frac{1}{|\mathrm{D}(u)|}(\mathrm{D}(u):\mathrm{D}(w)\, b\, dx}_{\mathbf{S}} + \int_{\partial\Omega} \kappa u\cdot w\, b\, ds.\end{aligned} \tag{3.3}$$

Concerning the term S, we have in fact:

$$\mathbf{S} = \lim_{\theta\to 0} \frac{g}{\theta}\int_\Omega (|\mathrm{D}(u+\theta w)| - |\mathrm{D}(u)|)\, b\, dx. \tag{3.4}$$

On the other hand, we have

$$
\begin{aligned}
|\,\mathrm{D}(u+\theta w)|^2 - |\,\mathrm{D}(u)|^2 &= \sum_{i,j}(\mathrm{D}_{i,j}(u+\theta w))^2 - \sum_{i,j}(\mathrm{D}_{i,j}(u))^2 \\
&= \sum_{i,j}\big[(\mathrm{D}_{i,j}(u))^2 + 2\theta\,\mathrm{D}_{i,j}(u)\,\mathrm{D}_{i,j}(w) + \theta^2(\mathrm{D}_{i,j}(w))^2 - (\mathrm{D}_{i,j}(u))^2\big].
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathbf{S} &= \lim_{\theta\to 0}\big(\frac{g}{\theta}\int_\Omega \frac{|\,\mathrm{D}(u+\theta w)|^2 - |\,\mathrm{D}(u)|^2}{|\,\mathrm{D}(u+\theta w)| + |\,\mathrm{D}(u)|}\,b\,dx\big) \\
&= g\int_\Omega \lim_{\theta\to 0}\big(\frac{1}{|\,\mathrm{D}(u)| + |\,\mathrm{D}(u+\theta w)|}\big) \times \lim_{\theta\to 0}\big(\sum_{i,j}\frac{2\theta\,\mathrm{D}_{i,j}(u)\,\mathrm{D}_{i,j}(w)\,b\,dx + \theta^2\,\mathrm{D}_{i,j}(w)}{\theta}\big)\,b\,dx \\
&= g\int_\Omega \frac{1}{|\,\mathrm{D}(u)|}\,\mathrm{D}(u):\mathrm{D}(w)\,b\,dx.
\end{aligned}
$$

By changing w into $-w$, we find the opposite inequality of (3.3), and so we get

$$
\begin{aligned}
&\int_\Omega \partial_t(bu)\cdot w\,dx + \int_\Omega bu\cdot\nabla u\cdot w\,dx + 2\mu\int_\Omega b\,\mathrm{D}(u):\mathrm{D}(w)\,dx \\
&\quad + \lambda\int_\Omega b\,\mathrm{div}\,u\,\mathrm{div}\,w\,dx + g\int_\Omega \frac{b}{|\,\mathrm{D}(u)|}\,\mathrm{D}(u):\mathrm{D}(w)\,dx + \int_{\partial\Omega}\kappa u\cdot w\,b\,ds = \int_\Omega bf\cdot w\,dx.
\end{aligned}
$$

Thus, we establish formally the equivalence between (3.2) and (1.3).

Let us now precise our definition of weak solution of System (1.3)-(1.4).

Definition 3.1. *We say that u is a solution of system* (1.3)-(1.4) *equipped with boundary and initial conditions given in* (1.5)-(1.6) *if u satisfies the following regularity*

$$
\begin{aligned}
u &\in L^\infty(0,T;H_b)\cap L^2(0,T;V_b), \\
\partial_t(bu) &\in L^2(0,T;V_b'),
\end{aligned}
$$

and the following variational inequality holds

$$
\begin{aligned}
&\int_\Omega f\cdot(v-u)\,b\,dx \le \int_\Omega \partial_t u\cdot w\;b\,dx + \int_\Omega u\cdot\nabla u\cdot(v-u)\,b\,dx + 2\mu\int_\Omega \mathrm{D}(u):\mathrm{D}(v-u)\,b\,dx \\
&\quad + \lambda\int_\Omega \mathrm{div}\,u\,\mathrm{div}(v-u)\,b\,dx + g\int_\Omega(|\,\mathrm{D}(v)| - |\,\mathrm{D}(u)|)\,b\,dx + \int_{\partial\Omega}\kappa u\cdot(v-u)\,b\,ds \qquad \textit{for all } v\in V_b.
\end{aligned} \tag{3.5}
$$

Remark 3.1. *The well definition of the right hand side will be discussed in the sequel as we take u and $v\in V_b$ (mainly in lemma 4.1). It is good to mention here that for left hand side term in* (3.5)*, one should pay attention for the regularity of bf, hence assuming the least regularity possible, i.e in the dual of V_b, we can replace the integral form by the dual representation: $\langle bf, v-u\rangle_{V_b',V_b}$. Though, this in fact doesn't affect the computations. Without loss of generality, we will assume in the sequel that bf is in $(L_b^2(\Omega))' = L_{b^{-1}}^2(\Omega)$, or equivalently f in $L_b^2(\Omega)$.*

4. Main Results

We state in this section the existence result of problem (3.5). The presence of b in the diffusion operator and the "b-incompressibility condition" make the weighted Sobolev spaces the ambient ones to prove existence within. In what follows, we introduce a priori estimates concerning the non linear term in lemma 4.1, and which will later on serve the well definition of the integral forms. Next, we exhibit in theorem 4.2 the existence and uniqueness results. We will rely in the proof on a Galerkin approximation technique. Several operators will be used in the latter method for which different properties will be given in lemma 4.1.

Lemma 4.1. *Let Ω be an open bounded Lipschitz domain of $\mathbb{R}^2$, and b satsifying Hypothesis (**II**). Then, for $u\in L^\infty(0,T;H_b)\cap L^2(0,T;V_b)$ and $v\in V_b$, we have:*

$$
\begin{aligned}
u &\in L^2(0,T;L_b^6(\Omega)), \\
(u\cdot\nabla)u &\in L^1(0,T;L_b^{3/2}(\Omega)),
\end{aligned}
$$

$$(u\cdot\nabla)u\cdot v\in L^1(0,T;L^1_b(\Omega)),$$

and there exists a positive real number C independent of u and v such that

$$||u||_{L^2(0,T;L^6_b(\Omega))}\le C||\nabla u||_{L^2(0,T;L^2_b(\Omega))},$$

$$\Big|\int_0^T\int_\Omega (u\cdot\nabla)u\,b\,dx\Big|\le C\,||\nabla u||^2_{L^2(0,T;L^2_b(\Omega))},$$

$$\Big|\int_0^T\int_\Omega (u\cdot\nabla)u\cdot v\,b\,dx\Big|\le C\,||\nabla u||^2_{L^2(0,T;L^2_b(\Omega))}||\nabla v||_{L^2_b(\Omega)}.$$

We define the following linear operators A and B such that

$$Au\in L^2(0,T;V_b')\quad\textit{and}\quad Bu\in L^2(0,T;V_b'),$$

where A and B are given by

$$\langle Au,v\rangle_{V_b',V_b}:=a(u,v)=2\mu\int_\Omega \mathrm{D}(u):\mathrm{D}(v)\,b\,dx+\lambda\int_\Omega \operatorname{div}u\operatorname{div}v\,b\,dx+\int_{\partial\Omega}\kappa u\cdot v\,b\,ds,$$

$$\langle Bu,v\rangle_{V_b',V_b}:=b(u,u,v)=\int_\Omega (u\cdot\nabla)u\cdot v\,b\,dx,$$

and b satisfies $b(u,u,u)=0$.

Proof. Here we just want to prove that $Au\in L^2(0,T;V_b')$ and $Bu\in L^2(0,T;V_b')$. The other properties are proved in [2]. Indeed, For $u\in L^\infty(0,T;H_b)\cap L^2(0,T;V_b)$ and $v\in V_b$, we estimate using Hölder's inequality

$$\begin{aligned}|\langle Au,v\rangle_{V_b',V_b}|&=\Big|2\mu\int_\Omega \mathrm{D}(u):\mathrm{D}(v)\,b\,dx+\lambda\int_\Omega \operatorname{div}u\operatorname{div}v\,b\,dx,+\int_{\partial\Omega}\kappa u\cdot v\,b\,ds\Big|\\&\le C\|u\|_{V_b}\|v\|_{V_b}.\end{aligned}$$

Since $u\in L^2(0,T;V_b)$, thus by duality we get

$$Au\in L^2(0,T;V_b').$$

As for the operator B, we estimate the nonlinear term using Hölder and Gagliardo-Nirenberg inequalities as follows

$$\begin{aligned}|\langle Bu,v\rangle_{V_b',V_b}|&=\Big|\int_\Omega (u\cdot\nabla)u\cdot v\,b\,dx\Big|=\Big|-\int_\Omega (u\cdot\nabla)v\cdot u\,b\,dx\Big|\\&\le C\|u\|^2_{L^4_b(\Omega)}\|\nabla v\|_{L^2_b(\Omega)}\\&\le C\|u\|_{L^2_b(\Omega)}\|\nabla u\|_{L^2_b(\Omega)}\|\nabla v\|_{L^2_b(\Omega)}\\&\le C\|u\|_{H_b}\|u\|_{V_b}\|v\|_{V_b}.\end{aligned}$$

Since $\|u\|_{H_b}\|u\|_{V_b}\in L^2(0,T)$, thus by duality we get

$$Bu\in L^2(0,T;V_b').$$

□

Remark 4.1. *Notice that, in [Proposition 3.1, [2]] , the first author proved a weighted version of Korn's inequality. However, some constraints on the weight b and the domain Ω were needed. More precisely, the author proved that if we exclude that case when the domain Ω is a disc, b is radial and not identically zero on the boundary, then we have*

$$\|\mathrm{D}(u)\|_{L^2_b(\Omega)}\ge\|u\|_{H^1_b}\qquad\textit{for all } u \textit{ in } V_b.$$

Nevertheless, he showed also that these assumptions are not required when studying the evolution problem. That is why in our case we won't suppose such assumptions. For more details, the reader is referred to [2].

