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Abstract: This paper deals with continuous boundary time-varying feedbacks for fixed-time stabilization of constant-parameter reaction-diffusion systems. The time of convergence can be prescribed and is independent of the initial condition of the system. The design of time-varying feedbacks is carried out by using the backstepping approach for which suitable characterizations for time-varying kernels are derived. Kernel solutions are given in terms of power series involving *the exponential complete Bell polynomials* for which we have exploited the *Faà di Bruno formula.* Moreover, by particularizing the characterization, one can recover kernel solutions in terms of the *generalized Laguerre polynomials* and *the modified Bessel functions.*

Keywords: Reaction-diffusion equations, time-varying feedbacks, fixed-time convergence, exponential Bell polynomials, generalized Laguerre polynomials

1. INTRODUCTION

Finite/fixed-time stabilization and estimation have been widely considered for many years in the framework of linear and nonlinear ordinary differential equations (ODEs) (Bhat and Bernstein (2000); Polyakov et al. (2015); Lopez-Ramirez et al. (2018)). The need to meet some performance, time constraints and precision has highly motivated the stabilization and estimation in finite/fixed time. The concept of fixed-time differs from finite-time whenever the settling time is independent of the initial condition. More recent contributions deal with *prescribed-time* stabilization for nonlinear ODEs in normal form Song et al. (2017). For partial differential equations (PDEs), on the other hand, finite/fixed-time concepts have become a relevant research area (see e.g. Polyakov et al. (2018), Pisano et al. (2011)). The motivations for finite/fixed-time control and estimation for PDEs are in the same way of those for the finite-dimensional systems. In particular, since many complex systems are described by PDEs (e.g. hydraulic networks, tubular chemical reactors, etc), convergence while meeting time constraints or just realizing the well-known separation principle are central issues, which can be coped when exploiting finite-time concepts. For hyperbolic PDEs, on the one hand, there are few contributions in the literature (e.g. Perrollaz and Rosier (2014); Coron et al. (2017)) for the stabilization in finite-time with boundary control; however for parabolic PDEs, on the other hand, the scenario is even more demanding and only few works have addressed some relevant issues on null controllability and finite-time stabilization (e.g. Coron and Nguyen (2017)) by making use of the backstepping approach to design time-varying feedbacks; although with discontinuous kernels. Based on that work, this topic is moved forward for continuous time-varying feedbacks in Espitia et al. (2018) and Espitia et al. (2019) where the backstepping approach with continuous time-varying kernels is studied. In both of these works, the main idea relies on selecting a suitable target system that meets the fixed-time stability property. To that end, a blow-up function in time is used as reaction term which makes the target systems fixed-time stable. The approach leads to build on time-varying kernels allowing closed-form analytical solutions. Furthermore, the bounded invertibility is guaranteed and the fixed-time stability property is preserved for the closed-loop system.

In this work, and as a part of the perspectives in Espitia et al. (2019), we consider target systems with more general blow up functions where the power degree is left arbitrary. The main contribution relies on more general characterizations of kernel solutions. It is proved that using the *Faà di Bruno formula* we can express the resulting kernels in rather closed-form in terms of the *the exponential complete Bell polynomials* which have plenty of properties and relations with special functions within the framework of hypergeometric functions and orthogonal polynomials. The fixed-time stability property is guaranteed for both the target system and the original system due to the

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bounded invertibility of the backstepping transformation. When particularizing the degree of blow up function, one can recover the main results in Espitia et al. (2019).

This paper is organized as follows. In Section 2, we present the problem and the backstepping approach with time-varying kernels. Section 3 presents some characterization for kernel solutions. Section 4 provides the main result on fixed-time stabilization. Section 5 provides a numerical example to illustrate the main result. Finally, conclusions and perspectives are given in Section 6.

