Robustness of linear time-varying systems with relaxed excitation

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Abstract

It is a well-known fact that linear time-varying systems with a persistently excited state matrix are exponentially converging and input-to-state stable with respect to additive perturbations. Recently, several relaxed conditions of persistent excitation have been presented [1], [2], which ensure an asymptotic convergence rate in the system. In the present work it is shown that these conditions are similar, and that under such a relaxed excitation only non-uniform in time input-to-state stability and integral input-to-state stability properties can be obtained. The results are illustrated by simulations for a problem of estimation in the linear regression model.

I. Introduction

The stability and robustness with respect to exogenous disturbances of control and estimation algorithms are the main issues studied within the control theory. For linear time-invariant systems it has been established that Hurwitz property of the state matrix is equivalent to both, exponential convergence/stability in the noise-free case and robustness with respect to external perturbations. For linear time-varying systems it is impossible for a general scenario to make a conclusion about convergence and robustness using the Hurwitz property of the state matrix (for each instant of time), then a Lyapunov function has to be found to justify the system performances [3]. Unfortunately, there is no method to design such a Lyapunov function in a common setting (for an example, in [4], [5] asymptotic and exponential stability of nonlinear time-varying systems is studied using averaging tools and assuming that there exists a Lyapunov function decreasing at an infinite increasing sequence of time instants), and that is in particularly why an interest to special classes of linear time-varying systems arises. As an example, to the linear time-varying systems whose state matrix is symmetric:

$$\dot{p}(t) = -\gamma R(t)R^{\top}(t)p(t) + b(t),\ t \geq t_0, \qquad (1)$$
$$p(t_0) \in \mathbb{R}^l,\ t_0 \in \mathbb{R},$$

where $p(t) \in \mathbb{R}^l$ is the state, the functions $R : \mathbb{R} \to \mathbb{R}^{l\times k}$ and $b : \mathbb{R} \to \mathbb{R}^l$ are (Lebesgue) measurable and locally essentially bounded; $\gamma > 0$ is a parameter. Such a class of dynamics appears in adaptive estimation and identification of linear regression models [6], and it constitutes the main object of investigation in the present paper. The peculiarity of this system consists in the requirement imposed on the matrix function R, which is usually assumed to be *persistently excited*. It is a well-known fact [3], that if R is persistently excited, then (1) is globally exponentially stable (GES) for $b(t) \equiv 0$, $t \geq t_0$ and input-to-state stable (ISS) for any essentially bounded b (these properties are independent or uniform in the initial time t_0).

It is worth to stress that verification of the condition of persistent excitation is rather complicated (especially *a priori*) in applications, which stimulates analysis the cases when such a restriction is not satisfied. Recently, several relaxed excitation notions have been proposed for R in [1], [2] implying only (not uniform) global asymptotic stability (GAS) of (1). The

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goal of this work is to compare these relaxed excitation conditions and to establish the corresponding relations with ISS and integral ISS (iISS) properties introduced in [7], [8], [9][1].

The paper outline is as follows. The notions of persistence of excitation and stability definitions are introduced in Section II. The relaxed excitation conditions from [1], [2] are discussed and compared in Section III. The corresponding ISS and iISS properties with respect to different perturbations (exogenous disturbances or measurement noises) are established in Section IV. The results of simulation for a parameter estimation problem are presented in Section V. Some concluding remarks are summarized in Section VI.

II. Preliminaries

Denote $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, where $\mathbb{R}$ is the set of real numbers. Let $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$ (for $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers) or the corresponding induced matrix norm. For a (Lebesgue) measurable and essentially bounded function $b : \mathbb{R} \to \mathbb{R}^l$ denote $\|b\|_\infty = \sup_{t\in\mathbb{R}} |b(t)|$.

Denote by $\mathcal{K}^+$ the class of continuous functions $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$, $\alpha \in \mathcal{K}^+$ belongs to the class $\mathcal{K}$ if $\alpha(0) = 0$ and the function is strictly increasing. The function $\alpha \in \mathcal{K}$ belongs to the class $\mathcal{K}_\infty$ if it increases to infinity. A continuous function $\mu : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{KL}$ if $\mu(\cdot, t) \in \mathcal{K}_\infty$ for each fixed $t \in \mathbb{R}_+$ and $\lim_{t\to+\infty} \mu(s, t) = 0$ for each fixed $s \in \mathbb{R}_+$.

A. Persistence of excitation condition

A (Lebesgue) measurable and square integrable matrix function $R : \mathbb{R} \to \mathbb{R}^{l\times k}$ with the dimension $l \times k$ verifies the (ℓ, ϑ)–Persistence of Excitation (PE) condition, if there exist constants $\ell > 0$ and $\vartheta > 0$ such that

$$\int_t^{t+\ell} R(s)R^\top(s)ds \geq \vartheta I_l$$

for any $t \in \mathbb{R}$, where I_l denotes the identity matrix of dimension $l \times l$.

Lemma 1. *[11] Assume that in the system (1) the function R is continuous, bounded by $\rho = \sup_{t\in\mathbb{R}} |R(t)| < +\infty$, and (ℓ, ϑ)–PE for some $\ell > 0$, $\vartheta > 0$. Then, for any $t_0 \in \mathbb{R}$ and any initial condition $p(t_0) \in \mathbb{R}^l$:*

$$|p(t)| \leq \sqrt{\zeta}\rho[e^{-0.5\gamma\zeta^{-1}(t-t_0)}|p(t_0)| + \gamma^{-1}\zeta\|b\|_\infty] \quad \forall t \geq t_0$$

for $\zeta = \gamma\eta^{-1}e^{2\eta\ell}$ and $\eta = -0.5\ell^{-1}\ln(1 - \frac{\gamma\vartheta}{1+\gamma^2\ell^2\rho^4})$.

Note that from the definition of PE property, in the conditions of the above lemma,

$$\vartheta \leq \ell\rho^2,$$

then $\frac{\gamma\vartheta}{1+\gamma^2\ell^2\rho^4} \in (0, 0.5]$ for any $\gamma > 0$, $\ell > 0$ and $\vartheta > 0$. Therefore, η and ζ are strictly positive and finite.

B. Robust stability properties

The system (1) is called nonuniform ISS [9] if there exist $\mu \in \mathcal{KL}$, $\sigma \in \mathcal{K}^+$and $\nu \in \mathcal{K}$ such that for all $t_0 \in \mathbb{R}$, all $p(t_0) \in \mathbb{R}^l$ and (Lebesgue) measurable and essentially bounded $b : \mathbb{R} \to \mathbb{R}^l$:

$$|p(t)| \leq \mu(\sigma(t_0)|p(t_0)|, t - t_0) + \nu(\|b\|_\infty) \quad \forall t \geq t_0.$$

The system (1) is called nonuniform iISS [9] if there exist $\mu \in \mathcal{KL}$, $\sigma \in \mathcal{K}^+$, $\alpha \in \mathcal{K}_\infty$ and $\nu \in \mathcal{K}$ such that for all $t_0 \in \mathbb{R}$, all $p(t_0) \in \mathbb{R}^l$ and (Lebesgue) measurable and integrable $b : \mathbb{R} \to \mathbb{R}^l$:

$$\alpha(|p(t)|) \leq \mu(\sigma(t_0)|p(t_0)|, t - t_0) + \int_{t_0}^t \nu(|b(s)|)ds \quad \forall t \geq t_0.$$

[1]The preliminary results in this direction have been presented in [10] without the main proofs.

