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Abstract The paper present a dilation symmetry based approach to expansion of regularity of nonlinear evolution equations. In particular, it is shown that a symmetry of an operator, which describes a right-hand side of a nonlinear evolution equation, is inherited by solutions of this equation. In the case of dilation symmetry, the latter implies that global-in-time existence of solutions for small initial data always imply global-in-time existence of solutions for large initial data. As an example, we consider the problem of expansion of regularity of the Navier-Stokes equations (in $\mathbb{R}^n$) accepting that the existence of global-in-time solutions for small initial data is already proven.

Keywords Dilation Symmetry · Nonlinear Evolution Equations

1 Introduction

According to the classical concept of homogeneity introduced by Leonhard Euler in 18th century, homogeneity is a sort of symmetry of an object (e.g. a function or a set) with respect to a group of transformations knows today as dilations. For example, a function f in $\mathbb{R}^n$ is homogeneous in the classical (standard) sense if it is symmetric with respect to a uniform dilation of an argument, i.e. there exists $\nu \in \mathbb{R}$ such that

$$f(e^s u) = e^{(\nu+1)s} f(u), \quad u \in \mathbb{R}^n, s \in \mathbb{R}.$$

Homogeneity of a function is inherited by other objects induced by this function. For example, the Euler's Homogeneous Function Theorem implies that

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any derivative of f is homogeneous too. Similarly, if $u(\cdot):[0,+\infty)\to\mathbb{R}^n$ is a classical solution of

$$\frac{du}{dt}=f(u),\quad t>0$$

with the initial condition $u(0)=u_0$ then $u_s(t):=e^s u(e^{\nu s}t)$ is defined on $[0,+\infty)$, then, due to symmetry, we derive $\frac{du_s}{dt}=e^{(\nu+1)s}f(u(e^{\nu s}t))=f(u_s(t))$, $t>0$, i.e. u_s is a classical solution of the same differential equation with the initial condition $u_s(0)=e^s u_0$, where $s\in\mathbb{R}$.

Let $\exists\varepsilon>0$ such that a classical solution of the differential equation exists on $[0,+\infty)$ for any initial value $u(0)=u_0\in B_\varepsilon:=\{u\in\mathbb{R}^n:\|u\|<\varepsilon\}$. Hence, exploiting the symmetry we derive existence of a classical solution for any initial condition $u(0)=u_0\in\bigcup_{s\in\mathbb{R}}e^s B_\varepsilon=\mathbb{R}^n$.

For non-linear evolution system, it may be simpler to prove existence and uniqueness of a regular solution for small initial data. In this paper we show that the dilation symmetry can be utilized for global expansion of regularity non-linear evolution equations in a Banach space $\mathbb{B}$ provided that a dilation group is properly introduced in $\mathbb{B}$. As an example, we consider the problem of expansion of regularity of the Navier-Stokes equations (in $\mathbb{R}^n$) accepting that the existence of global-in-time solutions for small initial data is already proven. In particular, we present a necessary and sufficient conditions for expansion of regularity of Navier-Stokes equation by means of dilation symmetry.

Mainly, the standard *notation* is utilized through the paper, e.g. $\mathbb{R}$ is the field of real numbers; $L^1_{loc}((0,T)\times\mathbb{R}^n,\mathbb{R})$ denotes the space of locally integrable functions $(0,T)\times\mathbb{R}^n\to\mathbb{R}$; $L^p(\mathbb{R}^n,\mathbb{R}^m)$, $1\le p\le+\infty$ is a Lebesgue space of function $\mathbb{R}^n\to\mathbb{R}^m$ with the norm $\|\cdot\|_p$; $C_c^\infty((0,T)\times\mathbb{R}^n,\mathbb{R}^m)$ is a space of smooth functions $(0,T)\times\mathbb{R}^n\to\mathbb{R}^m$ with compact support and $C_0^\infty([0,T)\times\mathbb{R}^n,\mathbb{R}^m)$ is a space of smooth functions which vanish at infinity, where $0<T\le\infty$. For composition of operators A, B we also use the notation $A\circ B$.

Let $L^p_\mu(\mathbb{R}^n,\mathbb{R}^m)$ denotes the following normed vector space of functions $\mathbb{R}^n\to\mathbb{R}^m$

$$L^p_\mu(\mathbb{R}^n,\mathbb{R}^m):=\{u:\|u\|_{p,\mu}<+\infty\},\quad \mu\in\mathbb{R}$$

$$\|u\|_{p,\mu}:=\left(\int_{\mathbb{R}^n}|x|^{\mu p}|u(x)|^p\,dx\right)^{1/p},\quad 0<p<\infty$$

$$\|u\|_{\infty,\mu}:=\text{ess sup}(|x|^\mu u(x)),\quad p=\infty,$$

which can be treated as a wighted L_p. The notation $\overset{a.e.}{\in}$ and $\overset{a.e.}{=}$ is utilized in order to indicate that an inclusion or identity is fulfilled almost everywhere on a domain.

2 Symmetry of nonlinear operators

2.1 Dilation group

Let $\mathbb{X}$ be a (linear) vector space and $\{\mathbf{d}(s)\}_{s\in\mathbb{R}}$ be a family of operators $\mathbf{d}(s):\mathbb{X}\to\mathbb{X}$. If

- $\mathbf{d}(0) = I$, where I is an identity operator (i.e. $Iz = z$ for all $z \in \mathbb{X}$);
- $\mathbf{d}(t+s)z = (\mathbf{d}(t) \circ \mathbf{d}(s))z = (\mathbf{d}(s) \circ \mathbf{d}(t))z$ for all $t, s \in \mathbb{R}, z \in \mathbb{X}$

then, by definition, $\mathbf{d}$ is a *group*. Using the group properties for $t = -s$ we derive $(\mathbf{d}(-s) \circ \mathbf{d}(s))z = (\mathbf{d}(s) \circ \mathbf{d}(-s))z = z$, the operator $\mathbf{d}(s)$ is invertible and $[\mathbf{d}(s)]^{-1} = \mathbf{d}(-s)$. Moreover, $\mathbf{d}(s)$ maps $\mathbb{X}$ onto $\mathbb{X}$ for any $s \in \mathbb{R}$. Indeed, suppose the contrary, i.e. $\exists z^* \in \mathbb{X}$ and such $s^* \in \mathbb{R}$ such that $z^* \notin \mathbf{d}(s^*)\mathbb{X}$. Since $u^* := \mathbf{d}(-s^*)z^* \in \mathbb{X}$ then $z^* = \mathbf{d}(s)\mathbf{d}(-s^*)z^* = \mathbf{d}(s)u^* \in \mathbf{d}(s)\mathbb{X}$.

Definition 1 A group $\mathbf{d}$ of operators on a normed vector space $\mathbb{X}$ is said to be a *dilation group* (or simply *dilation*) on $\mathbb{X}$ if $\mathbf{d}(s)\,\mathbf{0} = \mathbf{0}$ for any $s \in \mathbb{R}$ and the following *limit property* holds

$$\liminf_{s\to-\infty}\|\mathbf{d}(s)z\| = 0 \quad \text{and} \quad \limsup_{s\to+\infty}\|\mathbf{d}(s)z\| = \infty \quad \text{for} \quad z \neq \mathbf{0}\,.$$

The limit property given above specifies a group being a dilation in an abstract space. We refer the reader to [2] for more details about topological characterization of dilations.