Notations $\mathbb{R}^+$ will denote the set of nonnegative real numbers. The set of all functions $g : [0,1] \to \mathbb{R}$ such that $\int_0^1 g(x)^2 dx < \infty$ is denoted by $L^2((0,1),\mathbb{R})$ and is equipped with the norm $\|\cdot\|_{L^2((0,1),\mathbb{R})}$. $\Gamma(\cdot)$ denotes the Gamma function. $I_m(\cdot)$, $J_m(\cdot)$ with $m \in \mathbb{Z}$, denote the modified Bessel and (nonmodified) Bessel functions of the first kind, respectively. $L_m^{(\alpha)}(\cdot)$ denotes the generalized Laguerre polynomial. ${}_1F_1(a;b;p)$ denotes the (Kummer) confluent hypergeometric function. $(a)_k := a(a+1)(a+2)\cdots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}$ denotes the Pochhammer symbol or rising factorial. $\binom{n}{k} := \frac{n!}{k!(n-k)!}, k = 1,2,..,n$ denotes the binomial coefficients. Finally, $B_{n,k}(\cdot)$, $B_n(\cdot)$ denote the incomplete and complete exponential Bell polynomials, respectively.

2. PROBLEM DESCRIPTION

Let us consider the following scalar reaction-diffusion system with constant coefficients:

$$u_t(t,x) = \theta u_{xx}(t,x) + \lambda u(t,x) \tag{1}$$
$$u(t,0) = 0 \tag{2}$$
$$u(t,1) = U(t) \tag{3}$$

and initial condition:

$$u(0,x) = u_0(x) \tag{4}$$

where $\theta > 0$ and $\lambda \in \mathbb{R}$, $u : [0,T) \times [0,1] \to \mathbb{R}$ is the system state. The time $T > 0$ is given, and will be called from now *the prescribed time or fixed time*. In addition, $U(t) \in \mathbb{R}$ is the control input which will be a *time-varying feedback* having the functional form

$$U(t) = \mathcal{K}(t)[u(t,\cdot)](1) \tag{5}$$

with

$$\mathcal{K}(t)[u(t,\cdot)](1) = \int_0^1 k(1,y,t)u(t,y)dy \tag{6}$$

yet to be characterized.

The existence and uniqueness of classical solutions of (1)-(4) is assumed for initial conditions $u_0 \in H^2((0,1),\mathbb{R})$ satisfying the zero order compatibility conditions $u(0,0) = 0$ and $u(0,1) = U(0)$. Without imposing compatibility condition, solutions $u \in \mathcal{C}^0([0,T); L^2((0,1),\mathbb{R}))$ can be understood in the weak sense (see Coron and Nguyen (2017)).

2.1 Backstepping transformation and time-varying kernel equations

In this work, we build on the backstepping approach in order to come up with a boundary time-varying feedback that steers the state of the system (1)-(4) to zero in a prescribed time T. To that end, the invertible Volterra integral transformation is chosen to depend on time. It is given as follows,

$$\begin{aligned} w(t,x) &= u(t,x) - \int_0^x k(x,y,t)u(t,y)dy \\ &= \mathcal{K}(t)[u(t,\cdot)](x) \end{aligned} \tag{7}$$

The aim is to transform the system (1)-(4) into the following target system:

$$w_t(t,x) = \theta w_{xx}(t,x) - c(t)w(t,x) \tag{8}$$
$$w(t,0) = 0 \tag{9}$$
$$w(t,1) = 0 \tag{10}$$

with initial condition:

$$w_0(x) = u_0(x) - \int_0^x k(x,y,0)u_0(y)dy \tag{11}$$

where $w : [0,T) \times [0,1] \to \mathbb{R}$ is the state of the target system whose reaction term is a time-dependent parameter $c(t)$ that needs to be designed to achieve fixed-time stability.

Following the standard methodology to find the kernel equations and by taking into account the time-varying dependence, it can be shown that the original system is transformed into the target system with the kernel of the transformation (7) satisfying the following PDE system:

$$k_{xx}(x,y,t) - k_{yy}(x,y,t) = \gamma(t)k(x,y,t) + \tfrac{1}{\theta}k_t(x,y,t) \tag{12}$$
$$k(x,0,t) = 0 \tag{13}$$
$$\frac{d}{dx}k(x,x,t) = -\tfrac{1}{2}\gamma(t) \tag{14}$$

provided that

$$c(t) = -\lambda + \gamma(t)\theta \tag{15}$$

where k is defined on the domain $\mathcal{T} = \{(x,y,t) \in \mathbb{R}^2 \times [0,T) : 0 \le y \le x \le 1\}$ and γ can be chosen to be a smooth positive time-varying scalar function defined on $[0,T)$. Under a suitable characterization of γ, the fixed-time stabilization problem relates to the problem of solvability of kernel equations (12)-(14). Solving them and choosing $U(t)$ given in (5) we realize the backstepping transformation.