The system (1) is called uniform ISS or iISS if $\sigma(s) = const$ for all $s \in \mathbb{R}$.

For the case $b(t) \equiv 0$ for all $t \in \mathbb{R}$, the (uniformly) ISS or iISS system (1) is (uniformly) GAS at the origin.

III. Relaxed convergence conditions for the unforced case

First, lest us consider the conditions of asymptotic stability of (1) for the case $b(t) \equiv 0$ for all $t \in \mathbb{R}$. Then, to relax the PE condition we will assume that the parameters ℓ and ϑ are time-dependent:

Assumption 1. *Assume that for any $t_0 \in \mathbb{R}$ there exist sequences of positive numbers $\{t_k\}_{k=0}^{+\infty}$, $\{\ell_k\}_{k=0}^{+\infty}$ and $\{\vartheta_k\}_{k=0}^{+\infty}$ such that for all $k \geq 0$:*

$$\begin{aligned} t_{k+1} &\geq t_k + \ell_k, \\ \int_{t_k}^{t_k+\ell_k} R(s)R^\top(s)ds &\geq \vartheta_k I_l. \end{aligned}$$

A. Sufficient conditions of convergence

Of course such a hypothesis is rather generic, and there are several results relating this assumption and asymptotic stability of (1) under additional mild conditions:

Proposition 1. *[2] Let Assumption 1 be satisfied and*

$$\sum_{k=0}^{+\infty} \frac{\vartheta_k}{\gamma^{-1} + \gamma\left(\int_{t_k}^{t_k+\ell_k} |R(t)|^2 dt\right)^2} = +\infty. \tag{2}$$

Then the system (1) with $b(t) \equiv 0$ for all $t \in \mathbb{R}$ is GAS at the origin.

In [1] another condition of GAS for (1) has been established, which is based on the result given below that contains a small extension (the proof can be found in [10]):

Proposition 2. *Let $b(t) \equiv 0$ for all $t \in \mathbb{R}$ and $\Phi(t, t_0)$ be the transition matrix of (1) satisfying*

$$\frac{\partial}{\partial t}\Phi(t, t_0) = -\gamma R(t)R^\top(t)\Phi(t, t_0),\ t \geq t_0,\ \Phi(t_0, t_0) = I_l,$$

then for all $t \geq t_0$:

$$e^{-2\gamma \int_{t_0}^{t} |R(s)|^2 ds} I_l \leq \Phi^\top(t, t_0)\Phi(t, t_0) \leq I_l - \frac{\int_{t_0}^{t} R(s)R^\top(s)ds}{\gamma^{-1} + \gamma\left(\int_{t_0}^{t} |R(s)|^2 ds\right)^2}.$$

This proposition provides generic properties of the transition matrix of (1), from which the stability conditions can be deduced:

Corollary 1. *Let Assumption 1 be satisfied and*

$$\prod_{k=0}^{+\infty}\left(1 - \frac{\vartheta_k}{\gamma^{-1} + \gamma\left(\int_{t_k}^{t_k+\ell_k} |R(t)|^2 dt\right)^2}\right) = 0 \tag{3}$$

Then the system (1) with $b(t) \equiv 0$ for all $t \in \mathbb{R}$ is GAS at the origin.

Proof. Consider an auxiliary Lyapunov function $W(p) = p^\top p$, whose derivative for the system (1) has the form:

$$\begin{aligned} \frac{d}{dt}W(p(t)) &= -2\gamma p^\top(t)R(t)R^\top(t)p(t) \\ &= -2\gamma |R^\top(t)p(t)|^2, \end{aligned}$$

then the Lyapunov stability of the system (1) follows since $\frac{\partial}{\partial t}W(p(t)) \leq 0$, together with the property that

$$|\Phi(t,t_0)| \leq 1 \quad \forall t \geq t_0$$

for all $t_0 \in \mathbb{R}$. In addition, according to Assumption 1 and the estimate derived in Proposition 2 we obtain:

$$\begin{aligned} |\Phi(t_{k+\ell_k},t_k)| &\leq \sqrt{\left| I_l - \frac{\int_{t_k}^{t_k+\ell_k} R(s)R^\top(s)ds}{\gamma^{-1}+\gamma\left(\int_{t_k}^{t_k+\ell_k}|R(t)|^2dt\right)^2}\right|} \\ &\leq \sqrt{1-\frac{\vartheta_k}{\gamma^{-1}+\gamma\left(\int_{t_k}^{t_k+\ell_k}|R(t)|^2dt\right)^2}} \end{aligned}$$

for all $k \geq 0$. In order to prove asymptotic convergence to the origin it is necessary to ensure that

$$\lim_{t\to+\infty}|\Phi(t,t_0)| = 0$$

for any $t_0 \in \mathbb{R}$. Using the previous calculations and the properties of the transition matrix

$$\begin{aligned} \lim_{t\to+\infty}|\Phi(t,t_0)| &\leq \prod_{k=0}^{+\infty}|\Phi(t_{k+1},t_k)| \\ &\leq \prod_{k=0}^{+\infty}|\Phi(t_{k+1},t_{k+\ell_k})||\Phi(t_{k+\ell_k},t_k)| \\ &\leq \prod_{k=0}^{+\infty}|\Phi(t_{k+\ell_k},t_k)| \\ &\leq \sqrt{\prod_{k=0}^{+\infty}\left(1-\frac{\vartheta_k}{\gamma^{-1}+\gamma\left(\int_{t_k}^{t_k+\ell_k}|R(t)|^2dt\right)^2}\right)}, \end{aligned}$$

which gives the required conclusion under the introduced restrictions. □

Proposition 3. *The conditions (2) and (3) are equivalent.*

Proof. Since $0 < \vartheta_k \leq \int_{t_k}^{t_k+\ell_k}|R(t)|^2dt$, $\sup_{x\geq 0}\frac{x}{1+x^2} = 0.5$ and

$$\ln(1-s) \geq -\frac{s}{1-s}$$

for all $0 < s < 1$, by applying $\ln(\cdot)$ to both sides of (3) we conclude:

$$\begin{aligned} -\infty &= \sum_{k=0}^{+\infty}\ln\left(1-\frac{\vartheta_k}{\gamma^{-1}+\gamma\left(\int_{t_k}^{t_k+\ell_k}|R(t)|^2dt\right)^2}\right) \\ &\geq -\sum_{k=0}^{+\infty}\frac{\frac{\vartheta_k}{\gamma^{-1}+\gamma\left(\int_{t_k}^{t_k+\ell_k}|R(t)|^2dt\right)^2}}{1-\frac{\vartheta_k}{\gamma^{-1}+\gamma\left(\int_{t_k}^{t_k+\ell_k}|R(t)|^2dt\right)^2}}, \end{aligned}$$

and using the fact that

$$0 \leq \frac{\vartheta_k}{\gamma^{-1}+\gamma\left(\int_{t_k}^{t_k+\ell_k}|R(t)|^2dt\right)^2} \leq 0.5$$

for any $k \geq 0$ the last inequality can be rewritten as follows:

$$+\infty \leq \sum_{k=0}^{+\infty} \frac{\frac{\vartheta_k}{\gamma^{-1}+\gamma\left(\int_{t_k}^{t_k+\ell_k}|R(t)|^2dt\right)^2}}{1-\frac{\vartheta_k}{\gamma^{-1}+\gamma\left(\int_{t_k}^{t_k+\ell_k}|R(t)|^2dt\right)^2}}$$

$$\leq 2\sum_{k=0}^{+\infty} \frac{\vartheta_k}{\gamma^{-1}+\gamma\left(\int_{t_k}^{t_k+\ell_k}|R(t)|^2dt\right)^2},$$

which implies (2):

$$\sum_{k=0}^{+\infty} \frac{\vartheta_k}{\gamma^{-1}+\gamma\left(\int_{t_k}^{t_k+\ell_k}|R(t)|^2dt\right)^2} = +\infty.$$