Example 1 Let us recall a few well-know dilation groups in $\mathbb{R}^n$:

1) *Uniform dilation* (L. Euler, 18th century):

$$\mathbf{d}(s) = e^s I, \quad s \in \mathbb{R}$$

where I is the identity matrix $\mathbb{R}^{n\times n}$.
2) *Weighted dilation* [14]:

$$\mathbf{d}(s) = \begin{pmatrix} e^{r_1 s} & 0 & ... & 0 \\ 0 & e^{r_2 s} & ... & 0 \\ ... & ... & ... & ... \\ 0 & ... & ... & e^{r_n s} \end{pmatrix},$$

where $r_i > 0$, $i = 1, 2, ..., n$.
3) Geometric dilation (see e.g. [4], [12], [3]) is a flow generated by an Euler vector field [1].

The uniform dilation $\mathbf{d}(s) = e^s I \in \mathbb{B}, s \in \mathbb{R}$ is the simplest example of a dilation in any normed vector space. Let us consider a few other examples.

Example 2 Let $\mathbb{B}$ be a space of bounded uniformly continuous functions $\mathbb{R}^n \to \mathbb{R}^m$ with the supremum norm. A dilation group $\mathbf{d}$ in $\mathbb{B}$ can be defined as follows

$$(\mathbf{d}(s)z)(x) = e^{\alpha s} z(x + \beta s),$$

[1] A C^1 vector field $\nu : \mathbb{R}^n \to \mathbb{R}^n$ is called Euler if it is complete and $-\nu$ is globally asymptotically stable.

where $s\in\mathbb{R}$ is the group parameter, $z\in\mathbb{X}$, $x\in\mathbb{R}$ and $\alpha>0$ and $\beta\in\mathbb{R}$ are a constant parameters. Indeed, $\mathbf{d}(s)z\in\mathbb{X}$ if $z\in\mathbb{B}$, $s\in\mathbb{R}$ and for $v=\mathbf{d}(s)z$ we have

$$(\mathbf{d}(t)\circ\mathbf{d}(s)z)(x)=(\mathbf{d}(t)v)(x)=e^{\alpha t}v(x+\beta t)=e^{\alpha t}e^{\alpha s}z(x+\beta s+\beta t)=(\mathbf{d}(s+t)z)(x).$$

The limit property also holds since

$$\|\mathbf{d}(s)z\|=\sup_{x\in\mathbb{R}^n}|e^{\alpha s}z(x+s)|=e^{\alpha s}\sup_{x\in\mathbb{R}^n}|z(x+\beta s)|=e^{\alpha s}\|z\|.$$

The next lemma introduces the most common dilation in functional spaces.

Lemma 1 *The operator* $\mathbf{d}(s)$ *given by*

$$(\mathbf{d}(s)z)(x)=e^{\alpha s}z(e^{\beta s}x), \tag{1}$$

where $s\in\mathbb{R}$, z *is a function* $\mathbb{R}^n\to\mathbb{R}^m$, $x\in\mathbb{R}^n$ *and* $\alpha,\beta\in\mathbb{R}$ *are constant parameters, is*

– *a linear bounded invertible operator on* $L^p(\mathbb{R}^n,\mathbb{R}^m)$,

$$\|\mathbf{d}(s)z\|_p=e^{(\alpha-n\beta/p)s}\|z\|_p,\quad z\in L^p(\mathbb{R}^n,\mathbb{R}^m), s\in\mathbb{R},$$

– *a linear bounded invertible operator on* $L^p_\mu(\mathbb{R}^n,\mathbb{R}^m)$,

$$\|\mathbf{d}(s)z\|_{p,\mu}=e^{(\alpha-\beta(\mu+n/p))s}\|z\|_{p,\mu},\quad z\in L^p_\mu(\mathbb{R}^n,\mathbb{R}^m), s\in\mathbb{R},$$

where $0<p\leq\infty$. *The inverse operator is given by* $[\mathbf{d}(s)]^{-1}=\mathbf{d}(-s)$.

Proof Notice that $L^p=L^p_0$.

Let $1\leq p<\infty$. If $z\in L^p_\mu(\mathbb{R}^n,\mathbb{R}^m)$ then

$$\int_{\mathbb{R}^n}|x|^{\mu p}|z(x)|^p\,dx<+\infty$$

and

$$\int_{\mathbb{R}^n}|x|^{\mu p}|z(x)|^p\,dx=e^{n\beta s}\int_{\mathbb{R}^n}|e^{\beta s}x|^{\mu p}|z(e^{\beta s}x)|^p\,dx=$$
$$e^{((n+\mu p)\beta-\alpha p)s}\int_{\mathbb{R}^n}|x|^{\mu p}(\mathbf{d}(s)z)(x)|^p\,dx<+\infty.$$

Since $e^{((n+\mu p)\beta-\alpha p)s}>0$ for any $\alpha,\beta,p,s\in\mathbb{R}$ then $\mathbf{d}(s)z\in L^p_\mu(\mathbb{R}^n,\mathbb{R}^m)$ for any $s\in\mathbb{R}$. Obviously, $\mathbf{d}(s)$ is a linear operator on L^p_μ, i.e. $\mathbf{d}(s)(\mu_1 z_1+\mu z_2)=\mu_1\mathbf{d}(s)z_1+\mu_2\mathbf{d}(s)z_2$, for any $\mu_1,\mu_2\in\mathbb{R}$ and $z_1,z_2\in L^p_\mu(\mathbb{R}^n,\mathbb{R}^m)$. Moreover, the latter identities imply that

$$\|\mathbf{d}(s)z\|_{p,\mu}=e^{(\alpha-(n/p+\mu)\beta)s}\|z\|_p,\quad \|z\|_{p,\mu}:=\left(\int_{\mathbb{R}^n}|x|^{\mu p}|z(x)|^p\,dx\right)^{1/p}.$$

Hence, the operator $\mathbf{d}(s) : L^p(\mathbb{R}^n,\mathbb{R}^m) \to L^p(\mathbb{R}^n,\mathbb{R}^m)$ is bounded for any $s\in\mathbb{R}$.

Let $p=\infty$. If $z\in L^\infty_\mu(\mathbb{R}^n,\mathbb{R}^m)$ then

$$\text{ess sup}|z(x)| = \text{ess sup}(|e^{\beta s}x|^\mu |z(e^{\beta s}x)|) < +\infty$$

for any $\beta, s, \mu \in \mathbb{R}$ and $\|\mathbf{d}(s)z\|_\infty = e^{(\alpha-\beta\mu)s}\|z\|_\infty$ for any $s\in\mathbb{R}$. Therefore, $\mathbf{d}(s)$ is also a linear bounded operator on $z\in L^\infty(\mathbb{R}^n,\mathbb{R}^m)$.

Obviously, $(\mathbf{d}(s)\circ\mathbf{d}(-s))z = (\mathbf{d}(-s)\circ\mathbf{d}(s))z$ for any $z:\mathbb{R}^n\to\mathbb{R}^n$ and any $s\in\mathbb{R}$ and we derive $[\mathbf{d}(s)]^{-1}=\mathbf{d}(-s)$.

2.2 **d**-homogeneous operators

In this section we introduce a notion of **d**-homogeneous (symmetric with respect to a group **d**) operators in a vector space $\mathbb{X}$ and present a couple of examples.

Definition 2 An operator $F : \mathcal{D}(F) \subset \mathbb{X} \to \mathbb{X}$ is said to be **d**-homogeneous of degree $\mu\in\mathbb{R}$ if $\mathbf{d}(s)\mathcal{D}(F)\subset\mathcal{D}(F)$ for any $s\in\mathbb{R}$ and

$$(F\circ\mathbf{d}(s))u = e^{\mu s}(\mathbf{d}(s)\circ F)u \quad \text{for} \quad s\in\mathbb{R}, \quad u\in\mathcal{D}(F), \qquad (2)$$

where **d** is a group of invertible operators on $\mathbb{X}$.

