On a New Method to Build Group Equivariant Operators by Means of Permutants

Francesco Camporesi, Patrizio Frosini, Nicola Quercioli

On a new method to build group equivariant operators by means of permutants

Francesco Camporesi, Patrizio Frosini, and Nicola Quercioli

Department of Mathematics, University of Bologna, Italy
francesco.camporesi@studio.unibo.it, patrizio.frosini@unibo.it,
nicola.quercioli2@unibo.it

Abstract. The use of group equivariant operators is becoming more and more important in machine learning and topological data analysis. In this paper we introduce a new method to build G-equivariant non-expansive operators from a set Φ of bounded and continuous functions $\varphi : X \to \mathbb{R}$ to Φ itself, where X is a topological space and G is a subgroup of the group of all self-homeomorphisms of X.

Keywords: Natural pseudo-distance, filtering function, group action, group equivariant non-expansive operator, persistent homology group, topological data analysis

Introduction

In the last years the problem of data analysis has assumed a more and more relevant role in science, and many researchers have started to become interested in it from several different points of view. Some geometrical techniques have given their contribute to this topic, and persistent homology has proven itself quite efficient both for qualitative and topological comparison of data [5]. In particular, topological data analysis (TDA) has revealed important in managing the huge amount of data that surrounds us in the most varied contexts [3]. The use of TDA is based on the fact that in several practical situations the measurements of interest can be expressed by continuous $\mathbb{R}^m$-valued functions defined on a topological space, as happens for the weight of a physical body or a biomedical image [2]. However, for the sake of simplicity, in this work we will focus on real-valued functions. The continuity of the considered functions enables us to apply persistent homology, a theory that studies the birth and the death of k-dimensional holes when we move along the filtration defined by the sublevel sets of a continuous function from a topological space X to the real numbers. Interestingly, this procedure is invariant with respect to all homeomorphisms of X, that is if $g \in \mathrm{Homeo}(X)$, then φ and $\varphi \circ g$ induce on X two filtrations which have exactly the same topological properties under the point of view of persistent homology. For further and more detailed information about persistent homology, we refer the reader to [6].

The importance of group equivariance in machine learning is well-known (cf., e.g., [1,4,10,11]). The study of group equivariant non-expansive operators

(GENEOs) proposed in this work could be a first step in the path to establishing a link between persistence theory and machine learning. The ground idea is that the observer influences in a direct way the act of measurement, and that our analysis should be mainly focused on a good approximation of the observer rather than on a precise description of the data [7]. GENEOs reflect the way the information is processed by the observer, and hence they enclose the invariance the observer is interested in. In some sense, we could say that an observer can be seen as a collection of group equivariant non-expansive operators acting on suitable spaces of data. The choice of the invariance group G is a key point in this model. For example, in character recognition the invariance group should not contain reflections with respect to a vertical axis, since the symbols 'p' and 'q' should not be considered equal to each other, while this fact does not hold for the comparison of medieval rose windows.

The use of invariance groups leads us to rely on the concept of *natural pseudo-distance*. Let us consider a set Φ of continuous $\mathbb{R}$-valued functions defined on a topological space X and a subgroup G of the group $\mathrm{Homeo}(X)$ of all self-homeomorphisms of X. We assume that the group G acts on Φ by composition on the right. Now we can define the *natural pseudo-distance* d_G on Φ by setting $d_G(\varphi_1, \varphi_2) = \inf_{g \in G} \|\varphi_1 - \varphi_2 \circ g\|_\infty$, where $\|\cdot\|_\infty$ denotes the sup-norm. Although the natural pseudo-distance reflects our intent to find the best correspondence between two functions of Φ, unfortunately it leads to some practical limitations since it is difficult to compute, even when the group G has good properties.