And *vice versa*, the estimate (2) also implies (3), since using the inequality $1-x \leq e^{-x}$ that is satisfied for all $x \in \mathbb{R}$, and by applying $e^{-(\cdot)}$ to both sides of (2) we obtain:

$$0 = e^{-\sum_{k=0}^{+\infty} \frac{\vartheta_k}{\gamma^{-1}+\gamma\left(\int_{t_k}^{t_k+\ell_k}|R(t)|^2dt\right)^2}} = \prod_{k=0}^{+\infty} e^{-\frac{\vartheta_k}{\gamma^{-1}+\gamma\left(\int_{t_k}^{t_k+\ell_k}|R(t)|^2dt\right)^2}}$$

$$\geq \prod_{k=0}^{+\infty}\left(1-\frac{\vartheta_k}{\gamma^{-1}+\gamma\left(\int_{t_k}^{t_k+\ell_k}|R(t)|^2dt\right)^2}\right) \geq 0.$$

Thus, the conditions (2) and (3) are equivalent. □

Remark 1. The dependence on γ of both conditions, (2) and (3), proposed in [2] and [1], respectively, suggests that increasing the value of γ may not lead to acceleration of the rate of convergence, and there is an optimal choice of γ dependent on the regressor matrix $R(t)$.

B. Discussion on necessity of (2) and (3)

Based on the inequalities given in Proposition 2, the following bounds can be derived for stability analysis:

$$e^{-\gamma\int_{t_0}^{t}|R(s)|^2ds} \leq |\Phi(t,t_0)| \leq \sqrt{1-\frac{\vartheta_t}{\gamma^{-1}+\gamma\left(\int_{t_0}^{t}|R(s)|^2ds\right)^2}} \tag{4}$$

where it is assumed that

$$\int_{t_0}^{t} R(s)R^\top(s)ds \geq \vartheta_t I_l > 0.$$

The main drawback of this result is that such an upper estimate on $|\Phi(t,t_0)|$ is rather conservative. Indeed, assume that

$$R(t) = R_0(t) + \delta R(t),$$

where the first item is bounded,

$$|R_0(t)| \leq \rho \quad \forall t \in \mathbb{R}$$

for some $\rho \in \mathbb{R}_+$, and the second item, $\delta R(t)$, may be asymptotically growing. Intuitively, the appearance of an unbounded term $\delta R(t)$ may improve the rate of convergence in (1), since due to the system structure, an escape to infinity of $\delta R(t)$ cannot lead to the system instability. This conclusion is also illustrated via the improvement of the lower bound on $|\Phi(t,t_0)|$ given in (4), which starts to decrease faster with an unbounded term $\delta R(t)$. However, the upper bound on $|\Phi(t,t_0)|$ may be worse than one calculated for $R_0(t)$, *i.e.* the minimum eigenvalues of the matrices $\int_{t_0}^{t} R(s)R^\top(s)ds$ and $\int_{t_0}^{t} R_0(s)R_0^\top(s)ds$ can be the same, but the denominator in (4) for the case of $R(t)$ will go to infinity, in its turn implying that the upper bound approaches I_l. For the regressor $R_0(t)$ such an issue is not possible. In order to illustrate this idea, let us built an example exactly around this flaw:

Example. Let $l = 2$, $t_0 = 0$ and

$$R(t) = \begin{cases} [1\ 0]^\top & t \in [2k, 2k+1) \\ [0\ k]^\top & t \in [2k+1, 2k+2) \end{cases}$$

for any integer $k \geq 0$. Then, for $t \geq 2$ the value $n_t = floor(\frac{t}{2}) > 0$ (where $floor : \mathbb{R}_+ \to \mathbb{R}_+$ is the rounding operator to the smallest integer) is well defined and

$$\begin{aligned} \int_0^t R(s)R^\top(s)ds &\geq \sum_{k=0}^{n_t-1} \int_{2k}^{2k+1} R(s)R^\top(s)ds + \int_{2k+1}^{2k+2} R(s)R^\top(s)ds \\ &= \sum_{k=0}^{n_t-1} \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \geq n_t I_2, \\ \int_0^t |R(s)|^2 ds &\leq \sum_{k=0}^{n_t} \int_{2k}^{2k+1} 1 ds + \int_{2k+1}^{2k+2} k^2 ds \\ &= \sum_{k=0}^{n_t} (1+k^2), \end{aligned}$$

hence,

$$\lim_{t\to+\infty} |\Phi(t,0)| \leq \lim_{t\to+\infty} \sqrt{1 - \frac{n_t}{\gamma^{-1} + \gamma\left(\sum_{k=0}^{n_t}(1+k^2)\right)^2}} = 1$$

and the sufficient condition of convergence (2) (and, consequently, also (3)) fails to satisfy.

Note that this observation is also valid for the case of a persistently excited matrix $R(t)$, and that is why in all stability theorems on PE systems it is assumed that the regressor is additionally bounded [6], [11], [10]. Using the argumentation applied in the proof of Proposition 2, the upper estimate in (4) can be partially rewritten with respect to $R_0(t)$ slightly relaxing the issue:

Proposition 4. *Let $b(t) \equiv 0$, $R(t) = R_0(t) + \delta R(t)$ for all $t \in \mathbb{R}$, and $\Phi(t,t_0)$ be the transition matrix of (1) satisfying*

$$\frac{\partial}{\partial t}\Phi(t,t_0) = -\gamma R(t)R^\top(t)\Phi(t,t_0),\ t \geq t_0,\ \Phi(t_0,t_0) = I_l,$$

then for all $t \geq t_0$:

$$e^{-2\gamma\int_{t_0}^t |R(s)|^2 ds} I_l \leq \Phi^\top(t,t_0)\Phi(t,t_0) \leq I_l - \frac{\int_{t_0}^t R_0(s)R_0^\top(s)ds}{\gamma^{-1} + \gamma\int_{t_0}^t |R_0(s)|^2 ds \int_{t_0}^t |R(s)|^2 ds}.$$

