Eternal Domination in Grids

Fionn Mc Inerney, Nicolas Nisse, Stéphane Pérennes

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Fionn Mc Inerney[1], Nicolas Nisse[1], and Stéphane Pérennes[1]

Université Côte d'Azur, Inria, CNRS, I3S, France
Emails: {fionn.mc-inerney, nicolas.nisse, stephane.perennes}@inria.fr

Abstract. In the eternal domination game played on graphs, an attacker attacks a vertex at each turn and a team of guards must move a guard to the attacked vertex to defend it. The guards may only move to adjacent vertices on their turn. The goal is to determine the eternal domination number γ_{all}^{∞} of a graph which is the minimum number of guards required to defend against an infinite sequence of attacks.
This paper continues the study of the eternal domination game on strong grids $P_n \boxtimes P_m$. Cartesian grids $P_n \square P_m$ have been vastly studied with tight bounds existing for small grids such as $k \times n$ grids for $k \in \{2,3,4,5\}$. It was recently proven that $\gamma_{all}^{\infty}(P_n \square P_m) = \gamma(P_n \square P_m) + O(n+m)$ where $\gamma(P_n \square P_m)$ is the domination number of $P_n \square P_m$ which lower bounds the eternal domination number [Lamprou et al., CIAC 2017]. We prove that, for all $n, m \in \mathbb{N}^*$ such that $m \geq n$, $\lfloor \frac{n}{3} \rfloor \lfloor \frac{m}{3} \rfloor + \Omega(n+m) = \gamma_{all}^{\infty}(P_n \boxtimes P_m) = \lceil \frac{n}{3} \rceil \lceil \frac{m}{3} \rceil + O(m\sqrt{n})$ (note that $\lceil \frac{n}{3} \rceil \lceil \frac{m}{3} \rceil$ is the domination number of $P_n \boxtimes P_m$). Our technique may be applied to other "grid-like" graphs.

Keywords: Eternal Domination, Combinatorial Games, Graphs, Grids.

1 Introduction

The origins of the eternal domination game date back to the 1990's where the military strategy of Emperor Constantine for defending the Roman Empire was studied in a mathematical setting [1,23,21,22]. Roughly, a limited number of armies must be placed in such a way that an army can always move to defend against an attack by invaders.

Precisely, eternal domination is a 2-player game on graphs introduced in [6] and defined as follows. Initially, k guards are placed on some vertices of a graph $G = (V, E)$. Turn-by-turn, an *attacker* first chooses a vertex $v \in V$ to attack. Then, if no guard is occupying v or a vertex adjacent to v, then the attacker wins. Otherwise, one guard must move along an edge to occupy v if it is not already occupied, and the next turn starts. If the attacker never wins whatever be its sequence of attacks, then the guards win. So, clearly, there is no point in the attacker attacking an occupied vertex. The aim in eternal domination is to minimize the number of guards that must be used in order to win. Hence, let

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Due to a lack of space, several proofs have been omitted and can be found in [14].

$\gamma^\infty(G)$ be the minimum integer k such that there exists a strategy allowing k guards to win, regardless of what the attacker does [6].

In this paper, we consider the "all guards move" variant of eternal domination, proposed in [11], where, at their turn, every guard may move to a neighbour of its position (still satisfying that the attacked vertex is occupied by a guard at the end of the turn). Let $\gamma^\infty_{all}(G)$ be the minimum number of guards for which a winning strategy exists in this setting. By definition, $\gamma(G) \leq \gamma^\infty_{all}(G) \leq \gamma^\infty(G)$ for any graph G where $\gamma(G)$ is the minimum size of a dominating set in G[1].

Variants of the eternal domination game also differ in the fact that one or more guards may simultaneously occupy the same vertex. In the initial variant where a single guard is allowed to move each turn, this is not a strong constraint [6]. That is, imposing that a vertex cannot be occupied by more than one guard does not increase the number of guards required to win. In the case when multiple guards may move each turn, there are some graphs where this constraint increases the number of guards [18]. Let $\gamma^{*\infty}_{all}(G)$ be the minimum number of guards to win in G, moving several guards per turn, and in such a way that a vertex cannot be occupied by several guards.

Previous works mainly studied lower and upper bounds on $\gamma^\infty(G)$ and $\gamma^\infty_{all}(G)$ in function of other parameters of G, such as its domination number $\gamma(G)$ [11], independence number $\alpha(G)$[2] [6,11], and clique cover number $\theta(G)$[3] [6]. Notably, these results give the following inequalities $\gamma(G) \leq \gamma^\infty_{all}(G) \leq \alpha(G) \leq \gamma^\infty(G) \leq \theta(G)$ [6]. Particular graph classes have also been studied such as paths and cycles [11], trees [16], and proper interval graphs [5]. In particular, the class of grids and graph products has been widely studied [4,10,12,18,19,20,24].