The main result of this paper is given below.

Theorem 4.2. *(Existence of weak solution). We suppose that f and u_0 are the applied force and the initial datum given such that f lies in $L^2(0,T,L^2_b(\Omega))$ and u_0 belongs H_b. Assume that b satisfies Hypothesis (**II**), and that $\kappa(x)$ is in $L^\infty(\Omega)$, then there exists a unique vector field u such that*

$$u \in L^2(0,T;V_b) \quad \partial_t(bu) \in L^2(0,T;V_b') \tag{4.1}$$

satisfying

$$\int_\Omega f\cdot(v-u)\,b\,dx \le \int_\Omega \partial_t u\cdot(v-u)\,b\,dx + \int_\Omega u\cdot\nabla u\cdot(v-u)\,b\,dx + g\int_\Omega (|\,\mathrm{D}(v)| - |\,\mathrm{D}(u)|)\,b\,dx$$
$$+2\mu\int_\Omega \mathrm{D}(u):\mathrm{D}(v-u)\,b\,dx + \lambda\int_\Omega \operatorname{div} u \operatorname{div}(v-u)\,b\,dx + \int_{\partial\Omega}\kappa u\cdot(v-u)\,bdx \qquad \textit{for all } v\in V_b,$$

and the initial condition is defined in a weak sense:

$$\Big(\int_\Omega u\cdot v\,b\,dx\Big)(0) = \int_\Omega u_0\cdot v\,b\,dx \qquad \textit{for all} \quad v\in V_b.$$

Proof. In the same spirit of Lions and Duvaut's approach, our proof will be composed of three main steps. Let us denote by the operator j the integral form:

$$j(\psi) = g\int_\Omega |\,\mathrm{D}(\psi)|\,b\,dx.$$

(1) **Step1: Regularizing j**

In attempt to approximate the problem, we start first by regularizing the operator j which is derived from Bingham's singular term.

Let $\epsilon\in(0,1)$. We approximate j by a differentiable functional

$$j_\epsilon(\psi) = \frac{g}{1+\epsilon}\int_\Omega (|\,\mathrm{D}(\psi)|)^{1+\epsilon}\,b\,dx.$$

In fact, $j_\epsilon(\psi)$ is well defined since as we deal with a bounded domain and the fact that $\mathrm{D}(\psi)$ is in $L^2_b(\Omega)$, we infer that $j_\epsilon(\psi)<\infty$. The Gateaux differential of $j_\epsilon(\cdot)$ along v is given by

$$D_v(j_\epsilon(w)) := \lim_{\tau\to 0}\frac{j_\epsilon(w+\tau v) - j_\epsilon(w)}{\tau} = \frac{g}{1+\epsilon}\int_\Omega D_v(|\,\mathrm{D}(w)|^{1+\epsilon})\,b\,dx.$$

Then

$$\begin{aligned} D_v(|\,\mathrm{D}(w)|^{1+\epsilon}) &= \mathrm{D}_v((|\,\mathrm{D}(w)|^2)^{\frac{1+\epsilon}{2}}) \\ &= \frac{1+\epsilon}{2}(|\,\mathrm{D}(w)|^2)^{\frac{\epsilon-1}{2}} D_v(|\,\mathrm{D}(w)|^2) \\ &= \frac{1+\epsilon}{2}|\,\mathrm{D}(w)|^{\epsilon-1}\times 2\,\mathrm{D}(w):\mathrm{D}(v) \qquad \text{from (3.4)}. \end{aligned}$$

Therefore

$$D_v(j_\epsilon(w)) = g\int_\Omega |\,\mathrm{D}(w)|^{\epsilon-1}\,\mathrm{D}(w):\mathrm{D}(v)\,b\,dx \;<+\infty. \tag{4.2}$$

In fact, we claim that $|\,\mathrm{D}(w)|^{\epsilon-1}\,\mathrm{D}(w)$ belong to $L^2(0,T;H_b)$

$$\Big|\int_0^T\int_\Omega |\,\mathrm{D}(w)|^{\epsilon-1}\times \mathrm{D}(w)\,b\,dx\,dt\Big| \le \int_0^T\int_\Omega |\,\mathrm{D}(w)|^{\epsilon}\,b\,dx\,dt < \infty.$$

We adopt the notation in [12]

$$(j'_\epsilon(w),v) = g\int_\Omega |\,\mathrm{D}(w)|^{\epsilon-1}\,\mathrm{D}(w):\mathrm{D}(v)\,b\,dx. \tag{4.3}$$

It is good to mention here that $j'_\epsilon(w) \in L^2(0,T,V'_b)$ for all w in V_b. More precisely

$$\begin{aligned}\left|\int_0^T (j'_\epsilon(w),v)\,dt\right| &= \left|\int_0^T\int_\Omega g|\operatorname{D}(w)|^{\epsilon-1}\operatorname{D}(w):\operatorname{D}(v)\,b\,dx\,dt\right|\\ &\le C\int_0^T \|(\operatorname{D}(w))^\epsilon\|_{L^2_b(\Omega)}\|\operatorname{D}(v)\|_{L^2_b(\Omega)}\\ &\le C\|w\|_{L^2(0,T;V_b)}\|v\|_{L^2(0,T:V_b)}\\ &\le C.\end{aligned}$$

We have the following property

Lemma 4.3. *If w_ϵ converges weakly to w in $L^2(0,T;V_b)$, then*

$$\int_0^T j(w) \le \liminf_\epsilon \int_0^T j_\epsilon(w_\epsilon).$$

Proof. By Hölder's inequality, we have first

$$\begin{aligned}\int_0^T j(w_\epsilon)\,dt &= \int_0^T\int_\Omega |\operatorname{D}(w_\epsilon)|\,b\,dx\,dt\\ &\le \int_0^T \Big[\Big(\int_\Omega |\operatorname{D}(w_\epsilon)|^{1+\epsilon}\ b\ dx\Big)^{\frac{1}{1+\epsilon}}\Big(\int_\Omega b\,dx\Big)^{\frac{\epsilon}{1+\epsilon}}\Big]dt\\ &\le C\int_0^T \Big(\int_\Omega |\operatorname{D}(w_\epsilon)|^{1+\epsilon}\ b\ dx\Big)^{\frac{1}{1+\epsilon}}\ dt\\ &\le C\int_0^T \Big(\int_\Omega |\operatorname{D}(w_\epsilon)|\ b\ dx\Big)^{\frac{1}{1+\epsilon}}\ dt\\ &\le C\Big(\int_0^T j_\epsilon(w_\epsilon)\,dt\Big)^{\frac{1}{1+\epsilon}}.\end{aligned}$$

Thus

$$\Big(\int_0^T j(w_\epsilon)\,dt\Big)^{1+\epsilon} \le C\int_0^T j_\epsilon(w_\epsilon)\,dt.$$

As $v \longrightarrow j(v)$ is lower semi continuous for the weak topology of $L^2(0,T;V_b)$ (from the definition of weak convergence), then

$$\int_0^T j(w)\,dt \le \liminf_\epsilon \int_0^T j(w_\epsilon)\,dt.$$

Therefore, combining the above we get our result

$$\int_0^T j(w) \le \liminf_\epsilon \int_0^T j_\epsilon(w_\epsilon).$$

□

We render the problem into a new approximated one corresponding to j_ϵ such that:

$$\begin{aligned}&\int_\Omega \partial_t u_\epsilon\cdot v\,b\,dx + \int_\Omega (u_\epsilon\cdot\nabla)u_\epsilon\cdot v\,b\,dx + 2\mu\int_\Omega \operatorname{D}(u)_\epsilon:\operatorname{D}(v)\,b\,dx\\ &\quad+\lambda\int_\Omega \operatorname{div} u_\epsilon\cdot\operatorname{div} v\,b\,dx + \int_{\partial\Omega}\kappa(x)u_\epsilon\cdot v\,b\,ds + (j'_\epsilon(u_\epsilon),v) = \int_\Omega f\cdot(v-u)\,b\,dx \qquad \forall v\in V_b,\end{aligned} \tag{4.4}$$

where the representation $(j_\epsilon(\cdot),\cdot)$ stands for (4.3).