Proof. For any $t_0 \in \mathbb{R}$ and $p(t_0) \in \mathbb{R}^l$ denote $p(t) = \Phi(t,t_0)p(t_0)$ (the solution of (1) with initial condition $p(t_0)$ at initial time t_0) and consider an auxiliary Lyapunov function $W(p) = p^\top p$, whose derivative for the system (1) has the form:

$$\begin{aligned} \frac{d}{dt}W(p(t)) &= -2\gamma p^\top(t)R(t)R^\top(t)p(t) \\ &= -2\gamma|R^\top(t)p(t)|^2 \leq -2\gamma|R_0^\top(t)p(t)|^2 \end{aligned} \tag{5}$$

since $R(t)R^\top(t) = R_0(t)R_0^\top(t) + Q(t)$ and the matrix $Q(t) = \delta R(t)\delta R^\top(t) + \delta R(t)R_0^\top(t) + \delta R^\top(t)R_0(t)$ is symmetric and nonnegative definite. Then

$$\frac{d}{dt}W(p(t)) \geq -2\gamma|R(t)|^2 W(p(t))$$

implying

$$W(p(t)) \geq e^{-2\gamma\int_{t_0}^t |R(r)|^2 dr} W(p(t_0)),$$

which can be interpreted as

$$p^\top(t_0)\Phi^\top(t,t_0)\Phi(t,t_0)p(t_0) \geq p^\top(t_0)e^{-2\gamma\int_{t_0}^t |R(r)|^2dr}p(t_0)$$

providing the required lower estimate on $\Phi^\top(t,t_0)\Phi(t,t_0)$ due to an arbitrary choice of $p(t_0)$. From another side, since $0.5|a|^2 \leq |b|^2 + |a-b|^2$ for any $a,b \in \mathbb{R}^l$, for $a = R_0^\top(t)p(t_0)$ and $b = R_0^\top(t)p(t)$ we obtain:

$$0.5|R_0^\top(t)p(t_0)|^2 \leq |R_0^\top(t)p(t)|^2 + |R_0^\top(t)(p(t_0)-p(t))|^2. \tag{6}$$

Note that

$$\begin{aligned}
|R_0^\top(t)(p(t)-p(t_0))|^2 &= |R_0^\top(t)\int_{t_0}^t \dot{p}(s)ds|^2 \\
= \gamma^2\int_{t_0}^t p^\top(s)R(s)R^\top(s)dsR_0(t)R_0^\top(t)&\int_{t_0}^t R(s)R^\top(s)p(s)ds \\
\leq \gamma^2|R_0(t)|^2\int_{t_0}^t p^\top(s)R(s)R^\top(s)ds&\int_{t_0}^t R(s)R^\top(s)p(s)ds \\
\leq \gamma^2|R_0(t)|^2&\left(\int_{t_0}^t |R(s)||R^\top(s)p(s)|ds\right)^2 \\
\leq \gamma^2|R_0(t)|^2&\int_{t_0}^t |R(s)|^2ds\int_{t_0}^t |R^\top(s)p(s)|^2ds,
\end{aligned}$$

where the Cauchy-Schwarz inequality has been used on the last step. Integrating the obtained inequality we conclude that

$$\begin{aligned}
\int_{t_0}^t |R_0^\top(s)(p(s)-p(t_0))|^2ds &\leq \int_{t_0}^t \gamma^2|R_0(r)|^2\int_{t_0}^r |R(s)|^2ds \\
&\times\int_{t_0}^r |R^\top(s)p(s)|^2dsdr \\
&\leq \gamma^2\int_{t_0}^t |R_0(s)|^2ds\int_{t_0}^t |R(s)|^2ds\int_{t_0}^t |R^\top(s)p(s)|^2ds.
\end{aligned}$$

Next, integrating (6) the following relation is derived:

$$\begin{aligned}
0.5\int_{t_0}^t |R_0^\top(s)p(t_0)|^2ds &\leq \int_{t_0}^t |R_0^\top(s)p(s)|^2ds \\
&+\int_{t_0}^t |R_0^\top(s)(p(s)-p(t_0))|^2ds \\
&\leq \left[1+\gamma^2\int_{t_0}^t |R_0(s)|^2ds\int_{t_0}^t |R(s)|^2ds\right]\int_{t_0}^t |R^\top(s)p(s)|^2ds
\end{aligned}$$

since

$$\int_{t_0}^t |R_0^\top(s)p(s)|^2ds \leq \int_{t_0}^t |R^\top(s)p(s)|^2ds$$

due to the properties of the matrix $Q(t)$. Finally, integrating (5) we obtain:

$$\begin{aligned}
W(p(t)) - W(p(t_0)) &= -2\gamma\int_{t_0}^t |R^\top(s)p(s)|^2ds \\
&\leq -\frac{\gamma\int_{t_0}^t |R_0^\top(s)p(t_0)|^2ds}{1+\gamma^2\int_{t_0}^t |R_0(s)|^2ds\int_{t_0}^t |R(s)|^2ds},
\end{aligned}$$

or equivalently,

$$p^\top(t_0)\Phi^\top(t,t_0)\Phi(t,t_0)p(t_0) \le p^\top(t_0)\left(I_l - \frac{\gamma\int_{t_0}^t R_0(s)R_0^\top(s)ds}{1+\gamma^2\int_{t_0}^t |R_0(s)|^2 ds\int_{t_0}^t |R(s)|^2 ds}\right)p(t_0),$$

and since this inequality is satisfied for any $p(t_0)\in\mathbb{R}^l$, the desired upper estimated is substantiated. □

Therefore, in the case of an unbounded regressor $R(t)$, *i.e.* $\lim_{k\to+\infty}\int_{t_k}^{t_k+\ell_k}|R(t)|^2dt \to +\infty$, the conditions (2) and (3) are not necessary for convergence of the system (1).

IV. Robustness with respect to external inputs

Now, let us return to stability analysis of the perturbed system (1) with $b(t)\neq 0$.

Lemma 2. *Consider the system (1), then for any $t_0\in\mathbb{R}$ and any $p(t_0)\in\mathbb{R}^l$*

$$|p(t)| \le \sqrt{\left|I_l - \frac{\int_{t_0}^t R(s)R^\top(s)ds}{\gamma^{-1}+\gamma\left(\int_{t_0}^t |R(s)|^2ds\right)^2}\right|}\times\left(|p(t_0)| + e^{\gamma\int_{t_0}^t |R(s)|^2ds}\int_{t_0}^t |b(s)|ds\right)$$

for all $t\ge t_0$.

Proof. For any $t_0\in\mathbb{R}$ and $p(t_0)\in\mathbb{R}^l$ the corresponding solution of the system (1) can be calculated as follows:

$$p(t) = \Phi(t,t_0)\left(p(t_0) + \int_{t_0}^t \Phi^{-1}(s,t_0)b(s)ds\right)$$

for all $t\ge t_0$. From Proposition 2, if $b(t)\equiv 0$ for all $t\in\mathbb{R}$,

$$\Phi^\top(t,t_0)\Phi(t,t_0) \le I_l - \frac{\int_{t_0}^t R(s)R^\top(s)ds}{\gamma^{-1}+\gamma\left(\int_{t_0}^t |R(s)|^2ds\right)^2},$$

$$\left(\Phi^\top(t,t_0)\Phi(t,t_0)\right)^{-1} \le e^{2\gamma\int_{t_0}^t |R(s)|^2ds}I_l$$

for all $t\ge t_0$, then

$$|p(t)| \le |\Phi(t,t_0)|\left(|p(t_0)| + \int_{t_0}^t |\Phi^{-1}(s,t_0)||b(s)|ds\right)$$

and the needed estimate follows by a direct substitution. □

The latter result provides a generic auxiliary estimate on solutions of (1), which is used next to derive the stability conditions. To this end introduce a sequence

$$\rho_k = \int_{t_k}^{t_{k+1}} |R(t)|^2dt \quad \forall k\ge 0.$$

Theorem 1. *Consider the system (1) satisfying Assumption 1 with an additional constraint:*

$$t_{k+1} = t_k + \ell_k \quad \forall k\ge 0,$$

and assume that the property (3) is valid.