In this paper, we focus on the class of *strong grids* SG and provide an almost tight asymptotical value for $\gamma^\infty_{all}(SG)$. Our result also holds for $\gamma^{*\infty}_{all}(SG)$. Our main result is a new technique to prove upper bounds that we believe can be generalized to many other "grid-like" graphs.

1.1 Related Work

The "all guards move" variant of eternal domination was shown to be NP-complete in Hamiltonian split graphs [3]. Note that it is not known whether the problem of deciding γ^∞_{all} is in NP in general graphs. Moreover, given a graph G and an integer k as inputs, the problem of deciding if $\gamma^\infty(G) \leq k$ is coNP-hard [2].

Several graph classes have been studied. For a path P_n on n vertices, $\gamma^\infty_{all}(P_n) = \lceil \frac{n}{2} \rceil$ and for a cycle C_n on n vertices, $\gamma^\infty_{all}(C_n) = \lceil \frac{n}{3} \rceil$ [11]. In [16], the authors present a linear-time algorithm to determine $\gamma^\infty_{all}(T)$ for all trees T. It was proven that if G is a proper interval graph, then $\gamma^\infty_{all}(G) = \alpha(G)$ [5]. In the past few years, a lot of effort was put in by several authors to determine the eternal domination number of cartesian grids, $\gamma^\infty_{all}(P_n \Box P_m)$. Exact values were determined

[1] $D \subseteq V$ is a dominating set of G if every vertex is in D or adjacent to a vertex in D.
[2] $\alpha(G)$ is the maximum size of an independent set in G.
[3] $\theta(G)$ is the minimum number of complete subgraphs of G whose union covers $V(G)$.

for $2 \times n$ cartesian grids [12] and $4 \times n$ cartesian grids [4]. Asymptotical tight bounds for $3 \times n$ cartesian grids were obtained in [10] and improved in [20]. Finally, bounds for $5 \times n$ cartesian grids were given in [24]. The best known lower bound for $\gamma_{all}^{\infty}(P_n \Box P_m)$ for values of n and m large enough, is the domination number with the latter only being recently determined in [13]. The best known upper bound for $\gamma_{all}^{\infty}(P_n \Box P_m)$ was determined recently in [19], where it was shown that $\gamma_{all}^{\infty}(P_n \Box P_m) = \gamma(P_n \Box P_m) + O(n+m)$. Note that all the results discussed in this subsection also hold for $\gamma_{all}^{*\infty}$.

There are also many other variants of the game that exist and here we give a brief description and references for some of them. Recently, the eternal domination game and a variant have been studied in digraphs, including orientations of grids and toroidal strong grids [2]. Eternal total domination was studied in [17], where a total dominating set must be maintained by the guards each turn. The eviction model of eternal domination was studied in [15], where a vertex containing a guard is attacked each turn, which forces the guard to move to an adjacent empty vertex with the condition that the guards must maintain a dominating set each turn. The authors of the current paper studied a generalization of eternal domination, called the Spy game, in [7,8]. For more information and results on the original eternal domination game and its variants, see the survey [18].

1.2 Our results

The main result of this paper is that, for all $n, m \in \mathbb{N}^*$ such that $m \geq n$,

$$\left\lfloor \frac{n}{3} \right\rfloor \left\lfloor \frac{m}{3} \right\rfloor + \Omega(n+m) = \gamma_{all}^{\infty}(P_n \boxtimes P_m) = \left\lceil \frac{n}{3} \right\rceil \left\lceil \frac{m}{3} \right\rceil + O(m\sqrt{n}).$$

In [14], we show that this result also holds in the case when at most one guard may occupy each vertex.

Note that, in toroidal strong grids $C_n \boxtimes C_m$, the problem becomes trivial and $\gamma_{all}^{\infty}(C_n \boxtimes C_m) = \lceil \frac{n}{3} \rceil \lceil \frac{m}{3} \rceil$ for any n and m. However, in strong grids, border-effects make the problem much harder. The upper bound is proven by defining a set of specific configurations that each dominate the grid and are "invariant" to the movements required by the defined strategy to defend against attacks. That is, the attacks are separated into three types of attacks: horizontal, vertical, and diagonal, and the strategy defined gives the movement of the guards based on the type of attack. It is shown that in each of the three cases of attacks, the guards are able to move from their current configuration to another configuration in the set of configurations (so, it does not matter which configuration was the initial one and which new configuration the guards reach after their moves) and hence, the guards can defend against an infinite sequence of attacks.

The lower bound is proven by showing that, in any winning configuration in eternal domination, there are some vertices that are dominated by more than one guard, and/or some guards dominate at most 6 vertices. By double counting, this leads to the necessity of having $\Omega(n+m)$ extra guards compared to the classical domination (when $n \equiv 0 \pmod 3$ and $m \equiv 0 \pmod 3$).