(2) **Step 2: Seeking for a sequence (u_m) via Galerkin approximation**

Consider the canonical isomorphism $\wedge : V_b \longrightarrow V'_b$ with $\{w_1,...,w_m,..\}$ being the set of unit eigenfunctions of the operator $\wedge$, Define now the space $V_m = \operatorname{Span}\{w_1,\dots w_m\}$, where $\{w_1,\dots,w_m\}$ is a free and total family in H_b (it is permissible to choose the set as that since V_b is separable). In fact this isomorphism is proven for weighted Sobolev spaces by *Fröhlich* in bounded domains, check for instance [17]. Upon projecting our system on V_m, and according to the Cauchy theory in a finite space, we can construct a solution $u_m := u_{\epsilon m}$ in V_m for the approximated variational inequality (4.4)

$$
\begin{aligned}
&\int_\Omega \partial_t u_m \cdot w_j\, b\, dx + \int_\Omega (u_m \cdot \nabla u_m) \cdot w_j\, b\, dx + 2\mu \int_\Omega \mathrm{D}(u_m) : \mathrm{D}(w_j)\, b\, dx + \lambda \int_\Omega \operatorname{div} u_m \operatorname{div} w_j\, b\, dx \\
&+ \int_{\partial\Omega} \kappa(x) u_m \cdot w_j\, b\, ds + (j'_\epsilon(u_m), w_j) = \int_\Omega f \cdot w_j\, b\, dx, \qquad 1 \le j \le m,
\end{aligned} \tag{4.5}
$$

with $u_m(0)$ being the projection of u_0 on V_m, and u_m being defined in a space interval $[0, t_m]$. Since $u_m = \sum_{i=0}^{m} (u_m, w_i)_{V_b} w_i$, we multiply (4.5) by $(u_m, w_j)_{V_b}$ and sum over j, we get:

$$
\begin{aligned}
&\int_\Omega \partial_t u_m \cdot u_m\, b\, dx + \int_\Omega (u_m \cdot \nabla) u_m \cdot u_m\, b\, dx + 2\mu \int_\Omega |\,\mathrm{D}(u_m)|^2\, b\, dx + \lambda \int_\Omega |\operatorname{div} u_m|^2\, b\, dx \\
&\quad + \int_{\partial\Omega} \kappa(x) |u_m|^2 b\, ds + (j'_\epsilon(u_m), u_m(t))\, dx = \int_\Omega f \cdot u_m\, b\, dx.
\end{aligned} \tag{4.6}
$$

From Lemma 4.1, we deduce that the second term is equal to zero, and since

$$
(j'_\epsilon(u), u) = \int_\Omega g |\mathrm{D}(u)|^{\epsilon-1}\, \mathrm{D}(u) : \mathrm{D}(u)\, b\, dx = \int_\Omega |\mathrm{D}(u)|^{\epsilon+1}\, dx \ge 0.
$$

Notice that using Hölder's and Young's inequalities, we can estimate the term on the right hand of equation (4.6) as follows

$$
\begin{aligned}
\Big| \int_\Omega f \cdot u_m\, b\, dx \Big| &\le \|f\|_{L^2_b(\Omega)} \|u_m\|_{L^2_b(\Omega)} \\
&\le \eta \|f\|^2_{L^2_b(\Omega)} + C(\eta) \|u_m\|^2_{V_b},
\end{aligned}
$$

for some arbitrary $\eta > 0$. Now, after choosing η sufficiently small, we can deduce using weighted Korn's inequality from Remark 4.1 that

$$
\frac{d}{dt} \|u_m\|^2_{L^2_b(\Omega)} + \|u_m\|^2_{H^1_b(\Omega)} \le C \|f\|^2_{L^2_b(\Omega)}.
$$

Hence, we deduce

$$
u_m \in L^\infty(0, T; H_b) \cap L^2(0, T; V_b), \tag{4.7}
$$

and u_m remains in a bounded set of $L^\infty(0, T; H_b)$ and $L^2(0, T; V_b)$ uniformly with respect to m. The next step is to prove $(bu_m)' \in L^2(0, T; V'_b)$ uniformly, i.e for all m, $(bu_m)'$ remains in a bounded set of $L^2(0, T; V'_b)$. Introduce the projection operator $P_m : V_b \longrightarrow V_m$. Since $Au_m,\ Bu_m$ and $j'_\epsilon(u_m)$ belong to $L^2(0, T; V'_b)$, then, we can write from (4.5)

$$
\int_\Omega \partial_t(bu_m) \cdot w_j\, dx = \langle bf - Au_m - Bu_m - j'_\epsilon(u_m), w_j \rangle_{V'_b, V_b} \qquad \forall 1 \le j \le m.
$$

Since we have $P_m(\,\partial_t(bu_m)) = \partial_t(bu_m)$, then we can write

$$
\partial_t(bu_m) = P_m(bf - Au_m - Bu_m - j'_\epsilon(u_m)).
$$

Therefore, we get

$$
\begin{aligned}
b\partial_t u_m &= P_m(bf - Au_m - Bu_m - j'_\epsilon(u_m)) \quad \text{in } D'(0, T, V_m), \\
&:= P_m(k_m).
\end{aligned}
$$

Since k_m is bounded in $L^2(0, T; V'_b)$, then what remains to show is that

$$
\|P_m k_m\|_{V'_b} \le \|k_m\|_{V'_b}.
$$

Since $\wedge$ is an isomorphism from V_b to V_b', then we can deduce that $\lambda^{\frac{1}{2}} w_j$ constitute an orthogonal basis of V_b' for the norm $\|X\|_{V_b'} = \|\wedge^{-1} X\|_{V_b}$. Thus, we get that

$$\begin{aligned}\|P_m k_m\|_{V_b'} &= \sum_{j=1}^{m} (k_m, \lambda^{\frac{1}{2}} w_j)_{L_b^2} \\ &\leq \sum_{j=1}^{m} (k_m, \lambda^{\frac{1}{2}} w_j)_{L_b^2} \\ &\leq \|k_m\|_{V_b'}.\end{aligned}$$

Finally, we have our aimed result

$$bu_m' \in L^2(0,T;V_b'). \tag{4.8}$$

As a result of (4.7) and (4.8), we can deduce that up to a sequence (denoted again u_m), we have

$$\begin{aligned} u_m &\overset{*}{\rightharpoonup} u_\epsilon \quad \text{in } L^\infty(0,T;H_b), \\ u_m &\rightharpoonup u_\epsilon \quad \text{in } L^2(0,T;V_b), \\ bu_m' &\rightharpoonup bu_\epsilon \quad \text{in } L^2(0,T;V_b'). \end{aligned} \tag{4.9}$$

Now, in order to use compactness result we must apply a fractional estimate technique to have $||\tau_h u_m - u_m||_{L^2(0,T-h;L_b^2(\Omega))}$ converges to zero. Since it is a technical step, though, we refer the reader to [2] for a detailed proof. Thus we are able to use the compactness result and deduce that $u_m \to u_\epsilon$ strongly in $L^2(0,T,H_b)$. Now we are concerned in the convergence of terms in (4.5):

(3) **Step 3. Passage to the limit in** m:
Now, we will prove the convergence of the operators as $m \to 0$.

- **Convergence of $j_\epsilon'(u_m)$**
  $j_\epsilon'(u_m)$ is bounded uniformly in $L^2(0,T,V_b')$, hence
  $$j_\epsilon'(u_m) \rightharpoonup \xi \quad \text{in } L^2(0,T,V_b').$$
- **Convergence of the linear operator A**
  As for the operator A given by
  $$\langle Au_m, v\rangle_{V_b',V_b} = \int_\Omega 2\mu\, \mathrm{D}(u_m) : \mathrm{D}(v) + \lambda \operatorname{div} u_m \operatorname{div} v\, b\, dx + \int_{\partial\Omega} \kappa u_m \cdot v\, b\, ds,$$
  we have in fact that $u_m \rightharpoonup u_\epsilon$ in $L^2(0,T;V_b)$, then $\partial_i u_j \rightharpoonup \partial_i (u_\epsilon)_j$ in $L^2(0,T;L_b^2(\Omega))$, and hence
  $$\int_0^T \int_\Omega \mathrm{D}(u_m) : \mathrm{D}(v)\, b\, dx\, dt \to \int_0^T \int_\Omega \mathrm{D}(u_\epsilon) : \mathrm{D}(v)\, b\, dx\, dt,$$
  and
  $$\int_0^T \int_\Omega \operatorname{div} u_m \operatorname{div} v\, b\, dx\, dt \to \int_0^T \int_\Omega \operatorname{div} u_\epsilon \operatorname{div} v\, b\, dx\, dt.$$
  Before we treat the convergence on the boundary term, let us prove the following lemma.

  Lemma 4.4. *Let u and v be two sufficiently smooth vectors such that $bu \cdot n = 0$ and $bv \cdot n = 0$, then*

  $$(v \cdot \nabla(bu)) \cdot n = -\kappa bu \cdot v. \tag{4.10}$$

  Proof. Since we have $bu \cdot n = 0$ on $\partial\Omega$, then
  $$\begin{aligned} 0 = \frac{\partial}{\partial\tau}(bu \cdot n) &= \frac{\partial(bu)}{\partial\tau} \cdot n + bu \cdot \frac{\partial n}{\partial\tau} \\ &= (\tau \cdot \nabla(bu)) \cdot n + \kappa bu \cdot \tau. \end{aligned}$$
  Recall that κ is the curvature of $\partial\Omega$ where we have
  $$\frac{\partial n}{\partial\tau} := \frac{dn}{ds} = \kappa u \cdot \tau.$$

But v is parallel to τ, so (4.10) follows by linearity. □

Now back to prove the convergence of the boundary term, we notice first using (4.10) and the following identity

$$\nabla(bu) : \nabla^t v = \partial_i(bu_j)\,\partial_j v_i = \partial_j(v_i\,\partial_i(bu_j)) = \mathrm{div}(v\cdot\nabla(bu))$$

that we can write

$$\begin{aligned}\int_{\partial\Omega}\kappa(u_m-u_\epsilon)\cdot v\,b\,ds &= -\int_{\partial\Omega}(v\cdot\nabla(b(u_m-u_\epsilon)))\cdot n\,ds\\ &= -\int_\Omega \mathrm{div}(v\cdot\nabla(b(u_m-u_\epsilon)))\,dx\\ &= -\int_\Omega \nabla^t v : \nabla(b(u_m-u_\epsilon))\,dx\\ &= -\int_\Omega \nabla^t v : \nabla(u_m-u_\epsilon)\,b\,dx + \int_\Omega \nabla^t v : \frac{\nabla b}{b}\otimes(u_m-u_\epsilon)\,b\,dx.\end{aligned}$$

We easily remark that the first term on the right hand side of the above equality converges to zero because of the weak convergence of u_m to u_ϵ in V_b. For the second term, we need to use a Hardy's type inequality to show that

$$\frac{\nabla b}{b}\otimes(u_m-u_\epsilon)\quad\text{is uniformaly bounded in}\quad L^2_b(\Omega).$$

Indeed for $b=\rho^\alpha$, we have $\nabla b/b\sim 1/\rho$, then (see [13])

$$\begin{aligned}\int_\Omega\Big|\frac{\nabla b}{b}\otimes(u_m-u_\epsilon)\Big|^2 b\,dx &= \int_\Omega\Big|\frac{1}{\rho}(u_m-u_\epsilon)\Big|^2 b\,dx\\ &\le C\int_\Omega|\nabla(u_m-u_\epsilon)|^2 b\,dx.\end{aligned}$$

Thus we get the convergence of the boundary term.