However, the theory of group equivariant non-expansive operators makes available a method for the approximation of the natural pseudo-distance (cf. Theorem 1 in this paper). Moreover, in [8,9] it has been proven that under suitable hypotheses the space $\mathcal{F}(\Phi, G)$ of all GENEOs benefits from good computational properties, such as compactness and convexity. In order to proceed in the research about this space of operators, we devote this paper to introducing a new method to construct GENEOs by means of particular subsets of $\mathrm{Homeo}(X)$, called *permutants*. We underline that in our method we can treat the group of invariance as a variable. This is important because the change of the observer generally corresponds to a change of the invariance we want to analyze.

Our work is organized as follows. In Section 1 we start explaining the mathematical setting where our research will take place. In Section 2 we introduce our new method for the construction of group equivariant non-expansive operators. In particular, we show how specific subsets of $\mathrm{Homeo}(X)$ called permutants can help us in this procedure. In Section 3 we illustrate our method by giving two examples. Finally, in Section 4 we explore the limits of our approach by proving a result about permutants.

1 Our mathematical model

In this section we recall the mathematical model illustrated in [8]. Let us consider a (non-empty) topological space X, and the topological space $C^0_b(X, \mathbb{R})$ of the continuous bounded functions from X to $\mathbb{R}$, endowed with the topology induced by the sup-norm $\|\cdot\|_\infty$. Let Φ be a topological subspace of $C^0_b(X,\mathbb{R})$, whose elements represent our data. The functions in Φ will be called *admissible filtering functions* on the space X. We are interested in analyzing Φ by applying the invariance with respect to a subgroup G of the group $\mathrm{Homeo}(X)$ of all self-homeomorphisms of X. The group G is used to act on Φ by composition on the right, i.e. we assume that for every $\varphi \in \Phi$ and every $g \in G$ the map $\varphi \circ g$ is still in Φ. In other words, we consider the functions $\varphi, \varphi \circ g \in \Phi$ equivalent to each other for every $g \in G$.

A pseudo-metric that can be used to compare functions in this mathematical setting is the *natural pseudo-distance* d_G.

Definition 1. *We set* $d_G(\varphi_1,\varphi_2) := \inf_{g\in G}\max_{x\in X}|\varphi_1(x)-\varphi_2(g(x))|$ *for every* $\varphi_1,\varphi_2 \in \Phi$. *The function* d_G *is called the* natural pseudo-distance *associated with the group* G *acting on* Φ.

The previous pseudo-metric can be seen as the ground truth for the comparison of functions in Φ with respect to the action of the group G. Unfortunately, d_G is usually difficult to compute. However, a method to study the natural pseudo-distance via *G-equivariant non-expansive operators* is available.

Definition 2. *A G-equivariant non-expansive operator (GENEO) for the pair* (Φ, G) *is a function*

$$F : \Phi \longrightarrow \Phi$$

that satisfies the following properties:

1. $F(\varphi \circ g) = F(\varphi) \circ g, \quad \forall\ \varphi \in \Phi, \quad \forall\ g \in G;$
2. $\|F(\varphi_1) - F(\varphi_2)\|_\infty \leqslant \|\varphi_1 - \varphi_2\|_\infty, \quad \forall\ \varphi_1, \varphi_2 \in \Phi.$

The first property represents our request of equivariance with respect to the action of G, while the second one highlights the non-expansivity of the operator, since we require a control on the norm. We define $\mathcal{F}(\Phi, G)$ to be the set of all G-equivariant non-expansive operators for (Φ, G). Obviously $\mathcal{F}(\Phi, G)$ is not empty because it contains at least the identity operator.

Remark 1. The non-expansivity property implies that the operators in $\mathcal{F}(\Phi, G)$ are 1-Lipschitz and hence continuous. We highlight that GENEOs are not required to be linear, even though all the GENEOs exposed in this paper have this property.

The following key property holds, provided that X has nontrivial homology in degree k and Φ contains all the constant functions c from X to $\mathbb{R}$ such that there exists $\varphi \in \Phi$ with $c \leqslant ||\varphi||_\infty$ [8].