2 Preliminaries

We use classic graph-theory terminology [9]. Notably, given a graph $G = (V, E)$ and $S \subseteq V$, let $N(S) = \{v \in V \setminus S \mid \exists w \in S, \{v, w\} \in E\}$ denote the set of neighbours (not in S) of the vertices in S and let $N[S] = N(S) \cup S$ denote the *closed neighbourhood* of S. For $v \in V$, let $N(v) = N(\{v\})$ and $N[v] = N(v) \cup \{v\}$.

Let $n, m \in \mathbb{N}^*$ be such that $m \geq n$ and let the $n \times m$ strong grid, denoted by $SG_{n \times m}$, be the strong product $P_n \boxtimes P_m$ of an n-node path with an m-node path. Precisely, $SG_{n \times m}$ is the graph with the set of vertices $\{(i, j) \mid 1 \leq i \leq n, 1 \leq j \leq m\}$, and two vertices (i_1, j_1) and (i_2, j_2) are adjacent if and only if $\max\{|i_2 - i_1|, |j_2 - j_1|\} = 1$. That is, the vertices are identified by their Cartesian coordinates, *i.e.*, the vertex (i, j) is the vertex in *row* i and *column* j. The vertex $(1, 1)$ is in the *bottom-left corner* and the vertex (n, m) is in the *top-right corner*.

Definition 1. *The set of border vertices of* $SG_{n \times m}$ *is the set*

$$B = \bigcup_{1 \leq i \leq n, 1 \leq j \leq m} \{(1, j), (n, j), (i, 1), (i, m)\} \text{ of vertices of degree} \leq 5.$$

The set of pre-border vertices of $SG_{n \times m}$ *is the set* $PB = N(B)$.

Equivalently, PB is the set of border vertices of the strong grid induced by $V(SG_{n \times m}) \setminus B$.

We consider the turn-by-turn 2-player game in graphs called eternal domination. Each *turn*, each vertex of a graph $G = (V, E)$ may be occupied by one or more guards. Let $k \in \mathbb{N}^*$ be the total number of guards. The positions of the guards are formally defined by a multi-set C of vertices, called a *configuration*, where the number of occurrences of a vertex $v \in C$ corresponds to the number of guards at $v \in V$ and $k = |C|$. Each turn, given a current configuration $C = \{v_i \mid 1 \leq i \leq k\}$ of k guards, Player 1, the *attacker*, *attacks* a vertex $v \in V$. Then, Player 2 (the *defender*) may move each of its *guards* to a neighbour of their current position, thereby, achieving a new configuration $C' = \{w_i \mid 1 \leq i \leq k\}$ such that $w_i \in N[v_i]$ for every $1 \leq i \leq k$ (we then say that C' is *compatible* with C, which is clearly a symmetric relation). If $v \notin C'$, then the attacker *wins*, otherwise, the game goes on with a next turn (given the new configuration C').

A *strategy* for k guards is defined by an initial configuration of size k and by a function that, for every current configuration C and every attacked vertex $v \in V$, specifies a new configuration C' compatible with C. A strategy $\mathcal{S}$ for the guards is *winning* if, for every sequence of attacked vertices, the attacker never wins when the defender plays according to $\mathcal{S}$.

Our main contribution is the design of a winning strategy for $\gamma(SG_{n \times m}) + o(\gamma(SG_{n \times m}))$ guards in $SG_{n \times m}$, where $\gamma(SG_{n \times m}) = \lceil \frac{n}{3} \rceil \lceil \frac{m}{3} \rceil$ is the *domination number* of $SG_{n \times m}$. The next lemma is key for this winning strategy.

In our strategy, it will often be useful to move a guard from a node $u \in PB$ of the pre-border to another node $v \in PB$ such that u and v are not necessarily adjacent. For this purpose, the idea is to place a sufficient number of guards on the vertices of the border such that a "flow" of the guards on the border vertices will simulate the move of the guard from u to v in one turn.

Precisely, given a configuration C and $u, v \in V(SG_{n\times m})$ with $u \in C$, a guard is said to *jump* from u to v if the configuration $(C \setminus \{u\}) \cup \{v\}$ is compatible with C, *i.e.*, the guards, in one turn, can move to achieve the same configuration as C except that there is one guard less on u and one guard more on v. More generally, given $U \subset C$ and $W \subset V(SG_{n\times m})$, a set of guards is said to *jump* from U to W if the configuration $(C \setminus U) \cup W$ is compatible with the configuration C.