- **Convergence of the trilinear term B**

For the trilinear term, we remark that

$$b(u_m,u_m,v) = -b(u_m,v,u_m).$$

Using the fact that $H^1_b(\Omega)\hookrightarrow L^6_b(\Omega)\hookrightarrow L^4_b(\Omega)$ (since Ω is bounded, and b $\in A_{\frac{3}{2}}$ [19]), and due to the boundedness and strong convergence results proved for u_m, we obtain

$$\begin{aligned}&\Big|\int_0^T\Big(\int_\Omega(u_m\cdot\nabla)v\cdot u_m\,b\,dx - \int_\Omega(u_\epsilon\cdot\nabla)v\cdot u_\epsilon\,b\,dx\Big)\,dt\Big|\\ &=\Big|\int_0^T\Big(\int_\Omega(u_m-u_\epsilon)\cdot\nabla v\cdot u_m + \int_\Omega u_\epsilon\cdot\nabla v\cdot(u_m-u_\epsilon)\,b\,dx\Big)\,dt\Big|\\ &\le\int_0^T\|u_m\|_{L^4_b}\|\nabla v\|_{L^2_b}\|u_m-u_\epsilon\|_{L^4_b}\,dt + \int_0^T\|u_\epsilon\|_{L^4_b}\|\nabla v\|_{L^2_b}\|u_m-u_\epsilon\|_{L^4_b}\,dt\\ &\le c\int_0^T\|\nabla v\|_{H_b}\big(\|u_m\|_{V_b}+\|u_\epsilon\|_{V_b}\big)\big(\|\nabla u_m\|_{L^2_b}+\|\nabla u_\epsilon\|_{L^2_b}\big)^{\frac12}\|u_m-u_\epsilon\|^{\frac12}_{L^2_b}\,dt\\ &\le c\|u_m-u_\epsilon\|^{\frac12}_{L^2(0,T,L^2_b)}\underset{m\to 0}{\longrightarrow}0\,.\end{aligned}$$

- **Convergence of the integral containing u'_m**

Since we have $bu'_m \rightharpoonup bu'_\epsilon$ in $L^2(0,T,V'_b)$, thus

$$\int_0^T\int_\Omega\partial_t u_m\cdot v\,b\,dx\,dt \longrightarrow \int_0^T\int_\Omega\partial_t u_\epsilon\cdot v\,b\,dx\,dt \qquad \forall v\in V_b.$$

- **Convergence of the initial condition**
The space V_b can be viewed as

$$V_b = \overline{D_b(\Omega)}^{||.||_{H^1_b(\Omega)}}$$

Now, we choose the orthonormal base of V_b such that for all j, we have $w_j \in D_b(\Omega)$. In this case,

$$\int_\Omega (u_m \cdot \nabla) u_m \cdot w_j \, b \, dx \quad \text{is bounded in} \quad L^2(0,T)$$

since $(bu_m \cdot \nabla)u_m$ is bounded in $L^2(0,T;(L^1(\Omega))^2)$. Moreover we have

$$a(u_m, w_j) + b(u_m, u_m, w_j) \in L^2(0,T).$$

Thus Equation (4.5) shows that

$$\partial_t \langle u_m, w_j \rangle_b \quad \text{is bounded in} \quad L^2(0,T), \tag{4.11}$$

and hence $\int_\Omega u_m.w_j \, b \, dx$ is bounded in $H^1(0,T)$. On the other hand, we know that $H^1(0,T)$ is compact in $\mathcal{C}_u(0,T)$ (see Theorem III.2.34 in [7]) which yields

$$\int_\Omega u_m \cdot w_j \, b \, dx \to \int_\Omega u_\epsilon \cdot w_j \, b \, dx \ \text{ in } \ \mathcal{C}_u(0,T).$$

The above convergence holds also in $L^2(0,T)$ (due to strong convergence of u_m in $L^2(0,T;H_b)$). In particular

$$\Big(\int_\Omega u_m \cdot w_j \, b \, dx\Big)(0) \to \Big(\int_\Omega u_\epsilon \cdot w_j \, b \, dx\Big)(0).$$

Notice that, by definition, we have $u_m(0) = P_m(u_0)$, so we can write

$$\int_\Omega (u_m(0,x) - P_m(u_0(x))) \cdot w_j(x) \, b \, dx = 0, \tag{4.12}$$

Passing to the limit in Equation (4.12) yields to

$$\Big(\int_\Omega u_\epsilon \cdot w_j \, b \, dx\Big)(0) = \int_\Omega u_0 \cdot w_j \, b \, dx. \tag{4.13}$$

On other hand, by virtue of Equation (4.4), we have for all $v \in V_b$,

$$\partial_t \int_\Omega u_\epsilon \cdot v \, b \, dx = \partial_t \langle u_\epsilon, v \rangle_b \in L^1(0,T)$$

which yields that

$$\langle u_\epsilon, v \rangle_b \in C([0,T]).$$

Since $u_\epsilon \in L^\infty(0,T;H_b)$ and V_b is dense in H_b, we infer that (see Lemma 1.4 in [28]).

$$u_\epsilon \in C([0,T]; H_b - \text{weak}) \text{ and that } u_\epsilon(0) = u_0 \text{ in } C([0,T]; H_b - \text{weak}).$$

Now, it remains to show that $\xi = j'_\epsilon(u_\epsilon)$. By now we have got

$$\int_0^T \int_\Omega u'_\epsilon \cdot v \, b \, dx \, dt + \int_0^T \langle B u_\epsilon, v \rangle_{V'_b, V_b} \, dt + \int_0^T \langle A u_\epsilon, v \rangle_{V'_b, V_b} \, dt + \int_0^T (\xi, v) \, dt = \int_0^T \int_\Omega f \cdot v \, b \, dx \, dt. \tag{4.14}$$

We choose a function ϕ such that ϕ in $L^2(0,T,V_b)$, $b\phi'$ in $L^2(0,T,V'_b)$. Now, we introduce the following quantity

$$X_m = \int_0^T \langle b u'_m - b\phi', u_m - \phi \rangle_{V'_b, V_b} \, dt + \int_0^T (j'_\epsilon(u_m) - j'_\epsilon(\phi), u_m - \phi) \, dt$$
$$+ \int_0^T \langle A u_m - A\phi, u_m - \phi \rangle_{V'_b, V_b} \, dt.$$

One can show obviously that X_m is positive. For instance, we can rewrite X_m such that

$$\begin{aligned}X_m = \int_0^T & (j'_\epsilon(u_m), u_m) + \langle Au_m, u_m\rangle_{V_b', V_b} + \langle bu'_m, u_m\rangle_{V_b', V_b} \\ & - (j'_\epsilon(u_m), \phi) - (j'_\epsilon(\phi), u_m - \phi) \\ & - \langle Au_m, \phi\rangle_{V_b', V_b} - \langle A\phi, u_m - \phi\rangle_{V_b', V_b} \\ & - \langle bu'_m, \phi\rangle_{V_b', V_b} - \langle b\phi', u_m - \phi\rangle_{V_b', V_b}\, dt.\end{aligned}$$

Substituting (4.5) in the above expression

$$\begin{aligned}X_m = & \int_0^T \langle bf, u_m\rangle_{V_b', V_b} - (j'_\epsilon(u_m), \phi) - (j'_\epsilon(\phi), u_m - \phi) \\ & - \langle Au_m, \phi\rangle_{V_b', V_b} - \langle A\phi, u_m - \phi\rangle_{V_b', V_b} - \langle bu'_m, \phi\rangle_{V_b', V_b} - \langle b\phi', u_m - \phi\rangle_{V_b', V_b}\, dt.\end{aligned}$$

Then

$$\begin{aligned}X_m \underset{m\to 0}{\longrightarrow} & \int_0^T \langle bf, u_\epsilon\rangle_{V_b', V_b} - (\xi, \phi) - (j'_\epsilon(\phi), u_\epsilon - \phi) \\ & - \langle Au_\epsilon, \phi\rangle_{V_b', V_b} - \langle A\phi, u_\epsilon - \phi\rangle - \langle bu'_\epsilon, \phi\rangle_{V_b', V_b} - \langle b\phi', u_\epsilon - \phi\rangle_{V_b', V_b}\, dt \\ & := X_\epsilon.\end{aligned}$$