A lot of examples of **d**-homogeneous vector-field for $\mathbb{X}=\mathbb{R}^n$ can be found in control literature (see e.g. [14], [12], [3] and references therein). For example, the vector function

$$(x_1,x_2)\to\left(x_1+x_2^2,\ |x_2|\sin\left(\frac{x_1-x_2^2}{|x_1|+x_2^2}\right)\right), \quad (x_1,x_2)\in\mathbb{R}^2$$

is **d**-homogeneous of degree 0 with

$$\mathbf{d}(s)(x_1,x_2)\to(e^{2s}x_1, e^s x_2), \quad s\in\mathbb{R}.$$

All linear and lot of nonlinear models of mathematical physics are **d**-homogeneous under a proper selection of a dilation group (see e.g. [11]).

Notice that, the identity (2) can always be understood in the weak sense. *For shortness we omit* $\mathbb{R}^n$ *in the notations for* $\int_{\mathbb{R}^n}$, $L^2(\mathbb{R}^n,\mathbb{R}^n)$,$L^1_{loc}(\mathbb{R}^n,\mathbb{R}^n)$ *and* $C_0^\infty(\mathbb{R}^n,\mathbb{R}^n)$ *in the examples below.*

*Example 3 (**d**-homogeneity of the Laplace operator)*

Let us consider the Laplace operator

$$\Delta:\mathcal{D}(\Delta)\subset L^2\to L^2,$$

with the domain

$$\mathcal{D}(A)=\left\{u\in L^2 : \exists f \in L^1_{loc} \text{ such that} \int u \cdot \Delta\phi = \int f \cdot \phi, \quad \forall \phi\in C_c^\infty\right\}.$$

Let us show that Δ is $\mathbf{d}$-homogeneous of degree 2β provided that the dilation $\mathbf{d}$ is given by (1).

By Lemma 1, $\mathbf{d}$ is a group of linear invertible operators on C_c^∞ and, consequently (see the beginning of Section 2), $\mathbf{d}(s)$ maps C_c^∞ onto C_c^∞. Notice that if $\phi \in C_c^\infty$ then, obviously,

$$(\Delta\circ\mathbf{d}(s))\phi)(x)=e^{(\alpha+2\beta)s}(\Delta\phi)(e^{\beta s}x)=e^{2\beta s}((\mathbf{d}(s)\circ\Delta)\phi)(x), \quad s\in\mathbb{R}, \quad x\in\mathbb{R}^n.$$

In other words, the Laplace operator is $\mathbf{d}$-homogeneous as operator $C_c^\infty \to C_c^\infty$. Since C_c^∞ is dense in L^2 then it is $\mathbf{d}$-homogeneous as an operator $L^2 \to L^2$. Let us prove this claim more rigorously.

Let $u \in \mathcal{D}(\Delta)$ and $\Delta u = f \in L^1_{loc}$ (in the weak sense). Since $\mathbf{d}(s)f \in L^1_{loc}$ then using the change-of-variable theorem (see e.g. [6]) in the Lebesgue integral we derive

$$e^{2\beta s}\!\int\!(\mathbf{d}(s)f)\cdot\phi = e^{(\alpha+2\beta)s}\!\int\! f(e^{\beta s}x)\cdot\phi(x)dx = e^{(\alpha+(2-n)\beta)s}\!\int\! f(x)\cdot\phi(e^{-\beta s}x)dx =$$

$$e^{(2\alpha+(2-n)\beta)s}\!\int\! f\cdot\tilde{\phi} = e^{(2\alpha+(2-n)\beta)s}\!\int\! u\cdot\Delta\tilde{\phi} = e^{(2\alpha+(2-n)\beta)s}\!\int\! u\cdot(\Delta\circ\mathbf{d}(-s))\phi =$$

$$e^{(\alpha-n\beta)s}\int u(x)\cdot\Delta\phi(e^{-\beta s}x)dx = e^{\alpha s}\int u(e^{\beta s}x)\cdot\Delta\phi(x)dx = \int(\mathbf{d}(s)u)\cdot\Delta\phi,$$

where $\tilde{\phi}=\mathbf{d}(-s)\phi \in C_c^\infty$. Hence, $\mathbf{d}(s)u \in \mathcal{D}(\Delta)$ and $(\Delta\circ\mathbf{d}(s))u = e^{2\beta s}\mathbf{d}(s)f = e^{2\beta s}(\mathbf{d}(s)\circ\Delta)u$ (in the weak sense) for any $s \in \mathbb{R}, u \in \mathcal{D}(\Delta)$.

3 Symmetry of Evolution Equations

3.1 Linear evolution equation

The dilation symmetry of an operator is inherited by other objects induced (generated) by this operator.

Lemma 2 *Let a linear closed densely defined operator $A : \mathcal{D}(A) \subset \mathbb{B} \to \mathbb{B}$ generate a strongly continuous semigroup $\Phi := \{\Phi(t)\}_{t\geq 0}$ of linear bounded operators on $\mathbb{B}$ and $\mathbf{d}$ be a group of linear bounded invertible operators on $\mathbb{B}$. If the operator A is $\mathbf{d}$-homogeneous of degree μ then*

$$\Phi(t)\circ\mathbf{d}(s) = \mathbf{d}(s)\circ\Phi(e^{\mu s}t), \quad \forall t \geq 0, \quad \forall s \in \mathbb{R}. \tag{3}$$

Proof Since Φ is generated by A then $\Phi(e^{\mu s}t)u \in \mathcal{D}(A)$ for any $u \in \mathcal{D}(A)$ (see e.g. [8, page 5]).

Let $s \in \mathbb{R}$ and $u \in \mathcal{D}(A)$ be selected arbitrary. Since the operator A is $\mathbf{d}$-homogeneous then $\mathcal{D}(A)$ is invariant with respect to the transformation $\mathbf{d}(s)$, i.e. $\mathbf{d}(s)z \in \mathcal{D}(A), \forall z \in \mathcal{D}(A)$, and, consequently,

$$y_1(t) := (\Phi(t)\circ\mathbf{d}(s))u \in \mathcal{D}(A), \text{ and } y_2(t) := (\mathbf{d}(s)\circ\Phi(e^{\mu s}t))u \in \mathcal{D}(A), \quad t \geq 0.$$

Being generated by A the semigroup Φ satisfy (see e.g. [8, page 5])

$$\frac{d}{dt}\Phi(t)z = (A \circ \Phi(t))z = (\Phi(t) \circ A)z, \quad \forall t > 0, \quad \forall z \in \mathcal{D}(A).$$

Taking into account that A is $\mathbf{d}$-homogeneous of degree μ and $\mathbf{d}(s)$ is a linear bounded operator on $\mathbb{B}$, we derive

$$\tfrac{d}{dt}y_2(t)=e^{\mu s}(\mathbf{d}(s) \circ A \circ \Phi(e^{\mu s}t))u=(A \circ \mathbf{d}(s) \circ \Phi(e^{\mu s}t))u=Ay_2(t), \ \forall t>0.$$

On the other hand, we have

$$\tfrac{d}{dt}y_1(t) = (A \circ \Phi(t) \circ \mathbf{d}(s))u = Ay_1(t), \quad \forall t > 0.$$

Since $y_1(0) = y_2(0) = \mathbf{d}(s)u$ then due to uniqueness of the semigroup Φ generated by A (see [8, page 6]) we derive $y_1(t) = y_2(t)$ for all $t \geq 0$ and

$$(\Phi(t) \circ \mathbf{d}(s))u = (\mathbf{d}(s) \circ \Phi(e^{\mu s}t))u, \quad \forall t \geq 0, \quad \forall u \in \mathcal{D}(A).$$

Since $\Phi(t) \circ \mathbf{d}(s)$ and $\mathbf{d}(s) \circ \Phi(e^{\mu s}t)$ are bounded linear operators and $\mathcal{D}(A)$ is dense in $\mathbb{B}$ then the latter identity holds for all $u \in \mathbb{B}$ and all $t \geq 0$.