Lemma 1. *Let $\alpha, \beta \in \mathbb{N}^*$ such that $\beta \leq \alpha$. Let $U, W \subseteq PB$ be two subsets of pre-border vertices such that $|U| = |W| = \beta$. In any configuration C such that $U \subseteq C$ and C contains at least α occurrences of each vertex in B (i.e., each border vertex is occupied by at least α guards), then β guards may "jump" from U to W in one turn. Moreover, only guards in $U \cup B$ move.*

Proof. The proof is by induction on β. The inductive hypothesis is that if each vertex in B contains α guards, then $\beta \leq \alpha$ guards may "jump" from U to W in one turn such that at most β guards move off of each vertex $w \in B$ in this turn. For the base case, let us assume that $U = \{u\}$ and $W = \{w\}$. Let us show how 1 guard can "jump" from u to w in one turn. If $u = w$, the result trivially holds, so let $u \neq w$. Let $u' \in B$ (resp., w') be a neighbour of u (of w) that shares one coordinate with u (with w). Let $Q = (u' = v_0, v_1, \ldots, v_\ell = w')$ be a path from u' to w' induced by the border vertices. In one turn, a guard at u moves to u', for every $0 \leq i < \ell$, a guard at v_i moves to v_{i+1}, and a guard at v_ℓ moves to w.

Now, assume the inductive hypothesis holds for $\beta \geq 1$. If $\beta = \alpha$, we are done, so assume $\beta < \alpha$. Let $|U| = |W| = \beta + 1 \leq \alpha$ and let $u \in U$ and $w \in W$. By the inductive hypothesis, β guards may jump from $U \setminus \{u\}$ to $W \setminus \{w\}$ in one turn in such a way that, for every vertex $b \in B$, at most β guards move off of b during this turn. Since every vertex of B is occupied by $\alpha > \beta$ guards, at least one guard is unused on every vertex of B. Thus, it possible to use the same strategy as in the base case to make one guard jump from u to w on this same turn. $\square$

3 Upper bound strategy

This section is devoted to proving that for all $n, m \in \mathbb{N}^*$ such that $m \geq n$, $\gamma^{\infty}_{all}(SG_{n\times m}) = \lceil \frac{n}{3} \rceil \lceil \frac{m}{3} \rceil + O(m\sqrt{n})$.

Before considering the general case, let us first assume that $n-2 \equiv 0 \pmod 3$ and that there exists $k \in \mathbb{N}^*$ such that $k - 2 \equiv 0 \pmod 3$, and $m \equiv 0 \pmod k$. The $n \times m$ strong grid will be partitioned into *blocks* which are subgrids of size $n \times k$. More precisely, for all $1 \leq q \leq \frac{m}{k}$, the q^{th} block contains columns $(q-1)k + 1$ through qk of $SG_{n\times m}$.

3.1 Horizontal attacks

In this section, we only consider one block of $SG_{n\times m}$. W.l.o.g., let us consider the block $SG_{n\times k}$ induced by $\{(i, j) \mid 1 \leq i \leq n, 1 \leq j \leq k\}$. Let us first define a family of parameterized configurations for this block.

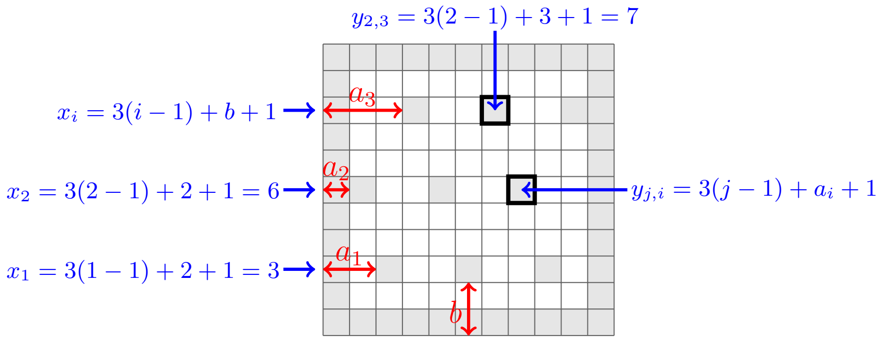

Fig. 1: $P_{11} \boxtimes P_{11}$ where the squares are vertices and two squares sharing a side and/or a corner are adjacent. Example of a configuration $C_H(X)$ where $X = (b = 2, a_1 = 2, a_2 = 1, a_3 = a_{\frac{n-2}{3}} = 3)$, there is one guard at each square in gray, and the white squares contain no guards.

Let $\mathcal{X} = \{(b, a_1, \ldots, a_{\frac{n-2}{3}}) \mid b \in \{1,2,3\}, a_i \in \{1,2,3\}$ for $i = 1, \ldots, \frac{n-2}{3}\}$.

Given $X = (b, a_1, \ldots, a_{\frac{n-2}{3}}) \in \mathcal{X}$, let $x_i(X) = 3(i-1)+b+1$, and $y_{j,i}(X) = 3(j-1)+a_i+1$ for every $1 \leq i \leq \frac{n-2}{3}$ and $1 \leq j \leq \frac{k-2}{3}$. We set $x_i = x_i(X)$ and $y_{j,i} = y_{j,i}(X)$ when there is no ambiguity. Intuitively, b will represent the *vertical shift* of the positions of the guards in configuration X. Similarly, for every $1 \leq i \leq \frac{n-2}{3}$, a_i represents the *horizontal shift* of the positions of the guards in row $x_i(X)$ in configuration X (see Figure 1).