Again, take $v = u_\epsilon$ in (4.14), and then substitute this latter in X_ϵ, we get

$$\begin{aligned}X_\epsilon = & \int_0^T \Big[\langle bu'_\epsilon, u_\epsilon\rangle_{V_b', V_b} + \langle Au_\epsilon, u_\epsilon\rangle_{V_b', V_b} + \langle \xi, u_\epsilon\rangle_{V_b', V_b} - \langle \xi, \phi\rangle_{V_b', V_b} - (j'_\epsilon(\phi), u_\epsilon - \phi) \\ & - \langle Au_\epsilon, \phi\rangle_{V_b', V_b} - \langle A\phi, u_\epsilon - \phi\rangle_{V_b', V_b} - \langle bu'_\epsilon, \phi\rangle_{V_b', V_b} - \langle b\phi', u_\epsilon - \phi\rangle_{V_b', V_b}\Big]\, dt \\ = & \int_0^T \Big[\langle bu'_\epsilon - b\phi', u_\epsilon - \phi\rangle_{V_b', V_b} + \langle Au_\epsilon - A\phi, u_\epsilon - \phi\rangle_{V_b', V_b} + (\xi - j'_\epsilon(\phi), u_\epsilon - \phi)\Big]\, dt.\end{aligned}$$

As $X_m \geq 0$, then $X_\epsilon \geq 0$. Take now $\phi = u_\epsilon - \theta\psi$ with $\theta \in \mathbb{R}$ in the expression of X_ϵ. This yields

$$X_\epsilon = \int_0^T \langle b\theta\psi,', \theta\psi\rangle_{V_b', V_b} + \langle A(\theta\psi), \theta\psi\rangle_{V_b', V_b} + (\xi - j'_\epsilon(u_\epsilon - \theta\psi), \theta\psi)\, dt \geq 0.$$

Dividing by θ we get

$$\int_0^T \theta\langle b\psi,', \psi\rangle_{V_b', V_b} + \theta\langle A(\psi), \psi\rangle_{V_b', V_b} + (\xi - j'_\epsilon(u_\epsilon - \theta\psi), \psi)\, dt \geq 0.$$

Take $\theta \to 0$, we obtain

$$\int_0^T (\xi - j'_\epsilon(u_\epsilon), \psi)\, dt \geq 0.$$

This is true for all ψ, hence:

$$\xi = j'_\epsilon(u_\epsilon).$$

Therefore in this step we have proved the existence of a sequence $(u_\epsilon)_\epsilon$ such that:

$$\begin{cases} u_\epsilon \text{ in, } L^\infty(0,T;H_b) \cap L^2(0,T;V_b), \\ \\ bu'_\epsilon \text{ in } L^2(0,T;V_b'), \\ \\ \int_0^T \langle bu'_\epsilon, v\rangle_{V_b', V_b} + \langle Bu_\epsilon, v\rangle_{V_b', V_b} + \langle Au_\epsilon, v\rangle_{V_b', V_b} + (j'_\epsilon(u_\epsilon), v) = \int_0^T \langle bf, v\rangle_{V_b', V_b} \\ \forall v \in L^2(0,T;V_b). \end{cases} \tag{4.15}$$

(4) **Step 4: Passage to the limit in ϵ.** From the previous steps, in particular (4.15), we conclude that there exists a subsequence of u_ϵ, that is still denoted the same, such that

$$\begin{aligned} u_\epsilon &\rightharpoonup u \text{ in } L^2(0,T,V_b), \\ u_\epsilon &\overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T,H_b), \\ bu'_\epsilon &\rightharpoonup bu' \text{ in } L^2(0,T,V_b'). \end{aligned}$$

Again, by compactness theory we get $u_\epsilon \to u$ strongly in $L^2(0,T;H_b)$. We introduce now the following quantity

$$Z_\epsilon = \int_0^T \langle bu'_\epsilon, v-u_\epsilon\rangle_{V_b',V_b} + \langle Au_\epsilon, v-u_\epsilon\rangle_{V_b',V_b} + \langle Bu_\epsilon, v-u_\epsilon\rangle_{V_b',V_b} + j_\epsilon(v) - j_\epsilon(u_\epsilon) - \langle bf, v-u_\epsilon\rangle_{V_b',V_b}\,dt.$$

Taking $v = u_\epsilon$ in (4.15) (which is allowed since $u_\epsilon \in L^2(0,T,V_b)$), and then substituting in Z_ϵ, we get

$$Z_\epsilon = \int_0^T j_\epsilon(v) - j_\epsilon(u_\epsilon) - (j'_\epsilon(u_\epsilon), v-u_\epsilon)\,dt.$$

In fact, since j_ϵ is coercive, the we get $Z_\epsilon \geq 0$. Hence

$$\int_0^T \Big[\langle bu'_\epsilon, v\rangle_{V_b',V_b} + \langle Au_\epsilon, v\rangle_{V_b',V_b} + \langle Bu_\epsilon, v\rangle_{V_b',V_b} + j_\epsilon(v) - \langle bf, v-u_\epsilon\rangle_{V_b',V_b}\Big]\,dt \geq \int_0^T \Big[\langle bu'_\epsilon, u_\epsilon\rangle_{V_b',V_b} + \langle Au_\epsilon, u_\epsilon\rangle_{V_b',V_b} + \underbrace{\langle Bu_\epsilon, u_\epsilon\rangle_{V_b',V_b}}_{0} + j_\epsilon(u_\epsilon)\Big]\,dt.$$

Now as we have $u_\epsilon \to u$ in $L^2(0,T,H_b)$, then there exists a subsequence such that $u_\epsilon(T) \to u(T)$ in H_b. Therefore:

$$\liminf_{\epsilon\to 0}\Big[\int_0^T \langle bu'_\epsilon, v\rangle_{V_b',V_b} + \langle Au_\epsilon, v\rangle_{V_b',V_b} + \langle Bu_\epsilon, v\rangle_{V_b',V_b} + j_\epsilon(v) - \langle bf, v-u_\epsilon\rangle_{V_b',V_b}\,dt\Big] \geq \frac{1}{2}\liminf_{\epsilon\to 0}\|u_\epsilon(T)\|^2_{L^2_b(\Omega)} - \frac{1}{2}\|u_0\|^2_{L^2_b(\Omega)} + \liminf_{\epsilon\to 0}\int_0^T \langle Au_\epsilon, u_\epsilon\rangle_{V_b',V_b}\,dt + \liminf_{\epsilon\to 0}\int_0^T j_\epsilon(u_\epsilon)\,dt.$$

Due to the fact that all terms on the right hand side are lower semi continuous (norms and linear continuous maps are lower semi continuous), and using lemma 4.3, we obtain

$$\int_0^T \langle bu', v\rangle_{V_b',V_b} + \langle Au, v\rangle_{V_b',V_b} + \langle Bu, v\rangle_{V_b',V_b} + j(v) - \langle bf, v-u\rangle_{V_b',V_b} \geq \frac{1}{2}\|u(T)\|^2_{L^2_b(\Omega)} - \frac{1}{2}\|u_0\|^2_{L^2_b(\Omega)} + \int_0^T \langle Au, u\rangle_{V_b',V_b}\,dt + \int_0^T j(u)\,dt.$$

Finally, we get

$$\int_0^T \Big(\langle bu', v-u\rangle_{V_b',V_b} + \langle Au, v-u\rangle_{V_b',V_b} + \langle Bu, v-u\rangle_{V_b',V_b} + j(v) - j(u) - \langle bf, v-u\rangle_{V_b',V_b}\Big)\,dt \geq 0 \quad \forall v \in L^2(0,T;V_b). \tag{4.16}$$

The last step is to prove that the above inequality holds not in integral form but almost everywhere in $(0,T)$, in particular, we have to prove

$$\langle bu'(t), v-u(t)\rangle_{V_b',V_b} + \langle Au(t), v-u(t)\rangle_{V_b',V_b} + \langle Bu(t), v-u(t)\rangle_{V_b',V_b} + j(v) - j(u(t)) \geq \langle bf, v-u(t)\rangle_{V_b',V_b}\,dt, \quad t\in(0,T), \quad v \in L^2(0,T;V_b). \tag{4.17}$$

Proving that in fact allows us tp prove the existence of solution for system (3.5). Let us fix a random function $v \in V_b$, and $t_0 \in (0,T)$. Define

$$\Theta_j = \left(t_0 - \frac{1}{j}, t_0 + \frac{1}{j}\right) \subset (0,T) \qquad j \text{ large enough.}$$

We introduce

$$\bar{v} = \begin{cases} v & t \in \Theta_j \\ u(t) & t \in [0,T]\backslash\Theta_j. \end{cases}$$

Hence taking $v=\bar{v}$, inequality (4.16) becomes

$$\underbrace{\int_{\Theta_j}\langle bu'+Au+Bu-bf,v\rangle_{V_b',V_b}\,dt}_{X} + \underbrace{\int_{\Theta_j}\Big[j(v)-j(u)-\langle bu'+Au-bf,u\rangle_{V_b',V_b}\Big]\,dt}_{Y}\geq 0.$$

Using the Lebesgue theorem in the differentiation of set functions, we have

$$X \underset{j\to 0}{\longrightarrow} \langle bu'(t_0)+Au(t_0)+Bu(t_0)-bf,v\rangle_{V_b',V_b},$$

and

$$Y \underset{j\to 0}{\longrightarrow} \langle bu'(t_0)+Au(t_0)-bf,u(t_0)\rangle_{V_b',V_b}+j(u(t_0))-j(v).$$

But t_0 is arbitrary in (0,T), thus we get

$$\begin{aligned}\langle bu'(t),v-u(t)\rangle_{V_b',V_b}+\langle Au(t),v-u(t)\rangle_{V_b',V_b}+\langle Bu(t),v-u(t)\rangle_{V_b',V_b}\\ +j(v)-j(u(t))\geq \langle bf,v-u(t)\rangle_{V_b',V_b},\end{aligned}\tag{4.18}$$

for all $t\in(0,T)$, for all $v\in L^2(0,T;V_b)$. By this we end the proof of existence.

