Trouver un trésor plus rapidement avec des conseils angulaires

Sébastien Bouchard, Yoann Dieudonné, Andrzej Pelc, Franck Petit

Trouver un trésor plus rapidement avec des conseils angulaires[†]

Sébastien Bouchard[1], Yoann Dieudonné[2], Andrzej Pelc[3], and Franck Petit[1]

[1]*Sorbonne Université, CNRS, INRIA, LIP6, F-75005 Paris, France*
[2]*Laboratoire MIS, Université de Picardie Jules Verne, Amiens, France*
[3]*Département d'informatique, Université du Québec en Outaouais, Gatineau, Canada*

Un touriste égaré souhaite retrouver son chemin vers la gare de train la plus proche. Heureusement pour lui, au cours de sa recherche, il rencontre de nombreuses personnes enclines à le renseigner. Toutefois, la plupart ne connaissent pas la direction exacte à suivre. N'ayant qu'une vague idée du chemin à prendre, lorsqu'elles sont interrogées, elles répondent en levant et en écartant simplement les bras pour indiquer un angle où la gare recherchée peut se trouver. Selon la précision du renseignement (c'est-à-dire de la largeur de l'angle), le touriste peut évidemment être amené à prendre des chemins l'écartant de la bonne direction.

De tels renseignements sont-ils de nature à amener notre touriste à destination en suivant un algorithme déterministe ? Et dans l'affirmative, après avoir parcouru quelle distance ? Le présent papier présente les réponses que nous avons apportées à ces deux questions dans un article récemment publié [BDPP18]. Le problème abordé ici, plus connu dans la littérature sous l'appellation de chasse au trésor, n'avait encore jamais été étudié en présence de conseils angulaires.

Mots-clefs : Chasse au trésor, agent mobile, algorithme déterministe

---

1 Introduction

1.1 The model and problem formulation

A mobile agent (modeling our lost tourist) equipped with a compass and a measure of length has to find an inert treasure (e.g., the train station) in the Euclidean plane. Both the agent and the treasure are modeled as points. In the beginning, the agent is at a distance at most $D > 1$ from the treasure, but knows neither the distance nor any bound on it. Finding the treasure means getting at distance at most 1 from it. In applications, from such a distance the treasure can be seen. The agent makes a series of moves. Each of them consists in moving straight in a chosen direction at a chosen distance. At the beginning and after each move the agent gets a hint consisting of a positive angle smaller than 2π whose vertex is at the current position of the agent and within which the treasure is contained. Within a series of moves, the angles of any two hints are not necessarily equal. We investigate the problem of how these hints permit the agent to lower the cost of finding the treasure, using a deterministic algorithm, where the cost is the worst-case total length of the agent's trajectory. It is well known from [BCR93] that the optimal cost of treasure hunt without hints is $\Theta(D^2)$.

1.2 Our results

In [BDPP18], we present three different results. First, we show that if all angles given as hints are at most π, then the cost of treasure hunt can be lowered to $O(D)$, which is optimal. The main difficulty of this result consists in handling all angles *at most* π. The case of all angles *strictly smaller* than π is simpler and can be derived from existing literature on ϕ-self approaching curves [AAI$^+$01]. Our real challenge here is in the

[†]This work was performed within Project ESTATE (Ref. ANR-16-CE25-0009-03), supported by French state funds managed by the ANR (Agence Nationale de la Recherche). It was also supported in part by NSERC discovery grant 8136 – 2013 and by the Research Chair in Distributed Computing of the Université du Québec en Outaouais.

fact that hints can be angles of size *exactly* π, in which case the design of a trajectory always leading to the treasure, while being cost-efficient in terms of traveled distance, is far from obvious. Next, if all angles are at most β, where $\beta < 2\pi$ is a constant unknown to the agent, then we prove that the cost is at most $O(D^{2-\varepsilon})$, for some $\varepsilon > 0$. Finally, we show that arbitrary angles smaller than 2π given as hints cannot be of significant help: using such hints the cost $\Theta(D^2)$ cannot be beaten.

For both our positive results, we presented in [BDPP18] deterministic algorithms achieving the above costs. Both algorithms work in phases "assuming" that the treasure is contained in increasing squares centered at the initial position of the agent. The common principle behind both algorithms is to move the agent to strategically chosen points in the current square, depending on previously obtained hints, and sometimes perform exhaustive search of small rectangles from these points, in order to guarantee that the treasure is not there. This is done in such a way that, in a given phase, obtained hints together with small rectangles exhaustively searched, eliminate a sufficient area of the square assumed in the phase to eventually permit finding the treasure.

In both algorithms, the points to which the agent travels and where it gets hints are chosen in a natural way, although very differently in each of the algorithms. The main difficulty is to prove that the distance travelled by the agent is within the promised cost. In the case of the first algorithm, it is possible to cheaply exclude large areas not containing the treasure, and thus find the treasure asymptotically optimally. For the second algorithm, the agent eliminates smaller areas at each time, due to less precise hints, and thus finding the treasure costs more. Due to the lack of space, Section 2 only describes the high level idea of the second algorithm (working with angles at most $\beta < 2\pi$) that we will call `TreasureHunt`.