Remark 1. The right-hand side of (12) contains the partial derivative of the kernel w.r.t time. For more general cases, where the reaction term is both space- and time-varying dependent, the well-posedness has been rigorously addressed in e.g. Colton (1977), Meurer and Kugi (2009) and Vazquez et al. (2008). Furthermore, the problem with time-dependent reaction term has been already addressed in Smyshlyaev and Krstic (2005) where power series solutions are obtained. As in Espitia et al. (2018, 2019), here we intend to exploit the power series representations. ∘

2.2 Inverse transformation and time-varying kernel equations

The inverse of the backstepping transformation is given by

$$\begin{aligned} u(t,x) &= w(t,x) + \int_0^x l(x,y,t)w(t,y)dy \\ &= \mathcal{L}(t)[w(t,\cdot)](x) \end{aligned} \tag{16}$$

whose kernel $l(x,y,t)$, can be shown to satisfy the following PDE system:

$$l_{xx}(x,y,t)-l_{yy}(x,y,t)=-\gamma(t)l(x,y,t)+\tfrac{1}{\theta}l_t(x,y,t) \quad (17)$$

$$l(x,0,t)=0 \quad (18)$$

$$\frac{d}{dx}l(x,x,t)=-\tfrac{1}{2}\gamma(t) \quad (19)$$

provided that $c(t)=-\lambda+\gamma(t)\theta$ and l is defined on the domain $\mathcal{T}=\{(x,y,t)\in\mathbb{R}^2\times[0,T):0\le y\le x\le 1\}$. As before, γ can be chosen to be a smooth positive time-varying scalar function defined on $[0,T)$. The time-varying feedback (5) can equivalently be written under the following functional form

$$U(t)=\mathcal{L}(t)[w(t,\cdot)](1) \quad (20)$$

with

$$\mathcal{L}(t)[w(t,\cdot)](1)=\int_0^1 l(1,y,t)w(t,y)dy \quad (21)$$

3. CHARACTERIZATIONS OF THE PDE KERNELS

3.1 Solution of the PDE kernel (12)-(14)

Let us choose γ in (12)-(14) to be the solution that satisfies the following scalar nonlinear ordinary differential equation:

$$\dot{\gamma}(t)=\frac{1+\epsilon}{\gamma_0 T}\gamma^{(2+\epsilon)/(1+\epsilon)}(t), \quad \gamma(0)=\gamma_0^{1+\epsilon}>0 \quad (22)$$

where $\epsilon>0$ is a design parameter, T is given and is going to be the prescribed time. The solution to (22) is as follows:

$$\gamma(t)=\frac{(\gamma_0 T)^{1+\epsilon}}{(T-t)^{1+\epsilon}} \quad (23)$$

This solution is monotonically increasing and blows up at time T. In order to characterize the solution of (12)-(14), let us briefly introduce some relevant results within the framework of special functions and advanced combinatorics.

Preliminaries on exponential Bell polynomials. The exponential (partial) Bell polynomials are defined by the series expansion (Comtet (1974)):

$$\sum_{n=k}^{\infty}B_{n,k}(x_1,x_2,...,x_{n-k+1})\frac{t^n}{n!}=\frac{1}{k!}\left(\sum_{m=1}^{\infty}x_m\frac{t^m}{m!}\right)^k$$

The exponential (complete) Bell polynomials are defined by

$$B_n(x_1,x_2,...,x_n)=\sum_{k=1}^{n}B_{n,k}(x_1,x_2,...,x_{n-k+1}), \quad B_0=1 \quad (24)$$

Let us point out some relevant properties as follows:

Property 1. (Homogeneity). The following identity holds for $n\ge k$ and $a,b\in\mathbb{R}$ (Comtet, 1974, Section 3.3):

$$B_{n,k}(abx_1,ab^2x_2,...,ab^{n-k+1}x_{n-k+1}) = a^k b^n B_{n,k}(x_1,x_2,...,x_{n-k+1}) \quad (25)$$