It is well-known (see e.g. [8, page 100]) that

$$u(t, u_0) = \Phi(t)u_0, \quad t \geq 0$$

is a unique solution of the linear evolution equation

$$\frac{du}{dt} = Au$$

with the initial condition $u(0) = u_0 \in \mathbb{B}$. The latter lemma, obviously, proves the symmetry of these solutions:

$$u(t, \mathbf{d}(s)u_0) = \mathbf{d}(s)u(e^{\mu s}t, u_0), \quad s \in \mathbb{R}$$

Below we prove this result for non-linear operators too.

3.2 Nonlinear evolution equation

Let us consider the non-linear evolution system

$$\frac{du}{dt} = Au + Gu, \quad t > 0, \tag{4}$$

where a densely defined closed linear operator

$$A : \mathcal{D}(A) \subset \mathbb{B} \to \mathbb{B}$$

generates a strongly continuous semigroup Φ of linear bounded operators on $\mathbb{B}$, and

$$G : \mathcal{D}(G) \subset \mathbb{B} \to \mathbb{B}$$

is a possibly nonlinear operator.

Definition 3 A continuous function $u : [0, T) \to \mathbb{B}$ is said to be

- a *mild* solution of the system (4) if $Gu \in L^1((0, T), \mathbb{B})$ and

$$u(t) = \Phi(t)u(0) + \int_0^t (\Phi(t - \tau) \circ G)u(\tau)\, d\tau, \qquad t \in (0, T); \tag{5}$$

- a *strong* solution of the evolution equation (4) if $u \in C([0, T), \mathbb{B})$, u is differentiable almost everywhere on $(0, T)$, $\frac{du}{dt}, Gu \in L^1((0, T), \mathbb{B})$ and (4) is satisfied almost everywhere on $(0, T)$;
- a *classical* solution of the evolution equation (4) if $u \in C([0, T), \mathbb{B})$, $\frac{du}{dt} \in C((0, T), \mathbb{B})$, $u(t) \in \mathcal{D}(A) \cap \mathcal{D}(G)$ for all $t \in (0, T)$ and (4) is satisfied on $(0, T)$.

The latter integral is understood in the sense of Bochner ([13, page 132]).

Theorem 1 *Let $\mathbf{d}$ be a group of linear bounded invertible operators on $\mathbb{B}$ and let A and G be $\mathbf{d}$-homogeneous operators of a degree $\mu \in \mathbb{R}$.*

If $u : [0, T) \to \mathbb{B}$ is a mild (or strong) solution of the evolution equation (4) *and*

$$u(t) \overset{a.e.}{\in} \mathcal{D}(G), \quad t \in (0, T),$$

then the function $u_s : [0, e^{-\mu s}T) \to \mathbb{B}$ defined as

$$u_s(t) = \mathbf{d}(s)u(e^{\mu s}t), \quad t \in [0, e^{-\mu s}T)$$

is also a mild (resp. strong) solution of the evolution equation (4) *and*

$$u_s(t) \overset{a.e.}{\in} \mathcal{D}(G), \quad t \in (0, e^{-\mu s}T),$$

for any $s \in \mathbb{R}$.

Moreover, the claim remains true for classical solutions and the above inclusions hold everywhere on $(0,T)$ and $(0, e^{-\mu s}T)$, respectively.

Proof If $u(t) \in \mathcal{D}(G)$ then due to $\mathbf{d}$-homogeneity of the operator G we have $\mathbf{d}(s)\mathcal{D}(G) \subset \mathcal{D}(G)$ and $u_s(e^{-\mu s}t) \in \mathcal{D}(G)$.

1) *The case of mild solutions.*

Since the $\mathbf{d}$-homogeneous operator A generates a strongly continuous semigroup Φ, then according Lemma 2 we have

$$\Phi(t) \circ \mathbf{d}(s) = \mathbf{d}(s) \circ \Phi(e^{\mu s}t), \quad \forall t \geq 0, \quad \forall s \in \mathbb{R}.$$

It is well known [13, page 134] that $K \int_0^t \xi(s)ds = \int_0^t K\xi(s)ds$ for any bounded linear operator $K : \mathbb{B} \to \mathbb{B}$ and any Bochner integrable function $\xi \in L^1((0,T), \mathbb{B})$. Hence, using $G(u) \in L^1((0,T), \mathbb{B})$ we derive

$$\mathbf{d}(s) \int_0^t G(u(\tau))d\tau = \int_0^t \mathbf{d}(s)G(u(\tau))d\tau = e^{-\mu s} \int_0^t G(\mathbf{d}(s)u(\tau))d\tau =$$

$$\int_0^{e^{-\mu s}t} G(\mathbf{d}(s)u(e^{\mu s}\tau))d\tau = \int_0^{e^{-\mu s}t} G(u_s(\tau))d\tau,$$

i.e. $G(u_s) \in L^1((0, e^{-\mu s}T), \mathbb{B})$. Similarly, we derive

$$\mathbf{d}(s)u(e^{\mu s}t) = (\mathbf{d}(s) \circ \Phi(e^{\mu s}t))u(0) + \int_0^{e^{\mu s}t} (\mathbf{d}(s) \circ \Phi(e^{\mu s}t - \tau) \circ G)u(\tau)\, d\tau =$$

$$(\Phi(t) \circ \mathbf{d}(s))u(0) + \int_0^{e^{\mu s}t} (\mathbf{d}(s) \circ \Phi(e^{\mu s}t - \tau) \circ G)u(\tau)\, d\tau =$$

$$(\Phi(t) \circ \mathbf{d}(s))u(0) + e^{\mu s} \int_0^t (\mathbf{d}(s) \circ \Phi(e^{\mu s}(t - \sigma)) \circ G)u(e^{\mu s}\sigma))\, d\sigma =$$

$$(\Phi(t) \circ \mathbf{d}(s))u(0) + e^{\mu s} \int_0^t (\Phi(t - \sigma) \circ \mathbf{d}(s) \circ G)u(e^{\mu s}\sigma))\, d\sigma =$$

$$(\Phi(t) \circ \mathbf{d}(s))u(0) + \int_0^t (\Phi(t - \sigma) \circ G)(\mathbf{d}(s)u(e^{\mu s}\sigma))\, d\sigma,$$

where the linearity of the operator $\Phi(t - \sigma)$ and the $\mathbf{d}$-homogeneity of the operator G are utilized on the last step. Therefore, we have shown that

$$u_s(t) = \Phi(t)u_s(0) + \int_0^t (\Phi(t - \sigma) \circ G)u_s(\sigma)\, d\sigma,$$

i.e. u_s is a mild solution of (4).