Uniqueness of solution. let $\mathrm{u}_1,\mathrm{u}_2$ be two solutions of the variational inequality (4.18). We take $v=\mathrm{u}_2(t)$ (resp $v=\mathrm{u}_1(t)$) as a test function in the variational inequality satisfied by u_1 (resp. u_2). Let $U=\mathrm{u}_1-\mathrm{u}_2$. Now, if we add both inequalities we get

$$\begin{aligned}\int_\Omega \partial_t U\cdot U\,b\,dx+2\mu\int_\Omega |\,\mathrm{D}(U)|^2\,b\,dx+\lambda\int_\Omega |\,\mathrm{div}(U)|^2\,b\,dx\\ -\int_\Omega(\mathrm{u}_1\cdot\nabla)\,\mathrm{u}_1\cdot U\,b\,dx+\int_\Omega(\mathrm{u}_2\cdot\nabla)\,\mathrm{u}_2\cdot U\,b\,dx\leq 0.\end{aligned}$$

Therefore, using Remark 4.1, and the fact that

$$\int_\Omega(u\cdot\nabla)v\cdot v\,b\,dx=0\quad\text{ for all }\quad u,v\quad\text{in }\ V_b,$$

we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega|U|^2\,b\,dx+2C_1\mu\|U\|_{V_b}^2 &\leq \int_\Omega(\mathrm{u}_1\cdot\nabla)\,\mathrm{u}_1\cdot U\,b\,dx-\int_\Omega(\mathrm{u}_2\cdot\nabla)\,\mathrm{u}_2\cdot U\,b\,dx\\ &=\int_\Omega(\mathrm{u}_1\cdot\nabla)U\cdot U\,b\,dx+\int_\Omega(\mathrm{u}_1\cdot\nabla)\,\mathrm{u}_2\cdot U\,b\,dx-\int_\Omega(\mathrm{u}_2\cdot\nabla)\,\mathrm{u}_2\cdot U\,b\,dx\\ &=\int_\Omega(U\cdot\nabla)\,\mathrm{u}_2\cdot U\,b\,dx,\end{aligned}\tag{4.19}$$

where C_1 is the constant resulting from Korn's inequality. We estimate the term on the right hand side of Inequality (4.19) using Weighted Gagliardo-Nirenberg inequality and Young's inequality as follows

$$\begin{aligned}\left|\int_\Omega(U\cdot\nabla)\cdot\mathrm{u}_2\cdot U\,b\,dx\right| &\leq \|U\|_{L^4_b(\Omega)}^2\|\nabla u_2\|_{L^2_b(\Omega)}\\ &\leq C\|\nabla u_2\|_{L^2_b(\Omega)}\|U\|_{V_b}\|U\|_{L^2_b(\Omega)}\\ &\leq C\|\nabla u_2\|_{L^2_b(\Omega)}\big(\frac{1}{2C_1\mu}\|U\|_{L^2_b(\Omega)}^2+\frac{C_1\mu}{2}\|U\|_{V_b}^2\big).\end{aligned}\tag{4.20}$$

From Inequality (4.19) and estimate (4.20), we deduce that

$$\frac{1}{2}\frac{d}{dt}\|U|_{L^2_b(\Omega)}^2\leq C\|\nabla u_2\|_{L^2_b(\Omega)}\|U\|_{L^2_b(\Omega)}^2.$$

which readily ensures that $U=0$ whence $\mathrm{u}_1-\mathrm{u}_2=0$ by applying a Gronwall's type inequality. Hence, the proof of uniqueness of solution is finished.

5. Newtonian fluids as a limit of Non-Newtonian fluids

Viscous lake systems for Newtonian fluids are asymptotically derived systems according to two successive approximations that are characterized by the smallness of non dimensional parameters. The first is the rigid lid approximation assuming that the typical deviation of the top of the fluid's surface from the mean level is much smaller than the typical depth. This smallness can be regarded also as a Froude number due to the dynamics of the physical case at hand. The second approximation is the shallow water approximation characterizing the ansatz of the smallness of the typical depth compared to the typical horizontal length, see for instance [20]. Nevertheless, in this section, we will prove the existence of a weak solution of the viscous lake system by passing to the zero limit of g (yield stress limit) in the variational inequality satisfied by the weak solution of Bingham system (1.3). Roughly speaking, one can naively conclude that the fluid will behave as a viscous liquid once g vanishes, yet for the mathematical justification we need to set on a rigorous proof. So, we will prove the following theorem.

Theorem 5.1. *Let Ω a bounded Lipschitz domain and b satisfying Hypothesis (**II**). Consider the Bingham model introduced in section 3. We suppose that all the parameters therein are fixed except for the the yield stress limit g which is assumed to be independent of other parameters and rendered to vary (consequently taken to zero). Denote by u_g the solution of system* (3.5) *proved in section 4. Then, there exits u in $L^\infty(0,T,H_b)\cap L^2(0,T,V_b)$ such that*

$$\begin{cases} u_g \to u \quad \text{weakly in} \quad L^\infty(0,T;H_b)\cap L^2(0,T;V_b), \\ bu_g' \to bu' \quad \text{weakly in} \quad L^2(0,T;V_b'). \end{cases} \tag{5.1}$$

where u is a weak solution of the viscous Lake system.

Proof. Let's recall the definition of the weak solution of viscous lake equations: $u\in L^2(0,T,V_b)\cap L^\infty(0,T,H_b)$ is said to be a weak solution of the viscous lake equations if it satisfies

$$\langle u', v\rangle_{L^2_b} + b(u,u,v) + a(u,v) = \langle bf, v\rangle_{V_b',V_b} \ \forall v\in V_b$$

From (4.2), we find that the weak solution u_g satisfies

$$\frac{1}{2}\frac{d}{dt}\|u_g\|^2_{L^2_b(\Omega)} + \mu\|u_g\|^2_{V_b} + \lambda\|\operatorname{div} u\|^2_{L^2_b(\Omega)} \le C.$$

Integrating in time, we get

$$\frac{1}{2}\|u_g(t)\|^2_{L^2_b(\Omega)} + \mu\int_0^t \|u_g\|^2_{V_b}\,dt + \lambda\int_0^t \|\operatorname{div} u\|^2_{L^2_b(\Omega)}\,t \le \frac{1}{2}\|u_g(0)\|^2_{L^2_b(\Omega)} + C \le C.$$

Thus, u_g remains in a bounded set with respect to g in $L^\infty(0,T;H_b)\cap L^2(0,T;V_b)$. In addition, from previous section, we have

$$bu'_{\epsilon g} \rightharpoonup bu'_g \ \in L^2(0,T;V_b'),$$

so, bu'_g is also bounded in $L^2(0,T;V_b')$ (the bound is uniform in g). Hence, there exists $u\in L^2(0,T;V_b)\cap L^\infty(0,T;H_b)$ such that

$$u_g \rightharpoonup u \in L^2(0,T;V_b),$$

$$u_g \overset{*}{\rightharpoonup} u \in L^\infty(0,T;H_b),$$

$$bu'_g \rightharpoonup bu' \in L^2(0,T;V_b').$$

Following the same compactness technique used in previous section, we get $u_g \longrightarrow u \in L^2(0,T;H_b)$, and the convergence of the corresponding operators follows as in section 4. Hence u_g satisfies

$$\langle bu'_g, v-u_g\rangle_{V_b',V_b} + a(u_g, v-u_g) + b(u_g,u_g,v-u_g) + j(v) - j(u_g) \ge \langle bf, v-u_g\rangle_{V_b',V_b}.$$

Consequently

$$\langle bu'_g, v\rangle_{V_b',V_b} + a(u_g,v) + b(u_g,u_g,v) + j(v) - j(u_g) - \langle bf, v-u_g\rangle_{V_b',V_b} \ge \frac{1}{2}\frac{d}{dt}\langle u_g,u_g\rangle_{L^2_b(\Omega)} + a(u_g,u_g).$$

Integrating in time gives

$$\int_0^t \langle bu_g', v\rangle_{V_b',V_b} + a(u_g,v) + b(u_g,u_g,v) + j(v) - j(u_g) - \langle bf, v-u_g\rangle_{V_b',V_b}\,dt$$
$$\geq \frac{1}{2}\|u_g(t)\|^2_{L^2_b(\Omega)} - \frac{1}{2}\|u_g(0)\|^2_{L^2_b(\Omega)} + \int_0^t a(u_g,u_g)\,dt.$$

Using the fact that RHS is lower semi continuous , and since $|\,\mathrm{D}(u_g)|$ and $|\,\mathrm{D}(v)|$ are bounded in $L^1(0,T;H_b)$, then

$$\lim_{g\to 0} j(v) = \lim_{g\to 0}\int_\Omega g|\,\mathrm{D}(v)|\,b\,dx = 0,$$

and

$$\lim_{g\to 0} j(u_g) = \lim_{g\to 0}\int_\Omega g|\,\mathrm{D}(u_g)|\,b\,dx = 0.$$

Therefore, for all t in $(0,T)$, we have

$$\begin{aligned}\liminf_{g\to 0}&\left[\int_0^t \langle bu_g', v\rangle_{V_b',V_b} + a(u_g,v) + b(u_g,u_g,v) - \langle bf, v-u_g\rangle_{V_b',V_b}\,dt\right]\\ &\geq \liminf_{g\to 0}\left[\frac{1}{2}\|u_g(t)\|^2_{L^2_b(\Omega)} - \frac{1}{2}\|u_0\|^2_{L^2_b(\Omega)} + \int_0^t a(u_g,u_g)\,dt\right].\end{aligned} \tag{5.2}$$