Theorem 1. *If $\mathcal{F}$ is the set of all G-equivariant non-expansive operators for the pair (Φ, G), then* $d_G(\varphi_1,\varphi_2) = \sup_{F\in\mathcal{F}} d_{match}(r_k(F(\varphi_1)), r_k(F(\varphi_2)))$, *where* $r_k(\varphi)$ *denotes the k-th persistent Betti number function with respect to the function* $\varphi : X \to \mathbb{R}$ *and* d_{match} *is the classical matching distance.*

Theorem 1 represents a strong link between persistent homology and the natural pseudo-distance via GENEOs. It establishes a method to compute d_G by means of G-equivariant non-expansive operators. As a consequence, the construction of GENEOs is an important step in the computation of the natural pseudo-distance. This fact justifies the interest for the result proven in Section 2.

2 A method to build GENEOs by means of permutants

In this section we introduce a new method for the construction of GENEOs, exploiting the concept of permutant. Let G be a subgroup of $\mathrm{Homeo}(X)$. We consider the conjugation map

$$\begin{aligned}\alpha_g : \mathrm{Homeo}(X) &\to \mathrm{Homeo}(X)\\ f &\mapsto g \circ f \circ g^{-1}\end{aligned}$$

where g is an element of G.

Definition 3. *A non-empty finite subset H of* $\mathrm{Homeo}(X)$ *is said to be a* permutant *for G if $\alpha_g(H) \subseteq H$ for every $g \in G$.*

Remark 2. The condition $\alpha_g(H) \subseteq H$, the finiteness of H and the injectivity of α_g imply that α_g is a permutation of the set H for every $g \in G$. Moreover, it is important to note that H is required neither to be a subset of the invariance group G, nor a subgroup of $\mathrm{Homeo}(X)$.

Remark 3. If H and K are two permutants for G, then also the union $H \cup K$ and the intersection $H \cap K$ are two permutants for G (provided that $H \cap K \neq \emptyset$).

If $H = \{h_1, \ldots, h_n\}$ is a permutant for G and $\bar{a} \in \mathbb{R}$ with $n|\bar{a}| \leqslant 1$, we can consider the operator $F_{\bar{a},H} : C_b^0(X, \mathbb{R}) \longrightarrow C_b^0(X, \mathbb{R})$ defined by setting

$$F_{\bar{a},H}(\varphi) \coloneqq \bar{a} \sum_{i=1}^{n} (\varphi \circ h_i).$$

The following statement holds.

Proposition 1. *If $F_{\bar{a},H}(\Phi) \subseteq \Phi$ then $F_{\bar{a},H}$ is a GENEO for (Φ, G).*

Proof. First of all we prove that $F_{\bar{a},H}$ is G-equivariant. Let $\tilde{\alpha}_g : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be an index permutation such that $\tilde{\alpha}_g(i)$ is the index of the image of h_i through the conjugacy action of g, i.e.

$$\alpha_g(h_i) = g \circ h_i \circ g^{-1} = h_{\tilde{\alpha}_g(i)}, \quad \forall\, i \in \{1, \ldots, n\}.$$

We obtain that

$$g \circ h_i = h_{\tilde{\alpha}_g(i)} \circ g.$$

Exploiting this relation we obtain that

$$\begin{aligned}F_{\bar{a},H}(\varphi\circ g) =&\bar{a}(\varphi\circ g\circ h_1+\cdots+\varphi\circ g\circ h_n)\\ =&\bar{a}(\varphi\circ h_{\tilde{\alpha}_g(1)}\circ g+\cdots+\varphi\circ h_{\tilde{\alpha}_g(n)}\circ g)\\ =&\bar{a}(\varphi\circ h_{\tilde{\alpha}_g(1)}+\cdots+\varphi\circ h_{\tilde{\alpha}_g(n)})\circ g.\end{aligned}$$

Since $\left\{h_{\tilde{\alpha}_g(1)},\ldots,h_{\tilde{\alpha}_g(n)}\right\}=\{h_1,\ldots,h_n\}$, we get

$$F_{\bar{a},H}(\varphi\circ g)=F_{\bar{a},H}(\varphi)\circ g,\quad \forall\,\varphi\in\Phi,\quad \forall\,g\in G.$$