2) *The case of strong and classical solutions.* Let u be a strong solution. Since $u(t)$ is differentiable almost everywhere and $\mathbf{d}(s)$ is a linear bounded operator then for $\tau = e^{\mu s}t$ we have

$$\tfrac{du_s(t)}{dt} \overset{a.e.}{=} e^{\mu s}\mathbf{d}(s)\tfrac{du(\tau)}{d\tau} \overset{a.e.}{=} e^{\mu s}\mathbf{d}(s)(Au(\tau)+G(u(\tau)) \overset{a.e.}{=} Au_s(t)+G(u_s(t)).$$

In the case of the classical solution the latter identity holds everywhere. Moreover, if $u(t) \in \mathcal{D}(A)$ then due to $\mathbf{d}$-homogeneity of the operator A we always have $u_s(t) \in \mathcal{D}(A)$. Finally, taking into account that the operator $\mathbf{d}(s)$ is bounded we conclude $u_s \in C([0,T),\mathbb{B})$ provided that $u \in C([0,T),\mathbb{B}$ and $\frac{d}{dt}u_s \in C((0,T),\mathbb{B})$ provided that $\frac{d}{dt}u \in C((0,T),\mathbb{B}$, i.e. u_s is also a classical solution.

Remark 1 The proven symmetry of solutions of the evolution equation (4) allows us to guarantee that a local result holds globally. For example, let us consider the initial value problem $u(0) = u_0$ for (4). If $\mathcal{D}(G) = \mathbb{B}$ then an existence of a solution for any $u_0 \in B(r)$ implies existence of solutions for all $u_0 \in \mathbb{B}$, where $B(r)$ is a ball in $\mathbb{B}$ of the radius $r > 0$. Indeed, using the limit property of the dilation $\mathbf{d}$ we derive that for any $u_0 \in \mathbb{B}$ there exists $s \in \mathbb{R}$ such that $\|\mathbf{d}(s)u_0\| < r$, i.e. $\mathbf{d}(s)u_0 \in \mathbb{B}(r)$. By Theorem 1, if $u(t, \mathbf{d}(-s)u_0)$ is a solution of the initial value problem $u(0) = \mathbf{d}(-s)u_0 \in \mathbb{B}(r)$ then $u_s(t, u_0) := \mathbf{d}(s)u(e^{\mu s}t, \mathbf{d}(-s)u_0)$ is also a solution of the evolution equation (4). Since $\mathbf{d}(s)u(0, \mathbf{d}(-s)u_0) = \mathbf{d}(s)\mathbf{d}(-s)u_0 = u_0$ then u_s is a solution of the initial value problem $u(0) = u_0 \in \mathbb{B}$ for (4).

Notice that if all solutions with $u_0 \in B(r)$ exist on $[0, +\infty)$ then all solutions with $u_0 \in \mathbb{B}$ also exist on $[0, +\infty)$. Below we show possible cases when such an approach allows to expand regularity of solutions of the Navier-Stokes equations provided that it is already proven for small initial data (see e.g. [5] and [7]).

Some other applications of a dilation symmetry in evolution equations, for example, to problems of mathematical control theory can be found in [11], [10].

3.3 Nonlinear implicit evolution equation

Let $\tilde{\mathbb{X}} = \mathbb{B} \times \mathbb{X}$, where $\mathbb{B}$ is Banach space and $\mathbb{X}$ is a linear (vector) space. Let us consider the nonlinear implicit evolution equation

$$\begin{aligned} \frac{du}{dt} &= Au + G(u,p), \\ \mathbf{0} &= Q(u,p), \end{aligned} \qquad t > 0, \tag{6}$$

where a densely defined closed linear operator

$$A : \mathcal{D}(A) \subset \mathbb{B} \to \mathbb{B}$$

generates a strongly continuous semigroup Φ of linear bounded operators on $\mathbb{B}$, and

$$G:\mathcal{D}(G)\subset\tilde{\mathbb{X}}\to\mathbb{B}\quad\text{and}\quad Q:\mathcal{D}(Q)\subset\tilde{\mathbb{X}}\to\mathbb{X}$$

are a (possibly) nonlinear operators.

Definition 4 A pair (u,p) with $u:[0,T)\to\mathbb{B}$ and $p:[0,T)$ is said to be

- a *mild* solution of the implicit evolution equation (6) if $u\in C([0,T),\mathbb{B})$, $G(u,p)\in L^1((0,T),\mathbb{B})$ and

$$\begin{aligned} u(t) &= \Phi(t)u(0)+\int_0^t(\Phi(t-\tau)\circ G)(u(\tau),p(\tau))\,d\tau, \\ \mathbf{0} &\overset{a.e.}{=} Q(u(t),p(t)), \end{aligned} \qquad t\in(0,T); \quad (7)$$

- a *strong* solution of the implicit evolution equation (6) if $u\in C([0,T),\mathbb{B})$, u is differentiable almost everywhere on $(0,T)$, $\frac{du}{dt}, G(u,p)\in L^1((0,T),\mathbb{B})$ and (6) is satisfied almost everywhere on $(0,T)$;
- a *classical* solution of the evolution equation (6) if $u\in C([0,T),\mathbb{B})$, $\frac{du}{dt}\in C((0,T),\mathbb{B})$, $u(t)\in\mathcal{D}(A)$ and $(u(t),p(t))\in\mathcal{D}(G)\cap\mathcal{D}(Q)$ for $t\in(0,T)$ and (6) is satisfied on $(0,T)$.

A symmetry of solutions of the implicit evolution equation (6) is also preserved provided that the operator, which defines its right-side is $\mathbf{d}$-homogeneous.

Theorem 2 *Let a group* $\mathbf{d}$ *of invertible operators on* $\tilde{\mathbb{X}}$ *be defined as follows*

$$\mathbf{d}(s)(u,p)=(\mathbf{d}_1(s)u,\mathbf{d}_2(s)p),\quad s\in\mathbb{R}, u\in\mathbb{B}, p\in\mathbb{X},$$

where $\mathbf{d}_1$ *is a group of linear bounded invertible operators on* $\mathbb{B}$*,* $\mathbf{d}_2$ *is a group of invertible operators on* $\mathbb{X}$ *such that*

$$\mathbf{d}_2(s)\,\mathbf{0}=\mathbf{0},\quad s\in\mathbb{R}.$$

Let us denote $F:=(G,Q)$ *and* $\mathcal{D}(F):=\mathcal{D}(G)\cap\mathcal{D}(Q)$*,*

$$F:\mathcal{D}(F)\subset\tilde{\mathbb{X}}\to\tilde{\mathbb{X}}.$$

Let A *be a* $\mathbf{d}_1$*-homogeneous operator of a degree* $\mu\in\mathbb{R}$ *and* F *be* $\mathbf{d}$*-homogeneous operator of the same degree* μ*.*

If the pair (u,p) *with* $u:[0,T)\to\mathbb{B}$ *and* $p:[0,T)\to\mathbb{X}$ *is a mild (or strong) solution of the implicit evolution equation* (6) *such that*

$$(u(t),p(t))\overset{a.e}{\in}\mathcal{D}(F),\quad t\in(0,T),$$

then the pair (u_s, p_s) *with* $u_s : [0, e^{-\mu s}T) \to \mathbb{B}$ *and* $p : [0,T) \to \mathbb{X}$ *defined as follows*

$$(u_s(t), p_s(t)) := \mathbf{d}(s)(u(e^{\mu s}t), p(e^{\mu s}t)) \quad \textit{with} \quad t \in [0, e^{-\mu s}T)$$

is also a mild (resp. strong) solution of the evolution equation (4) *and*

$$(u_s(t), p_s(t)) \overset{a.e}{\in} \mathcal{D}(F), \quad t \in (0, e^{-\mu s}T)$$

for any $s \in \mathbb{R}$.