Horizontal Configurations. Let us define the set $\mathcal{C}_H$ of configurations as follows. For every $X \in \mathcal{X}$, let $C_H(X) = B \cup \{(x_i(X), y_{j,i}(X)) \mid 1 \leq i \leq \frac{n-2}{3}, 1 \leq j \leq \frac{k-2}{3}\}$ be the configuration where there is one guard at every vertex of B and one guard at each vertex $(x_i(X), y_{j,i}(X)) = (3(i-1)+b+1, 3(j-1)+a_i+1)$ for every $1 \leq i \leq \frac{n-2}{3}$ and $1 \leq j \leq \frac{k-2}{3}$. See an example in Figure 1. Then,

$$\mathcal{C}_H = \{C_H(X) \mid X \in \mathcal{X}\}.$$

Note that $|C_H(X)| = \frac{(n-2)(k-2)}{9} + 2(n+k) - 4 = \kappa_H$ for every $X \in \mathcal{X}$. That is, any horizontal configuration uses κ_H guards.

Lemma 2. *Every configuration $C_H(X) \in \mathcal{C}_H$ is a dominating set of $SG_{n\times k}$.*

In this subsection, we limit the power of the attacker by allowing it to attack only some predefined vertices (this kind of attack will be referred to as a *horizontal attack*). For every configuration $C_H(X) \in \mathcal{C}_H$ and for any such attack, we show that the guards may be moved (in one turn) in such a way to defend the attacked vertex and reach a new configuration in $\mathcal{C}_H$.

Horizontal Attacks. Let $X = (b, a_1, \ldots, a_{\frac{n-2}{3}}) \in \mathcal{X}$ and $C_H(X) \in \mathcal{C}_H$. Let

$$A_H(X) = \{(x_i, y) \mid 1 \leq i \leq \frac{n-2}{3}, 1 \leq y \leq k\}.$$

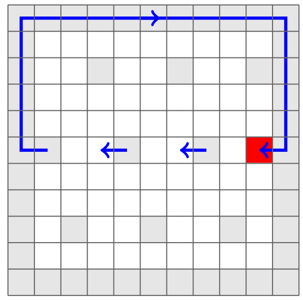

Fig. 2: Example of a horizontal attack at the red square. The arrows (in blue) show the movements of the guards in response to the attack.

A *horizontal attack with respect to* X is an attack at any vertex in $A_H(X)$, *i.e.*, an attack at any vertex of a row where some non-border vertex is occupied by a guard. Note that, for every vertex $v \in A_H(X)$, either v is occupied by a guard or there is a guard on the vertex to the left or to the right of v.

The next lemma proves that, from any horizontal configuration and against any horizontal attack (with respect to this current configuration), there is a strategy for the guards that defends against this attack and leads to a (new) horizontal configuration. Therefore, starting from any horizontal configuration, there is a strategy of the guards that wins against any sequence of horizontal attacks. See Figure 2 for a schematic representation of how the guards react to one of these attacks.

Lemma 3. *For any* $X \in \mathcal{X}$ *and any* $v \in A_H(X)$*, there exists* $X' \in \mathcal{X}$ *such that* $v \in C_H(X')$ *and configurations* $C_H(X)$ *and* $C_H(X')$ *are compatible. That is, in one turn, the guards may move from* $C_H(X)$ *to* $C_H(X')$ *and defend against an attack at* v*.*

3.2 Vertical attacks

In this section, we consider the entire strong grid $SG_{n\times m}$ partitioned into $\frac{m}{k}$ blocks $SG_{n\times k}$ with block q, for $1 \leq q \leq \frac{m}{k}$, being induced by $\{(i, j + (q-1)k) \mid 1 \leq i \leq n, 1 \leq j \leq k\}$. We first define a family of parameterized configurations for this graph. A configuration for the whole grid will be defined as the union of some configurations for each of the q blocks. Formally, for every $1 \leq q \leq \frac{m}{k}$, let

$$\mathcal{X}^q = \{(b^q, a_1^q, \ldots, a_{\frac{n-2}{3}}^q) \mid b^q \in \{1,2,3\}, a_i^q \in \{1,2,3\} \text{ for } i = 1, \ldots, \tfrac{n-2}{3} \text{ and } q = 1, \ldots, \tfrac{m}{k}\}.$$

Given $X^q = (b^q, a_1^q, \ldots, a_{\frac{n-2}{3}}^q) \in \mathcal{X}^q$, let $x_i^q(X^q) = 3(i-1) + b^q + 1$, and $y_{j,i}^q(X^q) = (q-1)k + 3(j-1) + a_i^q + 1$ for every $1 \leq i \leq \frac{n-2}{3}$, $1 \leq j \leq \frac{k-2}{3}$, and $1 \leq q \leq \frac{m}{k}$. We set $x_i^q = x_i^q(X^q)$ and $y_{j,i}^q = y_{j,i}^q(X^q)$ when there is no ambiguity.