It remains to show that $F_{\bar{a},H}$ is non-expansive:

$$\begin{aligned}\|F_{\bar{a},H}(\varphi_1)-F_{\bar{a},H}(\varphi_2)\|_\infty &= \left\|\bar{a}\sum_{i=1}^{n}(\varphi_1\circ h_i)-\bar{a}\sum_{i=1}^{n}(\varphi_2\circ h_i)\right\|_\infty\\ &= |\bar{a}|\left\|\sum_{i=1}^{n}(\varphi_1\circ h_i-\varphi_2\circ h_i)\right\|_\infty\\ &\leqslant |\bar{a}|\sum_{i=1}^{n}\|\varphi_1\circ h_i-\varphi_2\circ h_i\|_\infty\\ &= |\bar{a}|\sum_{i=1}^{n}\|\varphi_1-\varphi_2\|_\infty\\ &= n|\bar{a}|\|\varphi_1-\varphi_2\|_\infty\\ &\leqslant \|\varphi_1-\varphi_2\|_\infty\end{aligned}$$

for every $\varphi_1,\varphi_2\in\Phi$.

Remark 4. Obviously $H=\{id\}\subseteq \mathrm{Homeo}(X)$ is a permutant for every subgroup G of $\mathrm{Homeo}(X)$, but the use of Proposition 1 for this trivial permutant leads to the trivial operator given by a multiple of the identity operator on Φ.

Remark 5. If the group G is Abelian, every finite subset of G is a permutant for G, since the conjugacy action is just the identity. Hence in this setting, for any chosen finite subset $H=\{g_1,\ldots,g_n\}$ of G and any real number $\bar{a}$, such that $n|\bar{a}|\leqslant 1$, $F_{\bar{a},H}(\varphi)=\bar{a}(\varphi\circ g_1+\cdots+\varphi\circ g_n)$ is a G-equivariant non-expansive operator for (Φ,G), provided that $F_{\bar{a},H}$ preserves Φ.

Remark 6. The operator $F_{\bar{a},H}:\Phi\to\Phi$ introduced in Proposition 1 is linear, provided that Φ is linearly closed. Indeed, assume that a permutant $H=\{h_1,\ldots,h_n\}$ for G and a real number $\bar{a}$ such that $n|\bar{a}|\leqslant 1$ are given. Let us consider the associated operator $F_{\bar{a},H}(\varphi)=\bar{a}\sum_{i=1}^{n}(\varphi\circ h_i)$, and assume that

$F_{\bar{a},H}(\Phi) \subseteq \Phi$. If $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\varphi_1, \varphi_2 \in \Phi$, we have

$$\begin{aligned}
F_{\bar{a},H}(\lambda_1\varphi_1 + \lambda_2\varphi_2) =& \bar{a}\sum_{i=1}^{n}((\lambda_1\varphi_1 + \lambda_2\varphi_2) \circ h_i) \\
=& \bar{a}\sum_{i=1}^{n}(\lambda_1(\varphi_1 \circ h_i) + \lambda_2(\varphi_2 \circ h_i)) \\
=& \bar{a}\sum_{i=1}^{n}\lambda_1(\varphi_1 \circ h_i) + \bar{a}\sum_{i=1}^{n}\lambda_2(\varphi_2 \circ h_i) \\
=& \lambda_1\left[\bar{a}\sum_{i=1}^{n}(\varphi_1 \circ h_i)\right] + \lambda_2\left[\bar{a}\sum_{i=1}^{n}(\varphi_2 \circ h_i)\right] \\
=& \lambda_1 F_{\bar{a},H}(\varphi_1) + \lambda_2 F_{\bar{a},H}(\varphi_2).
\end{aligned}$$

3 Examples

In this section we give two examples illustrating our method to build GENEOs.