Moreover, the claim remains true for classical solutions and the above inclusions hold everywhere on $(0,T)$ *and* $(0, e^{-\mu s}T)$*, respectively.*

Proof Since the operator F is $\mathbf{d}$ homogeneous then

$$\mathbf{d}(s)\mathcal{D}(F) \subset \mathcal{D}(F)$$

and

$$G(\mathbf{d}_1(s)u, \mathbf{d}_2(s)p) = e^{\mu s}(\mathbf{d}_1(s) \circ G)(u,p), \quad \forall (u,p) \in \mathcal{D}(F), \forall s \in \mathbb{R},$$

$$Q(\mathbf{d}_1(s)u, \mathbf{d}_2(s)p) = e^{\mu s}(\mathbf{d}_2(s) \circ Q)(u,p), \quad \forall (u,p) \in \mathcal{D}(F), \forall s \in \mathbb{R}.$$

Hence, if $(u(t), p(t)) \in \mathcal{D}(F)$ we conclude $(u_s(e^{-\mu^s}t), p_s(e^{-\mu^s}t)) \in \mathcal{D}(F)$.

1) *The case of mild solutions.*

Since the $\mathbf{d}_1$-homogeneous operator A generates a strongly continuous semigroup Φ, then according Lemma 2 we have

$$\Phi(t)\mathbf{d}_1(s) = \mathbf{d}_1(s)\Phi(e^{\mu s}t), \quad \forall t \geq 0, \quad \forall s \in \mathbb{R}.$$

It is well known [13, page 134] that $K\int_0^t \xi(s)ds = \int_0^t K\xi(s)ds$ for any bounded linear operator $K : \mathbb{B} \to \mathbb{B}$ and any Bochner integrable function ξ. Hence, using $G(u,p) \in L^1((0,T), \mathbb{B})$ we derive

$$\mathbf{d}_1(s)\int_0^t G(u(\tau), p(\tau))d\tau = \int_0^t \mathbf{d}_1(s)G(u(\tau), p(\tau))d\tau =$$

$$e^{-\mu s}\int_0^t G(\mathbf{d}_1(s)u(\tau), \mathbf{d}_2(s)p(\tau))d\tau = \int_0^{e^{-\mu s}t} G(\mathbf{d}_1(s)u(e^{\mu s}\tau), \mathbf{d}_2(s)p(e^{\mu s}\tau))d\tau$$

$$= \int_0^{e^{-\mu s}t} G(u_s(\tau), p_s(\tau))d\tau,$$

i.e. $G(u_s, p_s) \in L^1((0, e^{-\mu s}T), \mathbb{B})$. Similarly, we derive

$$\mathbf{d}_1(s)u(e^{\mu s}t) = (\mathbf{d}_1(s) \circ \Phi(e^{\mu s}t))u(0) + \mathbf{d}_1(s)\int_0^{e^{\mu s}t} (\Phi(e^{\mu s}t - \tau) \circ G)(u(\tau), p(\tau))\, d\tau$$

$$= (\Phi(t) \circ \mathbf{d}_1(s))u(0) + \int_0^{e^{\mu s}t} (\mathbf{d}_1(s) \circ \Phi(e^{\mu s}t - \tau) \circ G)(u(\tau), p(\tau))\, d\tau =$$

$$(\Phi(t) \circ \mathbf{d}_1(s))u(0) + e^{\mu s} \int_0^{t} (\mathbf{d}_1(s) \circ \Phi(e^{\mu s}(t - \sigma)) \circ G)(u(e^{\mu s}\sigma), p(e^{\mu s}\sigma))\, d\sigma =$$

$$(\Phi(t) \circ \mathbf{d}_1(s))u(0) + e^{\mu s} \int_0^{t} (\Phi(t - \sigma) \circ \mathbf{d}_1(s) \circ G)(u(e^{\mu s}\sigma), p(e^{\mu s}\sigma))\, d\sigma =$$

$$(\Phi(t) \circ \mathbf{d}_1(s))u(0) + \int_0^{t} (\Phi(t - \sigma) \circ G)(\mathbf{d}_1(s)u(e^{\mu s}\sigma), \mathbf{d}_2(s)p(e^{\mu s}\sigma))\, d\sigma,$$

where the linearity of $\Phi(t-\sigma)$ and a homogeneity of the operator G is utilized on the last step. Therefore, we have shown that

$$u_s(t) = \Phi(t)u_s(0) + \int_0^{t} \Phi(t - \sigma)g(u_s(\sigma), p_s(\sigma))\, d\sigma.$$

Finally, since (u, p) is a mild solution on $[0, T)$ then

$$Q(u(t), p(t)) \overset{a.e.}{=} \mathbf{0}, \quad \forall t \in [0, T)$$

and using the identity $\mathbf{d}_2(s)\,\mathbf{0} = \mathbf{0}$ we derive

$$Q(u_s(\tau), u_s(\tau)) = Q((\mathbf{d}_1(s)u)(\tau), (\mathbf{d}_2(s)p)(\tau)) =$$

$$e^{\mu s}(\mathbf{d}_2(s) \circ Q)(u(e^{\mu s}\tau), p(e^{\mu s}\tau)) \overset{a.e.}{=} \mathbf{0}, \qquad \tau \in [0, e^{-\mu s}T).$$

2) *The case of strong and classical solutions.* Let (u, p) be a strong solution. Since $u(t)$ is differentiable almost everywhere and $\mathbf{d}(s)$ is a linear bounded operator then for $\tau = e^{\mu s}t$ we have

$$\tfrac{du_s(t)}{dt} \overset{a.e.}{=} e^{\mu s}\mathbf{d}(s)\tfrac{du(\tau)}{d\tau} \overset{a.e.}{=} e^{\mu s}\mathbf{d}(s)(Au(\tau) + G(u(\tau), p(\tau))) \overset{a.e.}{=} Au_s(t) + G(u_s(t), p_s(t)).$$

In the case of the classical solution the latter identity holds everywhere. Moreover, if $u(t) \in \mathcal{D}(A)$ then due to $\mathbf{d}$-homogeneity of the operator A we always have $u_s(t) \in D(A)$, i.e. u_s is a classical solution.

4 Example: On necessary and sufficient conditions of global existence solutions of Navier-Stokes Equations in $\mathbb{R}^n$

The Navier-Stokes equations,

$$\begin{aligned} \partial_t u &= \nu \Delta u - (u \cdot \nabla)u - \nabla p, \\ \mathbf{0} &= \mathrm{div} u \end{aligned}$$

where u denotes the velocity of a fluid, p denotes the scalar pressure and $\nu > 0$ denotes viscosity of the fluid, is the the classical model of the flow of an incompressible viscous fluid. Without loss of generality ([7, page 4])we can assume $\nu = 1$.

Below for shortness we omit $\mathbb{R}^n$ in the notations of $\int_{\mathbb{R}^n}$, L^p, C_c^∞ and C_0^∞ spaces if the context is clear.

The classical idea of analysis is to prove, initially, an existence and regularity of weak solutions and next to show that any weak solution is smooth. For Navier-Stokes equations this scheme has been realized by Jean Leray in 1938 (see, e.g. [5] and the recent review [7]). We follow the same way having in mind **d**-homogeneity (dilation symmetry) of Navier-Stokes equations (see e.g. [1, formula (1.5)]), which will guarantee a global expansion of all local results.