1.3 Related work

The problem of treasure hunt, i.e., searching for an inert target by one or more mobile agents was investigated under many different scenarios. The environment where the treasure is hidden may be a graph or a plane, and the search may be deterministic or randomized. One major and famous result about this problem is the early paper [BN70] showing that the best competitive ratio for deterministic treasure hunt on a line is 9. Another important result is the work [Lan10] in which the author presents an optimal algorithm to sweep a plane in order to locate an unknown fixed target, where locating means to get the agent originating at point O to a point P such that the target is in the segment OP. For the curious reader wishing to consider the problem of treasure hunt in greater depth, a good starting point is to go through [AG03, BCR93].

2 High level idea of Algorithm `TreasureHunt`

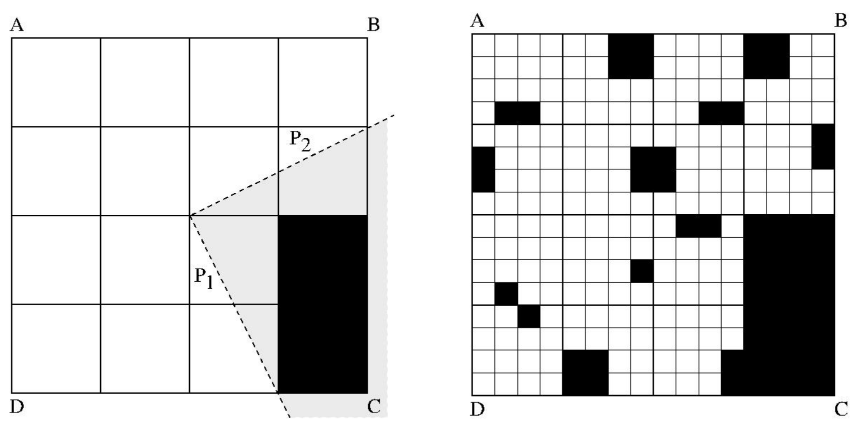
In order to properly present the intuition of our algorithm, we first need to provide two precisions. First, a hint given to the agent currently located at point a is formally described as an ordered pair (P_1,P_2) of half-lines originating at a such that the angle clockwise from P_1 to P_2 (including P_1 and P_2) contains the treasure. For a hint (P_1,P_2) we denote by $\overline{(P_1,P_2)}$ the complement of (P_1,P_2). Second, our algorithm heavily relies on the following notion of tiling. Given a square S with side of length $x > 0$, $Tiling(i)$ of S, for any non-negative integer i, is the partition of square S into 4^i squares with side of length $\frac{x}{2^i}$. Each of these squares, called *tiles*, is closed, i.e., contains its border, and hence neighboring tiles overlap in the common border. Having precised that, we are now ready to give the high level idea of our solution.

When executing Algorithm `TreasureHunt`, the agent proceeds in phases $j = 1,2,3,\ldots$, where in each phase j the agent "supposes" that the treasure is in the straight square centered at its initial position and of side length 2^j. The intended goal is to search each supposed square at relatively low cost, and to ensure the discovery of the treasure by the time the agent finishes the first phase for which the initially supposed square contains the treasure.

Let us consider a simpler situation in which the angle of every hint (P_1,P_2) is always equal to the bound β: the general case, when the angles may vary while being at most β, adds a level of technical complexity that is unnecessary to understand the intuition. In the considered situation, the angle of each excluded zone $\overline{(P_1,P_2)}$ is always the same as well. The following property holds in this case: there exists an integer i_β such that for every square S and every hint (P_1,P_2) given at the center of S, at least one tile of $Tiling(i_\beta)$ of S belongs to the excluded zone $\overline{(P_1,P_2)}$.

In phase j, the agent performs k steps: we will indicate later how the value of k should be chosen. At the beginning of the phase, the entire square S is white. In the first step, the agent gets a hint (P_1, P_2) at the center of S. By the above property, we know that $\overline{(P_1, P_2)}$ contains at least one tile of $Tiling(i_\beta)$ of S, and we have the guarantee that such a tile cannot contain the treasure. All points of all tiles included in $\overline{(P_1, P_2)}$ are painted black in the first step. This operation does not require any move, as painting is performed in the memory of the agent. As a result, at the end of the first step, each tile of $Tiling(i_\beta)$ of S is either black or white, in the following precise sense: a black tile is a tile all of whose points are black, and a white tile is a tile all of whose *interior* points are white.