Example 1. Let $X = \mathbb{R}$ and $\Phi \subseteq C_b^0(X, \mathbb{R})$. We consider the group G of all isometries of the real line, i.e. homeomorphisms of $\mathbb{R}$ of the form

$$g(x) = ax + b, \quad a, b \in \mathbb{R}, \quad a = \pm 1.$$

We also consider a translation $h(x) = x + t$ and its inverse transformation $h^{-1}(x) = x - t$, for some nonzero $t \in \mathbb{R}$. If g preserves the orientation, i.e. $a = 1$, the conjugation by g acts on $H := \{h, h^{-1}\}$ as the identity, while for $a = -1$ this conjugation exchanges the elements of H. We can conclude that H is a permutant for G. Therefore, Proposition 1 guarantees that the operator $F_{\frac{1}{2},H}(\varphi) = \frac{1}{2}(\varphi \circ h + \varphi \circ h^{-1})$ is a GENEO for (Φ, G), provided that $F_{\frac{1}{2},H}(\Phi) \subseteq \Phi$. We observe that the permutant used in this example is a subset but not a subgroup of $\mathrm{Homeo}(X)$.

Example 2. Let $X = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ and assume that Φ is the set of 1-Lipschitzian functions from X to $[0, 1]$. Let G and H be the group generated by reflection with respect to the line $x = 0$ and the group generated by the rotation ρ of $\pi/2$ around the point $(0, 0)$, respectively. It is easy to check that $H = \{id, \rho, \rho^2, \rho^3\}$ is a permutant for G and $F_{\frac{1}{4},H}(\Phi) \subseteq \Phi$. Therefore, Proposition 1 guarantees that the operator $F_{\frac{1}{4},H}(\varphi) = \frac{1}{4}\left(\varphi + \varphi \circ \rho + \varphi \circ \rho^2 + \varphi \circ \rho^3\right)$ is a GENEO for (Φ, G). We observe that the permutant used in this example is a subgroup of $\mathrm{Homeo}(X)$ but not a subgroup of G.

4 A result concerning permutants

When H contains only the identical homeomorphism, the operator $F_{\bar{a},H}$ is trivial, since it is the multiple by the constant $\bar{a}$ of the identical operator. This section highlights that in some cases this situation cannot be avoided, since non-trivial permutants for G are not available. In order to illustrate this problem, we need to introduce the concept of *versatile* group.

Definition 4. *Let G be a group that acts on a set X. We say that G is* versatile *if for every triple $(x, y, z) \in X^3$, with $x \neq z$, and for every finite subset S of X, at least one element $g \in G$ exists such that (1) $g(x) = y$ and (2) $g(z) \notin S$.*

Proposition 2. *Let X be a topological space and assume that $H = \{h_1, \ldots, h_n\}$ is a permutant for a subgroup G of* $\mathrm{Homeo}(X)$. *If G is versatile, then $H = \{id\}$.*

Proof. It is sufficient to prove that if H contains an element $h \neq id$, then G is not versatile. We can assume that $h \equiv h_1$. Since h_1 is different from the identity, a point $\bar{x} \in X$ exists such that $h_1(\bar{x}) \neq \bar{x}$. Let us consider the triple $(h_1(\bar{x}), \bar{x}, \bar{x})$ and the set $S = \{h_1^{-1}(\bar{x}), \ldots, h_n^{-1}(\bar{x})\}$. Suppose that $g \in G$ satisfies Property (1) with respect to the previous triple, that is $g(h_1(\bar{x})) = \bar{x}$. Since the conjugacy action of g on H is a permutation, we can find an element $h_2 \in H$ such that $h_2 = g \circ h_1 \circ g^{-1}$, so that $h_2(g(\bar{x})) = g(h_1(\bar{x})) = \bar{x}$ and hence $g(\bar{x}) = h_2^{-1}(\bar{x}) \in S$. Therefore, g does not satisfy Property (2), for $z = \bar{x}$. Hence we can conclude that no $g \in G$ exists verifying both Properties (1) and (2), i.e. G is not versatile.

Remark 7. Definition 4 immediately implies that if G, G' are two subgroups of $\mathrm{Homeo}(X)$, $G \subseteq G'$ and G is versatile, then also the group G' is versatile. For example, it is easy to prove that the group G of the isometries of the real plane is versatile. It follows that every group G' of self-homeomorphisms of $\mathbb{R}^2$ containing the isometries of the real plane is versatile. As a consequence of Proposition 2, every permutant for G' is trivial.