Due to the convergence of the LHS terms and the last term in the RHS of (5.2) (following the same strategy as in the proof of theorem (4.2) in section 4), and due to the lower semi-continuity of the norm operator for the topology of L^2_b, we obtain

$$\begin{aligned}\int_0^t \langle bu', v\rangle_{V_b',V_b} &+ a(u,v) + b(u,u,v) - \langle bf, v-u\rangle_{V_b',V_b}\,dt\\ &\geq \frac{1}{2}\|u(t)\|^2_{L^2_b(\Omega)} - \frac{1}{2}\|u_0\|^2_{L^2_b(\Omega)} + \int_0^t a(u,u)\,dt\\ &= \frac{1}{2}\int_0^t \frac{d}{dt}\langle u,u\rangle_{L^2_b(\Omega)}\,dt + \int_0^t a(u,u)\,dt.\end{aligned}$$

Thus, for all $v\in L^2(0,T,V_b)$, and for all t in $(0,T)$, we get

$$\int_0^t \langle bu', v-u\rangle_{V_b',V_b} + a(u,v-u) + b(u,u,v-u)\,dt \geq \int_0^t \langle bf, v-u\rangle_{V_b',V_b}\,dt.$$

Again, using Lebesgue theory in the differentiation of sets functions, we obtain for all $v\in L^2(0,T;V_b)$

$$\langle bu', v-u\rangle_{V_b',V_b} + a(u,v-u) + b(u,u,v-u) \geq \langle bf, v-u\rangle_{V_b',V_b} \quad \text{a.e in } [0,T]. \tag{5.3}$$

If we suppose that $\phi = \pm(u-v)\in L^2(0,T;V_b)$, then substituting v in (5.3) yields

$$\langle bu', \phi\rangle_{V_b',V_b} + a(u,\phi) + b(u,u,\phi) = \langle bf, \phi\rangle_{V_b',V_b} \quad \text{a.e in } [0,T],$$

which means that u satisfies the weak formulation of the viscous Lake system. Thus we end the proof.

6. Numerical Scheme

In this section, we propose in the one dimensional case a numerical scheme for the approximation of the studied model i.e. the system (1.3)-(1.4).

6.1. Semi-discrete scheme.

The system (1.3)-(1.4) can be rewritten as follows

$$\partial_t X + \partial_x F(X) - b\,\partial_x p = bf,$$
$$\partial_x X = 0,$$

where

$$X = (bu), \quad F(X) = bu^2.$$

Notice that in the 1d case, the definition of σ given by (1.4) reduces to

$$\sigma := \begin{cases} 2\mu b\,\partial_x u + gb\dfrac{\partial_x u}{|\,\partial_x u|} & \text{if } \partial_x u\neq 0,\\ |\sigma| < bg, & \text{if } \partial_x u = 0.\end{cases} \tag{6.1}$$

For the time discretisation, we denote $t^n = \sum_{k\leq n} \Delta t^k$ where the time steps Δt^k will be precised later though a CFL condition. Following [11], we use an operator splitting technique resulting in a two step scheme

$$\frac{X^{n+1/2} - X^n}{\Delta t^n} = -\,\partial_x F(X^n) + bf^n, \tag{6.2}$$

$$\frac{X^{n+1} - X^{n+1/2}}{\Delta t^n} = b\,\partial_x p^{n+1}, \tag{6.3}$$

where the quantity X^{n+1} satisfies the divergence free constraint

$$\partial_x(X^{n+1}) = 0. \tag{6.4}$$

The system (6.2)-(6.4) has to be completed with suitable boundary conditions that will be precised later, see paragraph 6.4.

More precisely, the prediction step (6.2) consists in the resolution of advection diffusion equation i.e.

$$X^{n+1/4} = X^n - \Delta t^n \partial_x F(X^n) + \Delta t^n\,\partial_x\Big(\mu b\,\partial_x\big(\frac{X^{n+1/4}}{b}\big)\Big), \tag{6.5}$$

$$X^{n+1/2} = X^{n+1/4} + \Delta t^n \partial_x(b\tilde{\sigma}^{n+1/4}) + \Delta t^n b f^n, \tag{6.6}$$

where

$$\tilde{\sigma}^{n+1/4} = \begin{cases} g\frac{\partial_x u^{n+1/4}}{|\,\partial_x u^{n+1/4}|} & \text{if } \partial_x u^{n+1/4} \neq 0 \\ |\tilde{\sigma}^{n+1/4}| \leq g & \text{else} \end{cases}$$

Notice that the definition of $\tilde{\sigma}^{n+1/4}$ will be precised hereafter. Notice also that in order to avoid a too restrictive CFL condition we propose an implicit discretisation of the linear viscosity term.

Concerning the correction step (6.3)-(6.4), inserting (6.4) into (6.3) gives the elliptic equation governing the pressure p^{n+1} under the form

$$\partial_x\big(b\,\partial_x p^{n+1}\big) = -\frac{1}{\Delta t^n}\,\partial_x X^{n+1/2}. \tag{6.7}$$

Thus, the numerical approximation of (1.3)-(1.4) consists in the numerical resolution of Eqs. (6.5)-(6.6), together with (6.7) and (6.3).

6.2. **Discrete scheme.** To approximate the solution $(X,p)^T$ of the system (1.3)-(1.4), we use a combined finite volume/finite element framework. We assume that the computational domain is discretized with I nodes x_i, $i = 1,\dots,I$. We denote C_i the cell $(x_{i-1/2}, x_{i+1/2})$ of length $\Delta x_i = x_{i+1/2} - x_{i-1/2}$ with $x_{i+1/2} = (x_i + x_{i+1})/2$. We denote $X_i^n = (b_i u_i^n)$ with

$$X_i^n \approx \frac{1}{\Delta x_i}\int_{C_i} X(x,t^n)dx,$$

the approximate solution at time t^n on the cell C_i. The pressure p is discretized on a staggered grid (in fact the dual mesh if we consider the 2d case)

$$p^n_{i+1/2} \approx \frac{1}{\Delta x_{i+1/2}}\int_{x_i}^{x_{i+1}} p(x,t^n)dx,$$

with $\Delta x_{i+1/2} = x_{i+1} - x_i$.

Hence, the discrete (in space and time) version of Eqs. (6.5)-(6.6),(6.7) is given by

$$\begin{aligned} X_i^{n+1/4} &= X_i^n - \frac{\Delta t^n}{\Delta x_i}\big(\mathcal{F}^n_{i+1/2} - \mathcal{F}^n_{i-1/2}\big) \\ &\quad + \mu\frac{\Delta t^n}{\Delta x_i^2}\Big(\frac{b_{i+1/2}}{b_{i+1}}X^{n+1/4}_{i+1} - \frac{b_{i+1/2}+b_{i-1/2}}{b_i}X_i^{n+1/4} + \frac{b_{i-1/2}}{b_{i-1}}X^{n+1/4}_{i-1}\Big), \end{aligned} \tag{6.8}$$

$$X_i^{n+1/2} = X_i^{n+1/4} + \frac{\Delta t^n}{\Delta x_i}\Bigg(b_{i+1/2}\big(\tilde{\sigma}^{n+1/4}_{i+1/2} + \tau^n_{i+1/2-}\big) - b_{i-1/2}\big(\tilde{\sigma}^{n+1/4}_{i-1/2} + \tau^n_{i-1/2+}\big)\Bigg), \tag{6.9}$$

$$X_i^{n+1} = X_i^{n+1/2} + \frac{\Delta t^n}{\Delta x_i} b_i\big(p^{n+1}_{i+1/2} - p^{n+1}_{i-1/2}\big), \tag{6.10}$$

$$b_{i+1}p^{n+1}_{i+3/2} - (b_{i+1}+b_i)p^{n+1}_{i+1/2} + b_i p^{n+1}_{i-1/2} = -\frac{\Delta x_i}{\Delta t^n}\big(X^{n+1/2}_{i+1} - X^{n+1/2}_i\big), \tag{6.11}$$

with

$$b_{i+1/2} = \min\{b_i, b_{i+1}\}, \tag{6.12}$$

and $\mathcal{F}^n_{i+1/2}$ is a numerical flux accounting for the advection term bu^2 and any classical flux e.g. full upwind, Rusanov... can be used. In the simulation results given at the end of this section, an upwind formula has been used and having the form

$$\mathcal{F}^n_{i+1/2} = u^n_{i+1/2}\left(\frac{b_{i+1/2}}{b_i}X^n_i \mathbf{1}_{u^n_{i+1/2}\geq 0} + \frac{b_{i+1/2}}{b_{i+1}}X^n_{i+1}\mathbf{1}_{u^n_{i+1/2}\leq 0}\right), \tag{6.13}$$

with $u^n_{i+1/2} = (u^n_{i+1}+u^n_i)/2$. The quantities $\tau^{n+1/4}_{i+1/2-}$ and $\tau^{n+1/4}_{i-1/2+}$ correspond to discretisation using the apparent topography technique (see [4]) of the source term f and we have

$$\tau^n_{i+1/2-} = (x_{i+1/2} - x_i)f^n_{i+1/2}, \qquad \tau^n_{i-1/2+} = (x_i - x_{i-1/2})f^n_{i-1/2}.$$