If

$$\sup_{k\geq 0}\left\{\ell_k e^{\gamma\rho_k}+\sum_{i=0}^{k}\prod_{j=i}^{k}\sqrt{1-\frac{\vartheta_j}{\gamma^{-1}+\gamma\rho_j^2}}\ell_i e^{\gamma\rho_i}\right\} \leq \eta_{t_0}<+\infty,$$

then for each $t_0\in\mathbb{R}$ *there exists a function* $\varrho_{t_0}\in\mathcal{KL}$ *such that*

$$|p(t)|\leq \varrho_{t_0}(|p(t_0)|,t-t_0)+\eta_{t_0}\|b\|_\infty \quad \forall t\geq t_0,\ \forall p(t_0)\in\mathbb{R}^l. \tag{7}$$

If

$$\sup_{k\geq 0} e^{\gamma\rho_k}\leq \beta_{t_0}<+\infty,$$

then for each $t_0\in\mathbb{R}$ *there exists a function* $\varrho_{t_0}\in\mathcal{KL}$ *such that*

$$|p(t)|\leq \varrho_{t_0}(|p(t_0)|,t-t_0)+\beta_{t_0}\int_{t_0}^{t}|b(t)|dt \quad \forall t\geq t_0,\ \forall p(t_0)\in\mathbb{R}^l. \tag{8}$$

Proof. Consider for the system (1) a Lyapunov function $W(p)=p^\top p$ with the time derivative:

$$\begin{aligned}\frac{d}{dt}W(p(t)) &= -2\gamma p^\top(t)R(t)R^\top(t)p(t)+2p^\top(t)b(t)\\ &\leq W(p(t))+|b(t)|^2.\end{aligned}$$

Obviously, for essentially or square-integrally bounded b, the solution $W(p(t))$ is well-defined for all $t\geq t_0$, which implies that a finite-time escape of trajectories of (1) is impossible and they are defined for all $t\geq t_0$ (the system is forward complete).

For all $k\geq 0$, using the estimate provided by Lemma 2 we obtain

$$|p(t)|\leq |p(t_k)|+e^{\gamma\rho_k}\int_{t_k}^{t}|b(s)|ds \quad \forall t\in[t_k,t_{k+1})$$

and

$$|p(t_{k+1})|\leq\sqrt{1-\frac{\vartheta_k}{\gamma^{-1}+\gamma\rho_k^2}}\left(|p(t_k)|+e^{\gamma\rho_k}\int_{t_k}^{t_{k+1}}|b(t)|dt\right).$$

Note that by definition $\vartheta_k\leq\rho_k$, then $\frac{\vartheta_k}{\gamma^{-1}+\gamma\rho_k^2}=\frac{\gamma\vartheta_k}{1+\gamma^2\rho_k^2}\in(0,0.5]$ for any $\gamma>0$ and $\vartheta_k>0$ since the function $\frac{s}{1+s^2}\in(0,0.5]$ for any $s>0$.

Let us consider the case when $\|b\|_\infty<+\infty$, then for all $k\geq 0$:

$$\begin{aligned}|p(t)|&\leq |p(t_k)|+\ell_k e^{\gamma\rho_k}\|b\|_\infty \quad \forall t\in[t_k,t_{k+1}),\\ |p(t_{k+1})|&\leq\sqrt{1-\frac{\vartheta_k}{\gamma^{-1}+\gamma\rho_k^2}}\left(|p(t_k)|+\ell_k e^{\gamma\rho_k}\|b\|_\infty\right).\end{aligned}$$

Recursively applying the last estimate for $k\geq 0$ the following inequality can be established:

$$\begin{aligned}|p(t_{k+1})|\leq&\prod_{i=0}^{k}\sqrt{1-\frac{\vartheta_i}{\gamma^{-1}+\gamma\rho_i^2}}|p(t_0)|\\ &+\sum_{i=0}^{k}\prod_{j=i}^{k}\sqrt{1-\frac{\vartheta_j}{\gamma^{-1}+\gamma\rho_j^2}}\ell_i e^{\gamma\rho_i}\|b\|_\infty.\end{aligned}$$

If

$$\sup_{k\geq 0}\left\{\ell_k e^{\gamma\rho_k}+\sum_{i=0}^{k}\prod_{j=i}^{k}\sqrt{1-\frac{\vartheta_j}{\gamma^{-1}+\gamma\rho_j^2}}\ell_i e^{\gamma\rho_i}\right\}\leq\eta_{t_0}$$

for some $\eta_{t_0}\in\mathbb{R}_+$, then a combination of these estimates leads to

$$|p(t)|\leq\prod_{i=0}^{k-1}\sqrt{1-\frac{\vartheta_i}{\gamma^{-1}+\gamma\rho_i^2}}|p(t_0)|+\eta_{t_0}\|b\|_\infty\quad\forall t\in[t_k,t_{k+1}).$$

Since under (3), for any $t_0\in\mathbb{R}$ there exists a decreasing function $\kappa_{t_0}:\mathbb{R}_+\to\mathbb{R}_+$ such that $\lim_{t\to+\infty}\kappa_{t_0}(t)=0$ and

$$\prod_{i=0}^{k-1}\sqrt{1-\frac{\vartheta_i}{\gamma^{-1}+\gamma\rho_i^2}}\leq\kappa_{t_0}(t-t_0)\quad\forall t\in[t_k,t_{k+1})\tag{9}$$

and for all $k\geq 0$, then selecting $\varrho_{t_0}(s,t)=\kappa_{t_0}(t)s$ we obtain the required function from class $\mathcal{KL}$ and the desired ISS-like estimate (7) is satisfied.

Now consider the case when $\int_{t_0}^{+\infty}|b(s)|ds<+\infty$, then for all $k\geq 0$:

$$\begin{aligned}|p(t_{k+1})|\leq\prod_{i=0}^{k}\sqrt{1-\frac{\vartheta_i}{\gamma^{-1}+\gamma\rho_i^2}}|p(t_0)|\\+\sum_{i=0}^{k}\prod_{j=i}^{k}\sqrt{1-\frac{\vartheta_j}{\gamma^{-1}+\gamma\rho_j^2}}e^{\gamma\rho_i}\int_{t_i}^{t_{i+1}}|b(t)|dt.\end{aligned}$$