Conclusions

In this paper we have illustrated a new method for the construction of group equivariant non-expansive operators by means of permutants, exploiting the algebraic properties of the invariance group. The procedure enables us to manage in a quite simple way Abelian groups, but our examples show that we can find permutants, and hence GENEOs, even in a non-commutative setting. The main goal of our study is to expand our knowledge about the topological space $\mathcal{F}(\Phi, G)$, possibly reaching a good approximation of this space and, consequently, a good approximation of the pseudo-natural distance d_G by means of Theorem 1. The more operators we know, the more information we get about the structure of $\mathcal{F}(\Phi, G)$, and this fact justifies the search for new methods to build GENEOs. Many questions remain open. In particular, a deeper study of the concept of permutant seems necessary, establishing conditions for the existence of non-trivial permutants and introducing constructive methods to build them. Furthermore, an extension of our approach to operators from a pair (Φ, G) to a different pair (Ψ, H) seems worth of further investigation. Finally, we should check if the idea described in this paper about getting GENEOs by a finite average based on the use of permutants could be generalized to "infinite averages" (and hence integrals) based on "infinite permutants".

Acknowledgment

The authors thank Marian Mrozek for his suggestions and advice. The research described in this article has been partially supported by GNSAGA-INdAM (Italy).

References

1. F. Anselmi, L. Rosasco, T. Poggio, *On invariance and selectivity in representation learning*, Information and Inference: A Journal of the IMA, 5(2), 134–158 (2016).
2. S. Biasotti, L. De Floriani, B. Falcidieno, P. Frosini, D. Giorgi, C. Landi, L. Papaleo, M. Spagnuolo, *Describing shapes by geometrical-topological properties of real functions*, ACM Computing Surveys, 40(4), 12:1–12:87 (2008).
3. G. Carlsson, *Topology and data*, Bull. Amer. Math. Soc. (N.S.), 46(2), 255–308 (2009).
4. T. Cohen, M. Welling, *Group equivariant convolutional networks*, Proceedings of the 33rd International Conference on Machine Learning, PMLR 48, 2990–2999 (2016).
5. H. Edelsbrunner, D. Morozov, *Persistent homology: theory and practice*, European Congress of Mathematics, 31–50 (2013).
6. H. Edelsbrunner, J. L. Harer, *Persistent homology—a survey*, Contemporary Mathematics, 453, 257–282 (2008).
7. P. Frosini, *Towards an observer-oriented theory of shape comparison*, Proceedings of the 8th Eurographics Workshop on 3D Object Retrieval, Lisbon, Portugal, A. Ferreira, A. Giachetti, and D. Giorgi (Editors), 5–8 (2016).
8. P. Frosini, G. Jabłoński, *Combining persistent homology and invariance groups for shape comparison*, Discrete & Computational Geometry, 55(2), 373–409 (2016).
9. P. Frosini, N. Quercioli, *Some remarks on the algebraic properties of group invariant operators in persistent homology*, Lecture Notes in Computer Science, Proceedings of the International Cross-Domain Conference, CD-MAKE 2017, Reggio, Italy, August 29-September 1, 2017, MAKE Topology, Springer, Cham, Holzinger A., Kieseberg P., Tjoa A M., Weippl E. (Eds.), LNCS 10410, 14–24 (2017).
10. D. Marcos, M. Volpi, N. Komodakis, D. Tuia, *Rotation equivariant vector field networks*, Proceedings of the 2017 IEEE International Conference on Computer Vision (ICCV), 5058–5067 (2017).
11. J. Masci, D. Boscaini, M. M. Bronstein, P. Vandergheynst, *Geodesic convolutional neural networks on Riemannian manifolds*, Proceedings of the 2015 IEEE International Conference on Computer Vision Workshop (ICCVW), IEEE Computer Society, 832–840 (2015).