In the second step, the agent repeats the painting procedure at a finer level. More precisely, the agent moves to the center of each white tile t of $Tiling(i_\beta)$ of S. When it gets a hint at the center of a white tile t, there is at least one tile of $Tiling(i_\beta)$ of t that can be excluded. As in the first step, all points of these excluded tiles are painted black. Note that a tile of $Tiling(i_\beta)$ of t is actually a tile of $Tiling(2i_\beta)$ of S. Moreover, each tile of $Tiling(i_\beta)$ of S is made of exactly 4^{i_β} tiles of $Tiling(2i_\beta)$ of S. Hence, as depicted in Figure 1, the property we obtain at the end of the second step is as follows: each tile of $Tiling(2i_\beta)$ of S is either black or white.



(a) At the end of a first step for a hint (P_1, P_2)

(b) At the end of a second step

FIGURE 1: White and black tiles at the end of the first and the second step of a phase, for square $S = ABCD$ and $i_\beta = 2$.

In the next steps, the agent applies a similar process at increasingly finer levels of tiling. More precisely, in step $2 < s \leq k$, the agent moves to the center of each white tile of $Tiling((s-1)i_\beta)$ of S and gets a hint that allows it to paint black at least one tile of $Tiling(s \cdot i_\beta)$ of S. At the end of step s, each tile of $Tiling(s \cdot i_\beta)$ of S is either black or white. We can show that at each step s the agent paints black at least $\frac{1}{4^{i_\beta}}$th of the area of S that is white at the beginning of step s.

After step k, each tile of $Tiling(k \cdot i_\beta)$ of S is either black or white. These steps permit the agent to exclude some area without having to search it directly, while keeping some regularity of the shape of the black area. The agent paints black a smaller area than excluded by the hints but a more regular one. This regularity enables in turn the next process in the area remaining white. Indeed, the agent subsequently executes a brute-force searching that consists in moving to each white tile of $Tiling(k \cdot i_\beta)$ of S in order to scan it using a procedure called `RectangleScan`: for every tile of area A, this procedure permits the agent to see all points of the tile at cost $O(A)$. If, after having scanned all the remaining white tiles, it has not found the treasure, the agent repaints white all the square S and enters the next phase. Thus we have the guarantee that the agent finds the treasure by the end of phase $\lceil \log_2 D \rceil + 1$, i.e., a phase in which the initially supposed square is large enough to contain the treasure.

At this point, a natural question arises: how much do we have to pay for all of this ?

In fact, the cost depends on the value that is assigned to k in each phase j. The value of k must be large enough so that the distance travelled by the agent during the brute-force searching is relatively small. At the same time, this value must be small enough so that the the distance travelled during the k steps is not too large. A good trade-off can be reached when $k = \lceil \log_{4^{i_\beta}} \sqrt{2^j} \rceil$. Indeed, we show that it is due to this carefully chosen value of k that we can beat the cost $\Theta(D^2)$ necessary without hints, and get a complexity of $O(D^{2-\varepsilon})$, where $\varepsilon = \frac{1}{2}(1 - \log_{4^{i_\beta}}(4^{i_\beta} - 1))$ is a positive real depending on i_β, and hence depending on the angle β.

3 Conclusion

Let us now give some concluding remarks on our work [BDPP18]. For hints that are angles at most π we gave a treasure hunt algorithm with optimal cost linear in D. For larger angles we showed a separation between the case where angles are bounded away from 2π, when we designed an algorithm with cost strictly subquadratic in D (the intuition of which is described in Section 2 of this paper), and the case where angles have arbitrary values smaller than 2π, when we showed a quadratic lower bound on the cost.

The optimal cost of treasure hunt with large angles bounded away from 2π remains open. In particular, the following questions seem intriguing:

— Is the optimal cost linear in D in this case, or is it possible to prove a super-linear lower bound on it?

— Does the order of magnitude of this optimal cost depend on the bound $\pi < \beta < 2\pi$ on the angles given as hints?

Références

[AAI+01] Oswin Aichholzer, Franz Aurenhammer, Christian Icking, Rolf Klein, Elmar Langetepe, and Günter Rote. Generalized self-approaching curves. *Discrete Applied Mathematics*, 109(1-2) :3–24, 2001.

[AG03] Steve Alpern and Shmuel Gal. *The Theory of Search Games and Rendezvous*. Kluwer Academic Publications, 2003.

[BCR93] Ricardo A. Baeza-Yates, Joseph C. Culberson, and Gregory J. E. Rawlins. Searching in the plane. *Inf. Comput.*, 106(2) :234–252, 1993.

[BDPP18] Sébastien Bouchard, Yoann Dieudonné, Andrzej Pelc, and Franck Petit. Deterministic treasure hunt in the plane with angular hints. In *29th International Symposium on Algorithms and Computation, ISAAC 2018, December 16-19, 2018, Jiaoxi, Yilan, Taiwan*, pages 48 :1–48 :13, 2018.

[BN70] Anatole Beck and D.J. Newman. Yet more on the linear search problem. *Israel J. Math.*, 8 :419–429, 1970.

[Lan10] Elmar Langetepe. On the optimality of spiral search. In *Proceedings of the Twenty-First Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2010, Austin, Texas, USA, January 17-19, 2010*, pages 1–12, 2010.