4.1 Dilation Symmetry of Navier-Stokes equation in $\mathbb{R}^n$

Let V be a set of the so-called *weakly divergence free* velocity fields:

$$V := \left\{ u \in L^2 : \int u \cdot \nabla \phi = 0, \ \forall \phi \in C_0^\infty \right\}.$$

Due to this the Navier-Stokes equations can be equivalently represented in the form of the **d**-homogeneous implicit evolution equation (6) with the operators

$$A = \Delta, \quad G(u,p) = -\nabla p - (u \cdot \nabla)u, \quad Q(u,p) = \mathrm{div} u$$

having the domains $\mathcal{D}(Q) = V \times L^1_{loc}(\mathbb{R}^n, \mathbb{R})$,

$$\mathcal{D}(A) = \left\{ u \in L^2 : \exists f \in L^1_{loc} \text{ such that} \int u \cdot \Delta \phi = \int f \cdot \phi, \quad \forall \phi \in C_0^\infty \right\},$$

$$\mathcal{D}(G) = \left\{ (u,p) \in V \times L^1_{loc} : \exists f \in L^1_{loc}, \int u \cdot (u \cdot \nabla)\phi - p \mathrm{div} \phi + f \cdot \phi = 0, \ \forall \phi \in C_0^\infty \right\}.$$

Lemma 3 *Let $\mathbf{d}(s) : \tilde{\mathbb{X}} \to \tilde{\mathbb{X}}$ with $\tilde{\mathbb{X}} := L^2 \times L^1_{loc}(\mathbb{R}^n, \mathbb{R})$ be defined as*

$$\mathbf{d}(s)(u,p) = (\mathbf{d}_1(s)u, \mathbf{d}_2(s)p), \quad s \in \mathbb{R}, u \in L^2, p \in L^1_{loc}(\mathbb{R}^n, \mathbb{R}),$$

where $\mathbf{d}_1$ is a group of linear bounded invertible operators on L^2,

$$(\mathbf{d}_1(s)u)(x) = e^s u(e^s x), \quad u \in L^2, x \in \mathbb{R}^n,$$

and $\mathbf{d}_2$ is a group of linear invertible operators on $L^1_{loc}(\mathbb{R}^n, \mathbb{R})$ given by

$$(\mathbf{d}_2(s)p)(x) = e^{2s} p(e^s x) \quad p \in L^1_{loc}, x \in \mathbb{R}^n.$$

*Then the operator $F : \mathcal{D}(F) \subset \tilde{\mathbb{X}} \to \tilde{\mathbb{X}}$ with $\mathcal{D}(F) = \mathcal{D}(G) \cap \mathcal{D}(Q)$ and $F = (G, Q)$ is **d**-homogeneous of degree 2.*

Proof To complete the proof we must show that

$$G(\mathbf{d}_1(s)u, \mathbf{d}_2(s)p) = e^{2s}(\mathbf{d}_1(s) \circ G)(u,p), \quad \forall (u,p) \in \mathcal{D}(G), \forall s \in \mathbb{R},$$

$$Q(\mathbf{d}_1(s)u, \mathbf{d}_2(s)p) = e^{2s}(\mathbf{d}_2(s) \circ Q)(u,p), \quad \forall (u,p) \in \mathcal{D}(Q), \forall s \in \mathbb{R}.$$

and the domain $\mathcal{D}(F)$ is invariant with respect to $\mathbf{d}(s), \forall s \in \mathbb{R}$.

1) According to the definition of Q, the identity $Q(u,p) = 0 \in L^1_{loc}$ (in the weak sense) means

$$\int u \cdot \nabla\phi = 0, \quad \forall \phi \in C_0^\infty.$$

and using a change-of-variable theorem in Lebesque integral we derive

$$e^{ns} \int \|u\|_2^4 u(e^s x) \cdot (\nabla\phi)(e^s x) = 0, \quad \forall \phi \in C_0^\infty.$$

By Lemma 1 we have $\mathbf{d}_1(s)u \in L^2$ for any $s \in \mathbb{R}$. Taking into account $\nabla(\mathbf{d}_1(s)\phi)(x) = e^{2s}(\nabla\phi)(e^s x)$ we obtain

$$e^{(n-3)s} \int (\mathbf{d}_1(s)u) \cdot (\nabla \mathbf{d}_1(s)\phi) = 0, \quad \forall \phi \in C_0^\infty.$$

Since $\mathbf{d}_1(s)$ maps C_0^∞ onto C_0^∞ then $\mathbf{d}_1(s)V \subset V$ for any $s \in \mathbb{R}$ and

$$(Q \circ \mathbf{d}(s))(u,p) = \mathbf{0} = e^{2s}\mathbf{d}_2(s)\,\mathbf{0} = e^{2s}(\mathbf{d}_2(s) \circ Q)(u,p)$$

1 for all $s \in \mathbb{R}$ and, at least, on $\mathcal{D}(Q)$.

2) The identity $G(u,p) = f \in L^1_{loc}$ (in the weak sense) means

$$\int u \cdot (u \cdot \nabla)\phi - p\,\mathrm{div}\phi + f \cdot \phi = 0, \quad \forall \phi \in C_0^\infty.$$

If $p \in L^1_{loc}$ and $u \in L^2$ then from Lemma 1 we derive $\mathbf{d}_1(s)u \in L^2$ and $\mathbf{d}_2(s)p \in L^1_{loc}$.

Using the change-of-variable theorem in the Lebesgue integral we derive

$$e^{ns} \int u(e^s x) \cdot (u(e^s x) \cdot (\nabla\phi)(e^s x)) + p(e^s x)\,(\mathrm{div}\phi)(e^s x) + f(e^s x) \cdot \phi(e^s x)\, dx = 0.$$

or, equivalently,

$$\int (\mathbf{d}_1(s)u) \cdot (\mathbf{d}_1(s)u \cdot \nabla)\mathbf{d}_1(s)\phi + (\mathbf{d}_2(s)p)\,\mathrm{div}(\mathbf{d}_1(s)\phi) + e^{2s}(\mathbf{d}_1(s)f) \cdot (\mathbf{d}_1(s)\phi) = 0,$$

for all $\phi \in C_0^\infty$. Since $\mathbf{d}_1(s)$ maps C_0^∞ onto C_0^∞ then we conclude that $\mathbf{d}(s)\mathcal{D}(G) \subset \mathcal{D}(G)$ and the identity

$$G(\mathbf{d}_1(s)u, \mathbf{d}_2(s)p) = e^s \mathbf{d}_1(s)f = e^{2s}\mathbf{d}_1(s)G(\mathbf{d}_1(s)u, \mathbf{d}_2(s)p)$$

holds for any $s \in \mathbb{R}$ and, at least, on $\mathcal{D}(G)$.

Therefore, F is, indeed, $\mathbf{d}$-homogeneous of degree 2.

The $\mathbf{d}$-homogeneity of the Laplace operator Δ is studied in Example 3. Therefore, the Navier-Stokes equation satisfy all conditions required for application of Theorem 2. In other words, if it has mild, strong or classical solutions defined on $[0, +\infty)$ for small initial values then it has, respectively, mild, strong or classical solutions defined on $[0, +\infty)$ for large initial values.