If

$$\sup_{k\geq 0}e^{\gamma\rho_k}\leq\beta_{t_0}$$

for some $\beta_{t_0}\in\mathbb{R}_+$, then since $\prod_{j=i}^{k}\sqrt{1-\frac{\vartheta_j}{\gamma^{-1}+\gamma\rho_j^2}}\leq 1$ for all $k\geq 0$ we obtain:

$$\begin{aligned}|p(t)|\ &\leq\ \prod_{i=0}^{k-1}\sqrt{1-\frac{\vartheta_i}{\gamma^{-1}+\gamma\rho_i^2}}|p(t_0)|+e^{\gamma\rho_k}\int_{t_k}^{t}|b(s)|ds\\&\quad+\sum_{i=0}^{k-1}\prod_{j=i}^{k-1}\sqrt{1-\frac{\vartheta_j}{\gamma^{-1}+\gamma\rho_j^2}}e^{\gamma\rho_i}\int_{t_i}^{t_{i+1}}|b(t)|dt\\&\leq\ \prod_{i=0}^{k-1}\sqrt{1-\frac{\vartheta_i}{\gamma^{-1}+\gamma\rho_i^2}}|p(t_0)|+\beta_{t_0}\int_{t_k}^{t}|b(s)|ds\\&\quad+\sum_{i=0}^{k-1}\beta_{t_0}\int_{t_i}^{t_{i+1}}|b(t)|dt\\&\leq\ \prod_{i=0}^{k-1}\sqrt{1-\frac{\vartheta_i}{\gamma^{-1}+\gamma\rho_i^2}}|p(t_0)|+\beta_{t_0}\int_{t_0}^{t}|b(s)|ds.\end{aligned}$$

Finally, under (3), for any $t_0\in\mathbb{R}$ there exists a decreasing to zero function $\kappa_{t_0}:\mathbb{R}_+\to\mathbb{R}_+$ such that the estimate (9) is satisfied, and for $\varrho_{t_0}(s,t)=\kappa_{t_0}(t)s$ the needed iISS-like estimate (8) follows. □

Thus, the conditions of non-uniform ISS property (7) are rather sophisticated and discussed below, while it is worth to stress here that the swap "conditions" of non-uniform iISS property (8) are very weak and satisfied, for example, if the system (1) is asymptotically stable for $b=0$ and

$$\int_{-\infty}^{+\infty}|R(t)|^2dt<+\infty$$

or

$$\sup_{k\geq 0} \ell_k < +\infty, \ \operatorname*{ess\,sup}_{t\geq 0} |R(t)| < +\infty.$$

A. Relation with PE conditions

Note that dependence of η_{t_0} on the initial time $t_0 \in \mathbb{R}$ is important in the formulation of Theorem 1, since if a uniform result would be obtained, *i.e.* there exist $\bar{\eta} \in \mathbb{R}_+$ such that $\sup_{t_0\in\mathbb{R}} \eta_{t_0} \leq \bar{\eta}$, then it implies GES of (1) (and, consequently, fulfillment of PE condition [12]), as it is shown in the following lemma:

Lemma 3. *[10] Assume that for the system (1) there exist constants $C_1 \geq 1$ and $C_2 > 0$ such that for all $t_0 \in \mathbb{R}$ and all initial conditions $p(t_0) \in \mathbb{R}^l$ we have*

$$|p(t)| \leq C_1|p(t_0)| + C_2\|b\|_\infty \quad \forall t \geq t_0. \tag{10}$$

Then the system (1) with $b(t) \equiv 0$ for all $t \in \mathbb{R}$ is uniformly GES.

Therefore, for the system (1), due to its structure, a bounded input-bounded state property (10) (such an inequality immediately follows from the definition of uniform ISS, for example) implies convergence with an exponential rate.

Corollary 2. *If system (1) with $b(t) \equiv 0$ for all $t \in \mathbb{R}$ is not uniformly GES, then the assumption of Lemma 3 is not satisfied, and therefore the system (1) is not uniformly ISS.*

The proof of Theorem 1 shows that for every $t_0 \in \mathbb{R}$ function $\rho_{t_0}(s, t-t_0)$ in (7) may be chosen as $\rho_{t_0}(s, t-t_0) = sk_{t_0}(t-t_0)$ with a decreasing function $k_{t_0}\colon \mathbb{R}_+ \to \mathbb{R}_+$. In such a case the following non-uniform counterpart of Lemma 3 can be formulated:

Proposition 5. *Assume for each $t_0 \in \mathbb{R}$ there exist a number $\eta_{t_0} > 0$ and a continuous function $\varrho_{t_0} \in \mathcal{KL}$ such that for all $p(t_0) \in \mathbb{R}^l$ in the system (1) we have (7). Then for every $t_0 \in \mathbb{R}$ system (1) with $b(t) \equiv 0$ for all $t \in \mathbb{R}$ is GES on $[t_0, \infty)$.*

Notice that this proposition implies that for every $t_0 \in \mathbb{R}$ the function R satisfies the PE condition on interval $[t_0, \infty)$, that is, there exist positive real numbers m_{t_0} and θ_{t_0} such that

$$\int_t^{t+m_{t_0}} R(s)R^\top(s)ds \geq \theta_{t_0} I_l$$

for all $t \geq t_0$. This condition is stronger than (3).

Proof. Fix a number $t_0 \in \mathbb{R}$. Pick up a number $\epsilon \in (0, \eta_{t_0}^{-1})$. Consider a solution $p(t)$ for any $p(t_0) \in \mathbb{R}^l$ of the system (1) with $b(t) \equiv 0$ for all $t \geq t_0$. Denote $y(t) = e^{\epsilon(t-t_0)}p(t)$. Then

$$\dot{y}(t) = -\gamma R(t)R(t)^\top y(t) + \epsilon y(t).$$

Pick up a number C such that $C > |p(t_0)|$. We are going to show that $|y(t)| < C$ for all $t \geq t_0$ and such a sufficiently big C. This implies global exponential stability of the system (1) with $b = 0$ on $[t_0, \infty)$. To this end assume that T is the first number bigger than t_0 such that $|y(T)| = C$. Then from (7) we have

$$C = |y(T)| \leq \varrho_{t_0}(|p(t_0)|, T-t_0) + \epsilon\eta_{t_0} C.$$

Denote $K = \sup_{t\geq t_0} \varrho_{t_0}(|p(t_0)|, t-t_0)$. Then

$$C \leq \frac{K}{1-\epsilon\eta_{t_0}}.$$

Therefore, for all C bigger than the constant in the right hand side such a positive number T does not exist. The lemma is proved. ∎

Remark 2. Notice that in Theorem 1 and this proposition, the stability conditions of the system (1) are dependent on $t_0 \in \mathbb{R}$ (the function ϱ_{t_0} and the parameter η_{t_0}). Hence, in both cases we have stability properties which are, in general, non uniform with respect to $t_0 \in \mathbb{R}$, and there is no contradiction with Lemma 3.

The result of Theorem 1 also provides an alternative estimate in the case of PE matrix function R:

Corollary 3. *Assume that in the system (1) the function R is (ℓ, ϑ)–PE for some $\ell > 0$, $\vartheta > 0$ and*

$$\mathop{ess\sup}_{t\geq 0}|R(t)| \leq \rho < +\infty.$$

Then, for any $t_0 \in \mathbb{R}$ and any initial condition $p(t_0) \in \mathbb{R}^l$:

$$|p(t)| \leq e^{-\theta(t-t_0)}|p(t_0)| + \frac{2-\xi}{1-\xi}\ell e^{\gamma\ell\rho^2}\|b\|_\infty \quad \forall t \geq t_0$$

for $\theta = \ell^{-1}\ln\xi^{-1}$ and $\xi = \sqrt{1 - \frac{\vartheta}{\gamma^{-1}+\gamma\ell^2\rho^4}}$.