That is, intuitively, b^q will represent the *vertical shift* of the positions of the guards in configuration X^q in the q^{th} block. Similarly, for every $1 \leq i \leq \frac{n-2}{3}$, a_i^q represents the *horizontal shift* of the positions of the guards in row $x_i(X)$ in configuration X^q in the q^{th} block.

Finally, let $\mathcal{Y} = \{(X^1, \ldots, X^{\frac{m}{k}}) \mid X^q \in \mathcal{X}^q \text{ for } q = 1, \ldots, \frac{m}{k}\}$.

Vertical Configurations. In order to properly define the following set of configurations, the following notation is used. For a set S of vertices in a configuration $\mathcal{C}$ and an integer $x > 0$, let $S^{[x]}$ be the multi-set of vertices that consists of x copies of each vertex in S. Intuitively, $S^{[x]}$ will be used to define a configuration where x guards occupy each vertex of S. Let us now define the set $\mathcal{C}_V$ of configurations as follows.

For every $Y = (X^1, \ldots, X^{\frac{m}{k}}) \in \mathcal{Y}$, let $C_V(Y) = B^{[\frac{k-2}{3}]} \cup \bigcup_{q=1}^{\frac{m}{k}} C_H(X^q)$ be the configuration obtained as follows. First, for any $1 \leq q \leq \frac{m}{k}$, guards are placed in configuration $C_H(X^q)$ in the q^{th} block. Then, $\frac{k-2}{3}$ guards are added to every border vertex. Note that overall, there are $\frac{k-2}{3} + 1$ guards at each vertex of B. See an example in Figure 3. Then, $\mathcal{C}_V = \{C_V(Y) \mid Y \in \mathcal{Y}\}$.

Note that $|C_V(Y)| = \frac{m}{k}\kappa_H + 2(\frac{k-2}{3})(n+m-2) = \kappa_V$ for every $Y \in \mathcal{Y}$. That is, any vertical configuration uses κ_V guards.

Lemma 4. *Every configuration $C_V(Y) \in \mathcal{C}_V$ is a dominating set of $SG_{n\times m}$.*

In this subsection, we limit the power of the attacker by allowing it to attack only some *vertical* vertices. For every configuration $C_V(X) \in \mathcal{C}_V$ and for any such attack, we show that the guards may be moved (in one turn) in such a way to defend the attacked vertex and reach a new configuration in $\mathcal{C}_V$.

Vertical Attacks. Let $Y = (X^1, \ldots, X^{\frac{m}{k}}) \in \mathcal{Y}$ and $C_V(Y) \in \mathcal{C}_V$. Let

$$\begin{aligned} A_V(Y) = &\{(x_i^q - 1, y_{j,i}^q), (x_i^q + 1, y_{j,i}^q) \mid 1 \leq i \leq \frac{n-2}{3}, 1 \leq j \leq \frac{k-2}{3}, 1 \leq q \leq \frac{m}{k}\} \\ &\cup \{(2, y_{j,n-1}^q) \mid 1 \leq j \leq \frac{k-2}{3}, 1 \leq q \leq \frac{m}{k} \text{ and } b^q = 3\} \\ &\cup \{(n-1, y_{j,2}^q) \mid 1 \leq j \leq \frac{k-2}{3}, 1 \leq q \leq \frac{m}{k} \text{ and } b^q = 1\} \end{aligned}$$

A *vertical attack with respect to* Y is an attack at any vertex in $A_V(Y)$, *i.e.*, an attack at any non-border vertex above or below a guard not on a border vertex. Moreover, if the vertical shift b^q of the q^{th} block equals 3, then some vertices of the second row of the q^{th} block may also be attacked (depending on the horizontal shift a_{n-1}^q). Finally, if the vertical shift b^q of the q^{th} block equals 1, then some vertices of the $(n-1)^{th}$ row of the q^{th} block may also be attacked (depending on the horizontal shift a_2^q).

Note that $A_V(Y) \cap C_V(Y) = \emptyset$, and $A_V(Y) \cap A_H(X^q) = \emptyset$ for any $X^q \in Y$, *i.e.*, any vertical attack with respect to Y is not a horizontal attack with respect to $X^q \in Y$ and vice versa.