4.2 Necessary and sufficient conditions for global existence of solutions in $\mathbb{R}^n$

Corollary 1 *Let $q_1, q_2 \in [1, \infty]$ and $r_1, r_2, \mu_1, \mu_2 \in \mathbb{R}$ and*

$$r_1(1 - \mu_1 - n/q_1) + r_2(1 - \mu_2 - n/q_2) \neq 0. \tag{8}$$

A mild (strong or classical) solution of the Navier-Stokes equations with arbitrary the initial data $u(0) = u_0 \in V \cap L^{q_1}_{\mu_1} \cap L^{q_2}_{\mu_2}$ exists on $[0, +\infty)$ if and only if there exist $\varepsilon > 0$ such that for any

$$u_0 \in \{u \in V \cap L^{q_1}_{\mu_1} \cap L^{q_2}_{\mu_2} : \|u\|^{r_1}_{q_1,\mu_1} \|u\|^{r_2}_{q_2,\mu_2} < \varepsilon\}$$

a mild (resp. strong or classical) solution (u, p) with the initial data $u(0) = u_0$ exists on $[0, +\infty)$.

Proof Let $\mathbf{d}_1$ be defined as in the proof of Lemma 3. Assume that for any $u_0 \in \{u \in V \cap L^{q_1}_{\mu_1} \cap L^{q_2}_{\mu_2} : \|u\|^{r_1}_{q_1,\mu_1} \|u\|^{r_2}_{q_2,\mu_2} < \varepsilon\}$ there exists a global in time mild (strong or classical) solution with $u(0) = u_0$ and let us show that for any $u_0 \in V \cap L^{q_1}_{\mu_1} \cap L^{q_2}_{\mu_2} : \|u_0\|^{r_1}_{q_1,\mu_1} \|u_0\|^{r_2}_{q_2,\mu_2} \geq \varepsilon$ the Navier-Stokes equations also have a mild(resp. strong or classical) solution on $[0, +\infty)$.

In the proof of Lemma 3 we have shown that $\mathbf{d}_1(s)u_0 \in V$ for any $s \in \mathbb{R}$, by Lemma 1 and $\mathbf{d}_1(s)u_0 \in L^{q_1}_{\mu_1}$ and for $\mathbf{d}_1(s)u_0 \in L^{q_2}_{\mu_1}$ any $s \in \mathbb{R}$. By Lemma 1 we also derive

$$\|\mathbf{d}_1(s)u_0\|_{q_1,\mu} = e^{s(1-\mu_1-n/q_1)} \|u_0\|_{q_1,\mu_1},$$

and

$$\|\mathbf{d}_1(s)u_0\|_{q_2,\mu} = e^{s(1-\mu_2-n/q_2)} \|u_0\|_{q_2,\mu_2}$$

and

$$\|\mathbf{d}_1(s)u_0\|^{r_1}_{q_1,\mu_1} \|\mathbf{d}_1(s)u_0\|^{r_1}_{q_2,\mu_2} = e^{s(r_1(1-\mu_1-\frac{n}{q_1})+r_2(1-\mu_2-\frac{n}{q_2}))} \|u_0\|^{r_1}_{q_1,\mu_1} \|u_0\|^{r_2}_{q_2,\mu_2}.$$

Since by assumption $r_1(1 - \mu_1 - n/q_1) + r_2(1 - \mu_2 - n/q_2) \neq 0$ then for any $u_0 \in V \cap L^{q_1}_{\mu_1} \cap L^{q_2}_{\mu_2}$ there exists $s_0 \in \mathbb{R}$ such that

$$\|\mathbf{d}_1(s)u_0\|^{r_1}_{q_1} \|\mathbf{d}_1(s)u_0\|^{r_2}_{q_2} < \varepsilon.$$

Hence, if a strong solution (u, p) with $u(0) = \mathbf{d}_1(s_0)u_0$ exists on $[0, +\infty)$ then by Theorem 2 the pair $(\tilde{u}, \tilde{p})$ given by

$$\tilde{u}(t, x) = e^{-s_0} u(e^{-2s_0}t, e^{-s_0}x), \tilde{p}(t, x) = e^{-2s_0} p(e^{-2s_0}t, e^{-s_0}x), t \in [0, +\infty), x \in \mathbb{R}^n$$

is also a strong solution of the Navier-Stokes equation. Since $u(0) = \mathbf{d}_1(s_0)u_0$ means that

$$u(0, x) = e^{s_0} u_0(e^{s_0}x), \quad x \in \mathbb{R}^n$$

then

$$\tilde{u}(0, x) = e^{-s_0} u(0, e^{-s_0}x) = u_0(x),$$

i.e. $\tilde{u}(0) = u_0 \in V \cap L^{q_1}_{\mu_1} \cap L^{q_2}_{\mu_2}$ and the proof is complete.

Notice that taking $\mu_1 = \mu_2 = 0$ we derive the usual L^{q_1} and L^{q_2} spaces in the latter corollary.

Let us consider also the well-known weak form (see e.g. [5] and the recent review [7]) of Navier-Stokes equations

$$\int u(0)\cdot\phi(0) + \int_0^T\!\!\int u\cdot(\phi_t + \Delta\phi) + p\,\mathrm{div}\phi = \int_0^T\!\!\int u\cdot(u\cdot\nabla)\phi, \quad \forall\phi\in C_0^\infty \tag{9}$$

with $u(t)\in V$ for $t\in(0,T)$.

Definition 5 [7, Definition 3.7] A pair (u,p) is said to be a solution of the Navier-Stokes equations on $[0,T)$ if $p\in L^1_{loc}((0,T)\times\mathbb{R}^n,\mathbb{R}))$, $u(t)\in V$, $t\in(0,T)$

$$u\in C([0,T),L^2)\cap C((0,T),L^\infty),$$

$\|u(t)\|_\infty$ is bounded as $t\to 0^+$ and (9) is satisfied.

According to [7] the considered solution can be treated as a mild solutions obtained using the so-called Oseen kernel.

Let us mention the following properties of the solutions introduced by Definition 5 (proven for $n=3$):

- *Global-in-time existence for small initial data* [7, Corollary 3.13 and Lemma 3.10]
  There exist $\varepsilon>0$ and $C>0$ such that for any
  $$u_0\in\{u\in V : \|u\|_2^2\|u\|_\infty<\varepsilon\} \tag{10}$$
  or
  $$u_0\in\{u\in V : \|u\|_2\|\nabla u\|_2<\varepsilon\} \tag{11}$$
  or
  $$u_0\in\{u\in V : \|u\|_2^{2(q-3)}\|u\|_q^q<\varepsilon\}, q>3 \tag{12}$$
  a strong solution (u,p) with the initial data $u(0)=u_0$ exists on $[0,+\infty)$ and $\|u(t)\|_\infty\le C\|u_0\|_\infty$.
- *Uniqueness of strong solutions* [7, Theorem 3.9]
  For any $u_0\in V\cap L^\infty$ a strong solution with $u(0)=u_0$ is unique.
- *Smoothness* [7, Corollary 3.3].
  If (u,p) is a strong solution of the Navier-Stokes equation then
  $$\partial_t^k\nabla^m u, \partial_t^k\nabla^m p\in C((0,T),L^2)\cap C((0,T),L^\infty), \quad \forall m,k\ge 0$$
  and, in particular, $u,p\in C^\infty(\mathbb{R}^3\times(0,T))$ constitute a classical solution of the Navier-Stokes equations on $(0,T)\times\mathbb{R}^3$.

To expand globally the regularity of the Navier-Stokes equation by means of dilation symmetry, the condition (9) must hold.

None of conditions (10), (11), (12) satisfy Corollary 1.

For instance, for $n \neq 3$ the conditions like (10), (11), (12) would be appropriate for application of Corollary 1.

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