Proof. Repeating the steps of the proof of Theorem 1 for $t_k = \ell k$, $\ell_k = \ell$ and $\vartheta_k = \vartheta$ for all $k \geq 0$ it is possible to obtain (in such a case $\rho_k = \ell\rho^2$):

$$|p(t)| \leq |p(t_k)| + \ell e^{\gamma\ell\rho^2}\|b\|_\infty \quad \forall t \in [t_k, t_{k+1}),$$

$$|p(t_{k+1})| \leq \prod_{i=0}^{k}\xi|p(t_0)| + \sum_{i=0}^{k}\prod_{j=i}^{k}\xi\ell e^{\gamma\ell\rho^2}\|b\|_\infty.$$

Since $\vartheta \leq \gamma\ell\rho^2$, then $\xi < 1$ and

$$\lim_{k\to+\infty}\prod_{i=0}^{k}\left(1 - \frac{\vartheta}{\gamma^{-1}+\gamma\ell^2\rho^4}\right) = \lim_{k\to+\infty}\xi^{2k} = 0$$

and the property (3) is satisfied (it has not been assumed in the formulation of the corollary). Next,

$$\sum_{i=0}^{k}\prod_{j=i}^{k}\xi = \sum_{i=0}^{k}\xi^k$$
$$\leq \sum_{i=0}^{+\infty}\xi^k = \frac{1}{1-\xi} < +\infty,$$

and for all $t \in [t_k, t_{k+1})$

$$|p(t)| \leq \xi^k|p(t_0)| + \frac{2-\xi}{1-\xi}\ell e^{\gamma\ell\rho^2}\|b\|_\infty.$$

Finally, for $-\theta \geq \ell^{-1}\ln\xi$ the inequality

$$e^{-\theta(k+1)\ell} \geq \xi^k$$

is verified for all $k \geq 0$, and the required uniform exponential ISS upper estimate follows. □

B. Robustness with respect to measurement noise

The conditions of ISS obtained for (1) in Theorem 1 are rather restrictive and related with PE property of the matrix R, in the case when b in (1) plays the role of exogenous disturbance. These conditions can be significantly relaxed considering the case with b representing a measurement noise:

$$\dot{p}(t) = -\gamma R(t)R^\top(t)\left(p(t) + b(t)\right),\ t_0 \in \mathbb{R},\ \gamma > 0, \tag{11}$$

where again $p(t) \in \mathbb{R}^l$ is the state, the functions $R : \mathbb{R} \to \mathbb{R}^{l\times k}$ and $b : \mathbb{R} \to \mathbb{R}^l$ are (Lebesgue) measurable and essentially bounded. For all $k \geq 0$ denote a sequence

$$\phi_k = \operatorname*{ess\,sup}_{t\in[t_k,t_{k+1})} |R(t)|^2,$$

then $\rho_k \leq \ell_k \phi_k$.

Corollary 4. *Consider the system (11) satisfying Assumption 1 with an additional constraint:*

$$t_{k+1} = t_k + \ell_k$$

for all $k \geq 0$, and assume that the property (3) is valid.

If

$$\gamma \sup_{k\geq 0}\left\{ \ell_k\phi_k e^{\gamma\rho_k} + \sum_{i=0}^{k}\prod_{j=i}^{k}\sqrt{1-\frac{\vartheta_j}{\gamma^{-1}+\gamma\rho_j^2}}\ell_i\phi_i e^{\gamma\rho_i}\right\} \leq \eta_{t_0} < +\infty$$

then for each $t_0 \in \mathbb{R}$ there exists a function $\varrho_{t_0} \in \mathcal{KL}$ such that for all $p(t_0) \in \mathbb{R}^l$ the estimate (7) is satisfied.

If

$$\gamma \sup_{k\geq 0} \phi_k e^{\gamma\rho_k} \leq \beta_{t_0} < +\infty,$$

then for each $t_0 \in \mathbb{R}$ there exists a function $\varrho_{t_0} \in \mathcal{KL}$ such that for all $p(t_0) \in \mathbb{R}^l$ the estimate (8) is satisfied.

Proof. Repeating the steps of the proof of Theorem 1 we obtain for all $k \geq 0$:

$$|p(t)| \leq |p(t_k)| + \gamma\phi_k e^{\gamma\rho_k}\int_{t_k}^{t}|b(s)|ds \quad \forall t\in[t_k,t_{k+1})$$

and

$$|p(t_{k+1})| \leq \sqrt{1-\frac{\vartheta_k}{\gamma^{-1}+\gamma\rho_k^2}}\left(|p(t_k)| + \gamma\phi_k e^{\gamma\rho_k}\int_{t_k}^{t_{k+1}}|b(t)|dt\right).$$

Similarly to Theorem 1, first let us consider the case when $\|b\|_\infty < +\infty$, then for all $k \geq 0$:

$$|p(t)| \leq |p(t_k)| + \gamma\ell_k\phi_k e^{\gamma\rho_k}\|b\|_\infty \quad \forall t\in[t_k,t_{k+1}),$$
$$|p(t_{k+1})| \leq \sqrt{1-\frac{\vartheta_k}{\gamma^{-1}+\gamma\rho_k^2}}\left(|p(t_k)| + \gamma\ell_k\phi_k e^{\gamma\rho_k}\|b\|_\infty\right).$$

Recursively applying the last estimate for $k \geq 0$ the following inequality can be established:

$$|p(t_{k+1})| \leq \prod_{i=0}^{k}\sqrt{1-\frac{\vartheta_i}{\gamma^{-1}+\gamma\rho_i^2}}|p(t_0)| + \gamma\sum_{i=0}^{k}\prod_{j=i}^{k}\sqrt{1-\frac{\vartheta_j}{\gamma^{-1}+\gamma\rho_j^2}}\ell_i\phi_i e^{\gamma\rho_i}\|b\|_\infty.$$

If

$$\gamma \sup_{k\geq 0}\left\{ \ell_k\phi_k e^{\gamma\rho_k} + \sum_{i=0}^{k}\prod_{j=i}^{k}\sqrt{1-\frac{\vartheta_j}{\gamma^{-1}+\gamma\rho_j^2}}\ell_i\phi_i e^{\gamma\rho_i}\right\} \leq \eta_{t_0}$$

for some $\eta_{t_0} \in \mathbb{R}_+$, then the claim of the corollary can be deduced repeating the arguments of the proof of Theorem 1.