Property 2. Relation to the *Lah numbers* (Comtet, 1974, Section 3.3):

$$B_{n,k}(1!,2!,..,(n-k+1)!)=\binom{n-1}{k-1}\frac{n!}{k!} \quad (26)$$

Property 3. Recurrence relation for the complete Bell polynomials (Bell, 1934, p. 270):

$$B_{n+1}(x_1,x_2,...,x_{n+1})=\sum_{k=0}^{n}\binom{n}{k}x_{n-k}B_k(x_1,x_2,...,x_k) \quad (27)$$

Let us recall the Faà di Bruno formula in terms of Bell polynomials as follows:

Definition 1. *[The Faà di Bruno formula] If g and h possess derivatives up to order n, then*

$$\frac{d^n}{dx^n}g(h(x))=\sum_{k=0}^{n}g^{(k)}(h(x))B_{n,k}(h',h'',...,h^{(n-k+1)}) \quad (28)$$

Theorem 1. *Let $T>0$ be given and γ satisfy* (22)*. The problem* (12)-(14) *has a well-posed solution on $\mathcal{T}$, given by*

$$k(x,y,t)$$
$$=-\frac{y}{2}\frac{\gamma^{1/(\epsilon+1)}(t)}{T\gamma_0\theta}\sum_{n=0}^{\infty}\frac{(\gamma^{1/(\epsilon+1)}(t)(x^2-y^2))^n}{4^n(T\gamma_0\theta)^n n!(n+1)!}$$
$$\times B_{n+1}\Big((\epsilon+1)_0(T\gamma_0\theta)\gamma^{\epsilon/(\epsilon+1)}(t),(\epsilon+1)_1(T\gamma_0\theta)\gamma^{\epsilon/(\epsilon+1)}(t),$$
$$...,(\epsilon+1)_{n+1}(T\gamma_0\theta)\gamma^{\epsilon/(\epsilon+1)}(t)\Big) \quad (29)$$

Proof. We recall that we look for a solution of the form Smyshlyaev and Krstic (2005):

$$k(x,y,t)=-\tfrac{y}{2}e^{-\theta\int_0^t\gamma(\tau)d\tau}f(z,t), \quad z=\sqrt{\tfrac{x^2-y^2}{\theta}} \quad (30)$$

Introducing (30) into (12)-(14), we straightforwardly obtain that $f(z,t)$ satisfies the following nonlinear parabolic PDE:

$$f_t(z,t)=f_{zz}(z,t)+\tfrac{3}{z}f_z(z,t) \quad (31)$$

with boundary conditions,

$$f_z(0,t)=0, \quad f(0,t)=\gamma(t)e^{\theta\int_0^t\gamma(\tau)d\tau}:=F(t) \quad (32)$$

where $F(t)$ is a C^∞ function (whose n-th derivative w.r.t time is denoted by $F^{(n)}(t):=\frac{d^n}{dt^n}F(t)$). Since we deal with an infinitely differentiable function $\gamma(t)$, the solution to (31)-(32) can be found in Polianin (2002) and it is as follows:

$$f(z,t)=\sum_{n=0}^{\infty}\frac{z^{2n}}{4^n n!(n+1)!}F^{(n)}(t) \quad (33)$$

The task is to characterize $F^{(n)}(t)$. To that end, note that $F(t)$ is just a product with composite functions. Therefore, using the general Leibniz rule, we have

$$F^{(n)}(t)=\sum_{k=0}^{n}\binom{n}{k}\gamma^{(n-k)}(t)\frac{d^k}{dt^k}e^{\theta\int_0^t\gamma(\tau)d\tau} \quad (34)$$

and then, by applying the *the Faà Di Bruno formula* (see (28)) to (34), we get

$$F^{(n)}(t)=\sum_{k=0}^{n}\binom{n}{k}\gamma^{(n-k)}\sum_{j=0}^{k}B_{k,j}(\theta\gamma,\theta\dot{\gamma},..,\theta\gamma^{(k-j)}) \times e^{\theta\int_0^t\gamma(\tau)d\tau} \quad (35)$$