It remains to define the quantities $\tilde{\sigma}^{n+1/4}_{i\pm 1/2}$ in Eq. (6.9) and we use the strategy proposed by Bouchut [4, paragraph 4.12.1]. The definition of σ given by (6.1) has to be understood as multivalued: σ can be any value in $[-bg, bg]$ when $\partial_x u = 0$. When $\partial_x u = 0$, Eqs. (6.5)-(6.6) become

$$X^{n+1/4} = X^n,$$
$$X^{n+1/2} = X^n + \Delta t^n \partial_x(b\tilde{\sigma}^{n+1/4} + b\tau^n),$$

with $\partial_x(b\tau^n) = f^n$ and we define

$$\tilde{\sigma}^{n+1/4}_{i+1/2} = -\mathrm{proj}_g\left(\tau^n_{i+1/2-} + \frac{1}{\Delta t^n}\Big(\frac{X^{n+1/4}_{i+1}}{b_{i+1}} - \frac{X^{n+1/4}_i}{b_i}\Big)\right), \tag{6.14}$$

where

$$\mathrm{proj}_g(X) = \begin{cases} X & \text{if } |X| \leq g \\ g\frac{X}{|X|} & \text{if } |X| > g \end{cases}$$

The formula (6.14) is consistent with the definition (6.1) because $|\tilde{\sigma}^{n+1/4}_{i+1/2}| \leq g$ and if $\frac{X^{n+1/4}_{i+1}}{b_{i+1}} \neq \frac{X^{n+1/4}_i}{b_i}$ then for Δt^n small enough the quantity

$$\frac{1}{\Delta t^n}\Big(\frac{X^{n+1/4}_{i+1}}{b_{i+1}} - \frac{X^{n+1/4}_i}{b_i}\Big),$$

will dominate the other giving

$$\tilde{\sigma}^{n+1/4}_{i+1/2} \approx g\mathrm{sign}\Big(\frac{X^{n+1/4}_{i+1}}{b_{i+1}} - \frac{X^{n+1/4}_i}{b_i}\Big).$$

6.3. **When $b \to 0$.** Thanks to the definition of $b_{i\pm 1/2}$ given by (6.12), Eqs. (6.8)-(6.10) well behave when b_i tends to zero. More precisely, when $b_i = 0$, Eqs. (6.8)-(6.10) reduce to $X^{n+1}_i = X^{n+1/2}_i = X^{n+1/4}_i = X^n_i = 0$. And we adopt the modified version of (6.11) under the form

$$b_{i+1}p^{n+1}_{i+3/2} - (b^\varepsilon_{i+1} + b^\varepsilon_i)p^{n+1}_{i+1/2} + b_i p^{n+1}_{i-1/2} = -\frac{\Delta x_i}{\Delta t^n}\big(X^{n+1/2}_{i+1} - X^{n+1/2}_i\big), \tag{6.15}$$

with $b^\varepsilon_i = \max\{b_i, \varepsilon\}$ and $0 < \varepsilon \ll 1$.

6.4. **Boundary conditions.** Boundary conditions have to be defined for Eqs. (6.8) and (6.11) at both side of the domain. And we have to face two difficulties

- Eq. (6.8) contains an hyperbolic part and a parabolic part whereas Eq. (6.11) is an elliptic equation,
- the boundary conditions applied to (6.11) have to be consistent with those applied (6.8).

The proposed solution has been adapted from [1] but notice that other solutions can be investigated since the coupling of the boundary conditions between a hyperbolic step and a parabolic/elliptic step is far from being obvious.

Let us consider the boundary at the entry of the domain i.e. at abscissa $x_{1/2}$. We assume the inflow is prescribed typically $b_0 u^n_0 = q_{in}(t^n)$ where $q_{in}(t^n)$ is a given quantity then the definition (6.13) can be used to define $\mathcal{F}^n_{1/2}$ under the form

$$\mathcal{F}^n_{1/2} = u^n_{1/2}\left(q_{in}(t^n)\mathbf{1}_{u^n_{1/2}\geq 0} + X^n_1 \mathbf{1}_{u^n_{1/2}\leq 0}\right),$$

with $u^n_{1/2} = (u^n_1 + q_{in}(t^n)/b_1)/2$ and assuming $b_0 = b_1 > 0$. For the parabolic part of Eq. (6.8) we use Dirichlet boundary conditions defined by $q_{in}(t^n)$.

Since the inflow $q_{in}(t^n)$ is prescribed, it is natural to assume Neumann boundary at the entry for p in Eq. (6.11) i.e. $\partial_x p|_0(t^{n+1}) = 0$.

Now we consider the boundary at the exit of the domain i.e. at abscissa $x_{I+1/2}$. Assuming Neumann boundary conditions, we define

$$\mathcal{F}^n_{I+1/2} = u^n_I X^n_I,$$

and $\partial_x X|_{I+1/2}(t^{n+1/4}) = 0$ with $b_{I+1} = b_I$ for the parabolic part. In this context, it is convenient to assume Dirichlet boundary at the exit for p in Eq. (6.11) i.e. $p|_{I+1}(t^{n+1}) = 0$.

For a more complete justification of the choices for the boundary conditions, the reader can refer to [1].

6.5. **Simulation results.** We present now some simulations results for the model (1.3)-(1.4) with the numerical scheme (6.8)-(6.11) where we have chosen the source term f

$$f = g\,\partial_x b,$$

mimicking the effects of the slope over the fluid rheology.

We consider a fluid domain defined by $x \in [0, x_{max}]$ with $x_{max} = 20$ meters and $I = 500$ nodes and a domain profile (see Fig. 1) defined by

$$b(x) = 1 - \frac{1}{2}\tanh\Big(x - \frac{x_{max}}{2}\Big) + 0.3e^{-(x-15)^2} - \frac{1}{2}e^{-(x-6)^2/2},$$

the fluid is initially at rest i.e.

$$u^0_i = 0, \qquad \forall i \in I.$$

As mentioned in paragraph 6.4, the inflow is prescribed at the entry $x = 0$ with

$$q_{in}(t) = 2 + \sin(2\pi t/T),$$

the simulations are carried out over the time interval $(0, T)$ with $T = 20$ seconds.

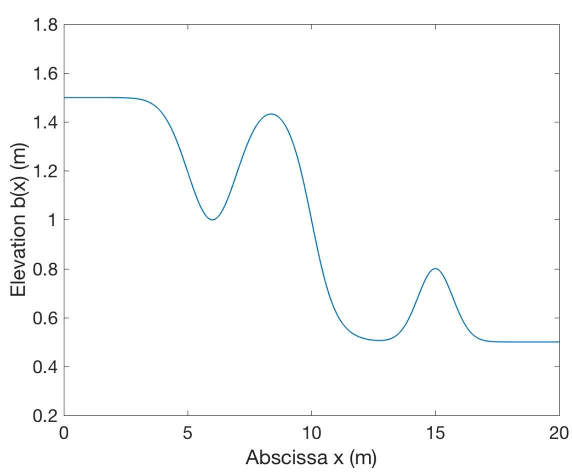
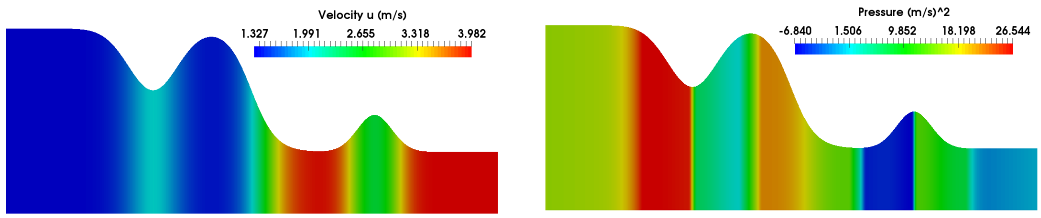


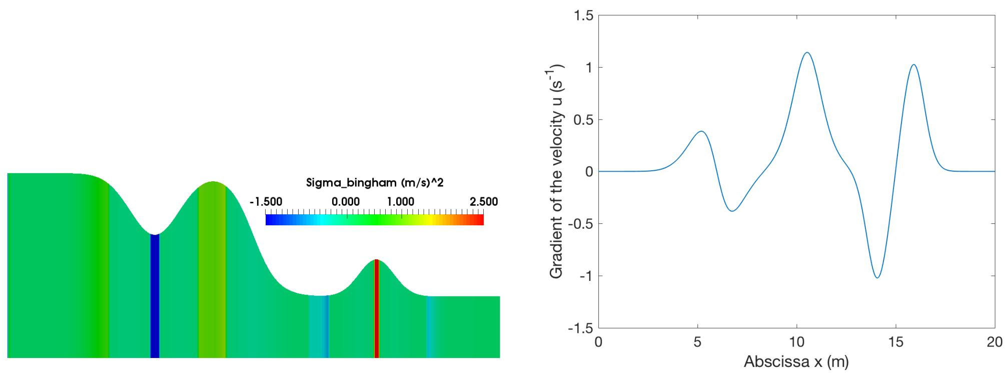
FIGURE 1. Chosen profile for $b(x)$.

The simulated velocity profile at time $t = T/2$ is depicted over Fig. 2-*(a)* whereas the variations of the simulated pressure is given over Fig. 2-*(b)*. The variations of $\tilde{\sigma}$ appear over Fig.3-*(b)*. Near the boundaries, the gradient of the velocity is very small $\partial_x u \approx 0$, see Fig. 3-*(b)*.



(a) *(b)*

FIGURE 2. *(a)* variations of the velocity u and *(b)* variations of the pressure p in the fluid domain at time $t = T/2$.



(a) *(b)*

FIGURE 3. *(a)* variations of the quantity $\tilde{\sigma}$ and *(b)* variations of $\partial_x u$ at time $t = T/2$.

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