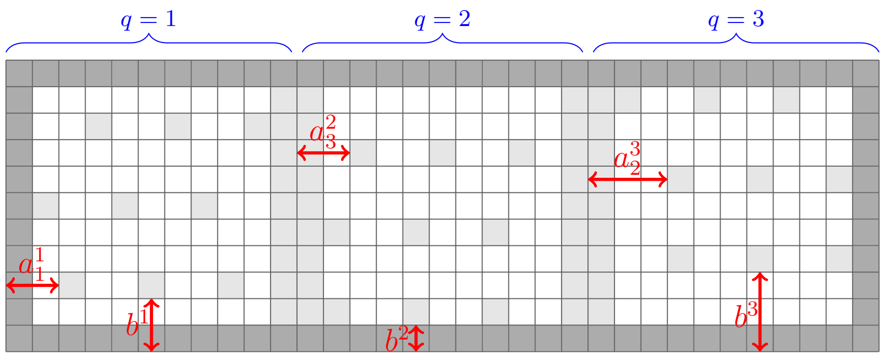

Fig. 3: A configuration $C_V(Y)$ where $k = 11$, $Y = (X^1, X^2, X^3)$, $X^1 = (2, 2, 1, 3)$, $X^2 = (1, 1, 1, 2)$, $X^3 = (3, 3, 3, 1)$, there are $(k - 2)/3 + 1 = 4$ guards at each square in dark gray, 1 guard at each square in light gray, and the white squares contain no guards.

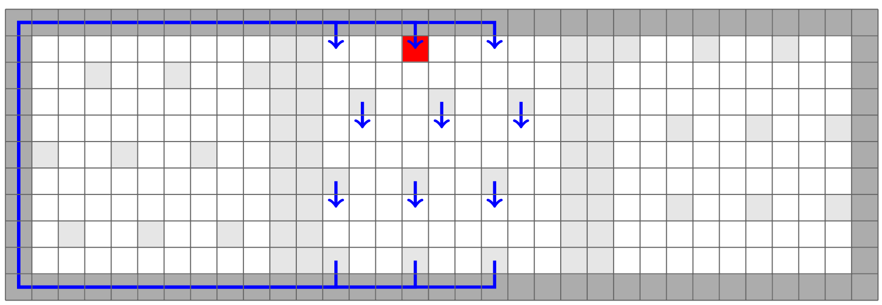

Fig. 4: Example of a vertical attack at the red square and how the guards react.

The next lemma proves that, from any vertical configuration and against any vertical attack (with respect to this current configuration), there is a strategy for the guards that defends against this attack and leads to a (new) vertical configuration. Therefore, starting from any vertical configuration, there is a strategy of the guards that wins against any sequence of vertical attacks. See Figure 4.

Lemma 5. *For any $Y \in \mathcal{Y}$ and any $v \in A_V(Y)$, there exists $Y' \in \mathcal{Y}$ such that $v \in C_V(Y')$ and configurations $C_V(Y)$ and $C_V(Y')$ are compatible. That is, in one turn, the guards may move from $C_V(Y)$ to $C_V(Y')$ and defend against an attack at v.*

3.3 Diagonal attacks

The same $n \times m$ strong grid $SG_{n\times m}$, notations, and configurations for the guards used in subsection 3.2 will be used here. In this subsection, we limit the power

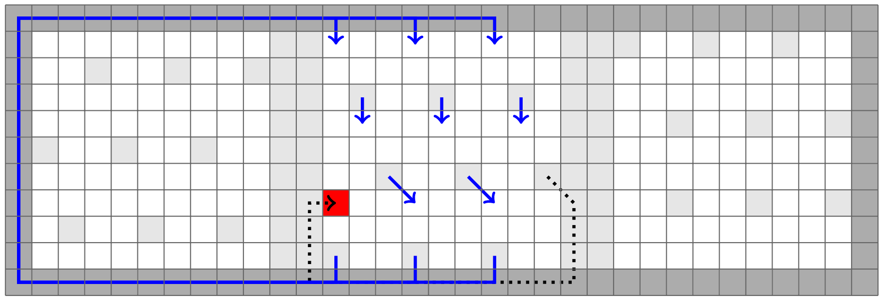

Fig. 5: Example of a diagonal attack at the red square. The dotted arrow in black is to differentiate between the different guards jumping.

of the attacker by allowing it to attack only some *diagonal* vertices. For every configuration $C_V(X) \in \mathcal{C}_V$ and for any such attack, we show that the guards may be moved (in one turn) in such a way to defend the attacked vertex and reach a new configuration in $\mathcal{C}_V$.

Diagonal Attacks. Let $Y = (X^1, \ldots, X^{\frac{m}{k}}) \in \mathcal{Y}$ and $C_V(Y) \in \mathcal{C}_V$. Let $A_D(Y) = V(SG_{n\times m}) \setminus (B \cup A_H(Y) \cup A_V(Y))$. That is, $A_D(Y)$ covers all possible attacks that are neither horizontal nor vertical.

A *diagonal attack with respect to* Y is an attack at any vertex in $A_D(Y)$. Note that, for every vertex $v \in A_D(Y)$, there is a guard on a vertex adjacent to v and neither in the same column nor in the same row as v.