By (22) and computing iteratively its $(k-j)$-th derivative, (35) is reformulated as follows:

$$F^{(n)}(t) = \sum_{k=0}^{n} \binom{n}{k} \frac{(1+\epsilon)_{n-k}}{(T\gamma_0)^{n-k}} \gamma (\gamma^{1/(\epsilon+1)})^{n-k} \times \sum_{j=0}^{k} B_{k,j}\Big(\theta\gamma, \theta\tfrac{(1+\epsilon)}{T\gamma_0}\gamma\gamma^{1/(\epsilon+1)}, \theta\tfrac{(1+\epsilon)(2+\epsilon)}{(T\gamma_0)^2}\gamma(\gamma^{1/(\epsilon+1)})^2, \ldots, \theta\tfrac{(1+\epsilon)_{k-j}}{(T\gamma_0)^{k-j}}\gamma(\gamma^{1/(\epsilon+1)})^{k-j}\Big) e^{\theta\int_0^t \gamma(\tau)d\tau}$$

From Property 1 and the definition of the complete Bell polynomials (24), we get

$$F^{(n)}(t) = \sum_{k=0}^{n} \binom{n}{k} \frac{(1+\epsilon)_{n-k}}{(T\gamma_0)^n} \gamma\gamma^{n/(\epsilon+1)} \times B_k\Big((1+\epsilon)_0 (T\gamma_0\theta)\gamma^{\epsilon/(1+\epsilon)}, (1+\epsilon)_1 (T\gamma_0\theta)\gamma^{\epsilon/(1+\epsilon)}, \ldots, (1+\epsilon)_k (T\gamma_0\theta)\gamma^{\epsilon/(1+\epsilon)}\Big) e^{\theta\int_0^t \gamma(\tau)d\tau} \tag{36}$$

which can further be rewritten as follows:

$$F^{(n)}(t) = \frac{\gamma^{(n+1)/(\epsilon+1)}}{(T\gamma_0)^{n+1}\theta} \sum_{k=0}^{n} \binom{n}{k} (1+\epsilon)_{n-k} (T\gamma_0\theta)\gamma^{\epsilon/(\epsilon+1)} \times B_k\Big((1+\epsilon)_0 (T\gamma_0\theta)\gamma^{\epsilon/(1+\epsilon)}, (1+\epsilon)_1 (T\gamma_0\theta)\gamma^{\epsilon/(1+\epsilon)}, \ldots, (1+\epsilon)_k (T\gamma_0\theta)\gamma^{\epsilon/(1+\epsilon)}\Big) e^{\theta\int_0^t \gamma(\tau)d\tau} \tag{37}$$

Then, by the recurrence relation in Property 3, we finally obtain the characterization for $F^{(n)}(t)$. That is,

$$F^{(n)}(t) = \frac{\gamma^{(n+1)/(\epsilon+1)}}{(T\gamma_0)^{n+1}\theta} \times B_{n+1}\Big((1+\epsilon)_0 (T\gamma_0\theta)\gamma^{\epsilon/(1+\epsilon)}, (1+\epsilon)_1 (T\gamma_0\theta)\gamma^{\epsilon/(1+\epsilon)}, \ldots, (1+\epsilon)_{n+1} (T\gamma_0\theta)\gamma^{\epsilon/(1+\epsilon)}\Big) e^{\theta\int_0^t \gamma(\tau)d\tau} \tag{38}$$

Hence, replacing it into (33) and by virtue of (30), we obtain (29). This concludes the proof. •

Let us particularize the case when $\epsilon = 1$ from which it is possible to obtain closed-form solutions for the kernel in terms of generalized Laguerre polynomials and subsequently, in terms of the modified Bessel functions. Thus, we can alternatively but more rigorously recover the results of Espitia et al. (2018) and Espitia et al. (2019)

Corollary 1. *Let $T > 0$ be given and γ satisfy* (22)*. If $\epsilon = 1$, then the problem* (12)-(14) *has a well-posed solution on $\mathcal{T}$, given by*

$$k(x,y,t) = -\frac{y}{2}\gamma(t) \sum_{n=0}^{\infty} \frac{(\gamma^{1/2}(t)(x^2-y^2))^n}{4^n (T\gamma_0\theta)^n (n+1)!} \times L_n^{(1)}\Big(-(T\gamma_0\theta)\gamma^{1/2}(t)\Big) \tag{39}$$

Before we get into the proof, let us first introduce some intermediary results from the theory of hypergeometric functions.