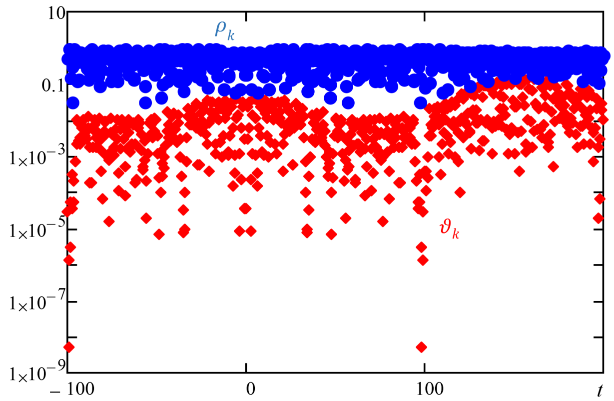

Figure 1. The parameters of excitation

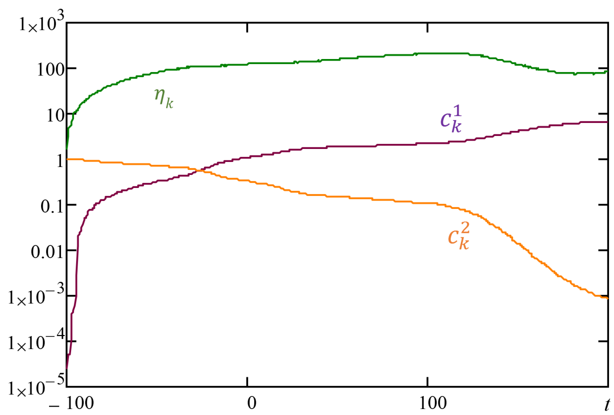

Figure 2. The conditions of asymptotic convergence and ISS

Now consider the case when $\int_{t_0}^{+\infty} |b(s)|ds < +\infty$, then for all $k \geq 0$:

$$|p(t_{k+1})| \leq \prod_{i=0}^{k} \sqrt{1 - \frac{\vartheta_i}{\gamma^{-1} + \gamma\rho_i^2}} |p(t_0)|$$

$$+\gamma \sum_{i=0}^{k} \prod_{j=i}^{k} \sqrt{1 - \frac{\vartheta_j}{\gamma^{-1} + \gamma\rho_j^2}} \phi_i e^{\gamma\rho_i} \int_{t_i}^{t_{i+1}} |b(t)|dt.$$

If

$$\gamma \sup_{k\geq 0} \phi_k e^{\gamma\rho_k} \leq \beta_{t_0}$$

for some $\beta_{t_0} \in \mathbb{R}_+$, then the estimate (8) follows the same arguments as in the proof of Theorem 1. □

V. Simulations

Verification of these theoretical results for the scalar case can be found in [10] (together with the analytical estimates improved for the case $l = 1$).

In this work let us consider the problem of estimation in the linear regression model:

$$x(t) = R^\top(t)\theta,\ t \geq t_0,\ t_0 \in \mathbb{R},$$
$$y(t) = x(t) + w(t),$$

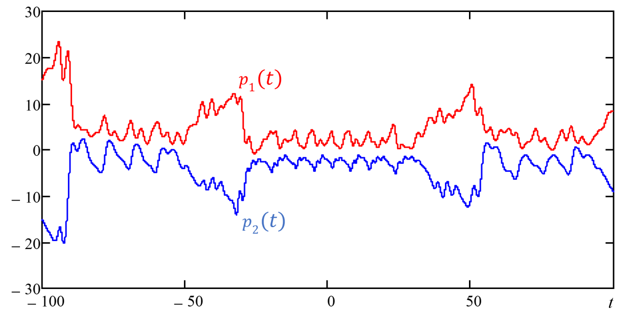

Figure 3. The results of simulation with $\ell = 2$

where $x(t) \in \mathbb{R}$ is the model output, $\theta \in \mathbb{R}^l$ is the vector of unknown constant parameters that is necessary to estimate, $R : \mathbb{R} \to \mathbb{R}^l$ represents the regressor function (usually assumed to be bounded and known), $y(t) \in \mathbb{R}$ is the signal available for measurements with a measurement noise $w : \mathbb{R} \to \mathbb{R}$. The noise $w(t)$ may also represent, for example, the time-varying deviations of $\theta(t)$ (if it is not a constant), then $w(t)$ is proportional to the derivative $\dot{\theta}(t)$. The most popular solution to solve this problem is the linear estimation algorithm [6]:

$$\dot{\hat{\theta}}(t) = \gamma R(t)\left(y(t) - R^\top(t)\hat{\theta}(t)\right), \ \gamma > 0,$$

where $\hat{\theta}(t) \in \mathbb{R}^n$ is the estimate of θ, and the dynamics of its estimation error $p(t) = \theta - \hat{\theta}(t)$ takes the form of (1) for

$$b(t) = -\gamma R(t) w(t).$$

Therefore, the ability of estimation of the value of θ is crucially dependent on the form of excitation of the regressor $R(t)$.

For an illustration let $l = 2$ and

$$R(t) = \begin{bmatrix} \sin(t) \\ \sin(g(t)t) \end{bmatrix}, \ \gamma = 1,$$

where $g : \mathbb{R} \to [1, 2]$ is a slowly varying function modeling the frequency of the second sinusoidal signal in R. Obviously, if $g(t) = 1$ then there is no excitation in the system (an estimation is impossible), and if $g(t) = 2$ then the system is well excited. By choosing

$$g(t) = 2 - \sin^2\left(\frac{t}{20\pi}\right)$$

we ensure that the frequency $g(t)$ will approach both limits, 1 and 2, infinitely many times and the conventional PE property is not satisfied. To check the conditions (2) and (3) let us select

$$\ell_k = 0.5, \ t_0 = -100, \ t_{k+1} = t_k + \ell_k \quad \forall k \geq 0,$$

and calculate

$$\rho_k = \int_{t_k}^{t_{k+1}} |R(t)|^2 dt, \ \vartheta_k = \lambda_{\min}\left(\int_{t_k}^{t_{k+1}} R(s)R^\top(s)ds\right),$$

where $\lambda_{\min}(\cdot)$ corresponds to the minimal eigenvalue of the argument matrix. The resulting values are plotted in Fig. 1. Next, let us construct the sequences

$$c_k^1 = \sum_{i=0}^{k} \frac{\vartheta_i}{\gamma^{-1} + \gamma\rho_i^2},$$

$$c_k^2 = \prod_{i=0}^{k}\left(1 - \frac{\vartheta_i}{\gamma^{-1} + \gamma\rho_i^2}\right),$$

which according to (2) and (3) have to diverge and converge, respectively. These values are plotted in Fig. 2, from which we can conclude that the conditions (2) and (3) are verified. Since the sequence ρ_k is ultimately bounded, following the result of Theorem 1 we can claim that such a system admits iISS property with respect to additive perturbation b. To verify ISS property let us calculate

$$\eta_k = \ell_k e^{\gamma \rho_k} + \sum_{i=0}^{k} \prod_{j=i}^{k} \sqrt{1 - \frac{\vartheta_j}{\gamma^{-1} + \gamma \rho_j^2}} \ell_i e^{\gamma \rho_i},$$

whose boundedness according to Theorem 1 implies the desired robustness property. The sequence η_k is also shown in Fig. 2, from which we observe its boundedness and can resume the ISS of (1) in the considered case. In order to validate this conclusion, the results of simulation of the system for

$$b(t) = \begin{bmatrix} 1 + \sin(1.5t) \\ -1 \end{bmatrix}$$

and the initial conditions

$$p(t_0) = \begin{bmatrix} 15 \\ -15 \end{bmatrix}$$

are presented in Fig. 3, which confirm the evaluated stability property of the system.

VI. Conclusion

For a time-varying linear system under a relaxed notion of PE, several relations with ISS and iISS properties are established. It is shown that iISS is rather natural and it is satisfied almost always in this class of systems (non uniformly GAS in the noise-free case), and especially with respect to measurement noises. All proposed conditions are sufficient, and investigation of necessary counterpart can be a direction of future research.

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