The next lemma proves that, from any vertical configuration and against any diagonal attack (with respect to this current configuration), there is a strategy for the guards that defends against this attack and leads to a (new) vertical configuration. Therefore, starting from any vertical configuration, there is a strategy of the guards that wins against any sequence of diagonal attacks. See Figure 5.

Lemma 6. *For any* $Y \in \mathcal{Y}$ *and any* $v \in A_D(Y)$*, there exists* $Y' \in \mathcal{Y}$ *such that* $v \in C_V(Y')$ *and configurations* $C_V(Y)$ *and* $C_V(Y')$ *are compatible. That is, in one turn, the guards may move from* $C_V(Y)$ *to* $C_V(Y')$ *and defend against an attack at* v*.*

3.4 Upper Bound in Strong Grids

Note that, for any $Y = (X^1, \ldots, X^{\frac{m}{k}}) \in \mathcal{Y}$, $A_D(Y) \cup A_V(Y) \cup \bigcup_{q=1}^{\frac{m}{k}} A_H(X^q) \cup B = V(SG_{n\times m})$. That is, any attack by the attacker in $SG_{n\times m}$ is either an attack at an occupied vertex or a horizontal, vertical or diagonal attack. Hence, lemmas 3,5, and 6 hold for any possible attack, which leads to our main theorem.

Theorem 1. *For all $n, m \in \mathbb{N}^*$ such that $m \geq n$,*

$$\gamma^{\infty}_{all}(SG_{n\times m}) = \left\lceil \frac{n}{3} \right\rceil \left\lceil \frac{m}{3} \right\rceil + O(m\sqrt{n}) = (1+o(1))\gamma(SG_{n\times m}).$$

Sketch of Proof. Let k be the integer closest to $\sqrt{n}$ such that $k-2 \equiv 0 \pmod 3$. $O(m + n\sqrt{n})$ guards suffice to place one guard at every vertex of some rows and columns so that it can be assumed that n and m satisfy $n-2 \equiv 0 \pmod 3$ and $m \equiv 0 \pmod k$. Let $Y \in \mathcal{Y}$ be any configuration. The guards initially occupy the configuration $C_V(Y)$. By Lemma 4, the guards occupy a dominating set. We show that, for an attack at any vertex v, there is $Y' \in \mathcal{Y}$ such that $v \in C_V(Y')$ and $C_V(Y')$ is compatible with $C_V(Y)$. Indeed, the guards respond to attacks according to their type, *i.e.*, horizontal, vertical or diagonal. Since $k = \Theta(\sqrt{n})$, the strategy uses $\kappa_V = \lceil \frac{n}{3} \rceil \lceil \frac{m}{3} \rceil + O(m\sqrt{n})$ guards. ⋄

4 Lower Bound in Strong Grids

So far, the best lower bound for $\gamma^{\infty}_{all}(SG_{n\times m})$ was the trivial lower bound $\gamma(SG_{n\times m})$. In this section, we slightly increase this lower bound, reducing the gap with the new upper bound of the previous section.

Theorem 2. *For all $n, m \in \mathbb{N}^*$, $\gamma^{\infty}_{all}(SG_{n\times m}) = \lfloor \frac{n}{3} \rfloor \lfloor \frac{m}{3} \rfloor + \Omega(n+m)$.*

Sketch of Proof. If n and m are divisible by 3, there is a unique minimum dominating set of $SG_{n\times m}$ and each vertex is dominated by exactly one guard in this dominating set. The idea of the proof is that, in any winning configuration, some vertices are dominated by more than one guard, and/or some guards dominate at most 6 vertices. Indeed, this is because if there is a 4×5 subgrid that includes 5 border vertices with only one guard in it, then the attacker can win in at most two turns. By double counting, this leads to the necessity of having $\Omega(n+m)$ extra guards compared to the classical domination. ⋄

5 Further Work

Our results in the strong grid leave the open problem of tightening the bounds. Also, for which other grid graphs can our techniques used in obtaining the upper bound be applied? The technique of considering subgrids where only certain attacks are permitted and packing the borders of these subgrids as well as the entire grid with guards should allow to prove that $\gamma^{\infty}_{all}(G) = \gamma(G) + o(nm)$ for many types of $n \times m$ grids G. This should be true since, for all Cayley graphs H obtainable from abelian groups, $\gamma^{\infty}_{all}(H) = \gamma(H)$ [11], and many grid graphs can be represented as Cayley graphs obtained from abelian groups which are truncated. This truncation may increase the number of guards needed but our technique should permit the additional $o(nm)$ guards to suffice. Lastly, as mentioned in the introduction, it is known that given a graph G and an integer k as inputs and asking whether $\gamma^{\infty}_{all}(G) \leq k$ is NP-hard in general [3] but the exact complexity of the decision problem is open.

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