Kummer confluent hypergeometric functions and generalized Laguerre polynomials. The Kummer confluent hypergeometric function ${}_1F_1(a;b;p)$ is defined as follows:

$${}_1F_1(a;b;p) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{p^k}{k!} \tag{40}$$

The Kummer function has the following property:

Property 4.

$$e^p\, {}_1F_1(a;b;-p) = {}_1F_1(b-a;b;p) \tag{41}$$

A relation between the polynomials involving the Lah numbers and the Kummer function is as follows (Qi and Guo, 2018, Theorem 1):

Property 5.

$$\sum_{k=1}^{n} \binom{n-1}{k-1} \frac{n!}{k!} p^k = n! e^{-p}\, {}_1F_1(n+1;2;p) \tag{42}$$

The generalized Laguerre polynomial expressed in terms of (40) is

$$L_k^{(\alpha)}(p) = \binom{k+\alpha}{k} {}_1F_1(-k;\alpha+1;p) \tag{43}$$

Proof.[of Corollary 1] From (36) with $\epsilon = 1$, we have that $F^{(n)}(t)$ in (32) is given by

$$F^{(n)}(t) = \frac{\gamma\gamma^{n/2}}{(T\gamma_0)^n} \sum_{k=0}^{n} \binom{n}{k} (2)_{n-k} e^{\theta\int_0^t \gamma(\tau)d\tau} \times \sum_{j=0}^{k} (T\gamma_0\theta)^j (\gamma^{1/2})^j B_{k,j}\left((2)_0, (2)_1, (2)_2, \ldots, (2)_{k-j}\right) \tag{44}$$

Note that $B_{k,j}\left((2)_0, (2)_1, (2)_2, \ldots, (2)_{k-j}\right) = B_{k,j}\left(1, 2!, 3!, \ldots, (k-j+1)!\right)$. Therefore, applying Properties 2, 4, 5 and (43) it holds that

$$F^{(n)}(t) = n! \frac{\gamma\gamma^{n/2}}{(T\gamma_0)^n} \Bigg((n+1) + ((T\gamma_0\theta)\gamma^{1/2}) \times \sum_{k=1}^{n} \frac{(n+1-k)}{k} L_{k-1}^{(1)}\left(-(T\gamma_0\theta)\gamma^{1/2}\right) \Bigg) e^{\theta\int_0^t \gamma(\tau)d\tau} \tag{45}$$

For the next developments we need the following intermediate result (Coffey, 2011, Proposition 1):

$$\sum_{j=1}^{n} \frac{L_{j-1}^{(1)}}{j}(p) = \frac{1}{p}(1 - L_n^{(0)})(p) \tag{46}$$

along with the following well-known recurrence relations of the generalized Laguerre polynomials (Szego, 1975, Section 5.1) :

$$p\sum_{k=0}^{n} L_k^{(\alpha)}(p) = (n+\alpha+1)L_n^{(\alpha)}(p) - (n+1)L_{n+1}^{(\alpha)}(p) \tag{47}$$

and

$$L_n^{(\alpha)}(p) = L_n^{(\alpha+1)}(x) - L_{n-1}^{(\alpha+1)}(x) \tag{48}$$

Hence, from (45) combined with (46) yields

$$F^{(n)}(t) = n! \frac{\gamma\gamma^{n/2}}{(T\gamma_0)^n} \Bigg((n+1) L_n^{(0)}\left(-(T\gamma_0\theta)\gamma^{1/2}\right) - (T\gamma_0\theta)\gamma^{1/2} \sum_{k=1}^{n} L_{k-1}^{(1)}\left(-(T\gamma_0\theta)\gamma^{1/2}\right) \Bigg) e^{\theta\int_0^t \gamma(\tau)d\tau} \tag{49}$$

Shifting the index in the above sum and using the first recurrence relation (47), we have

$$F^{(n)}(t) = n!\frac{\gamma\gamma^{n/2}}{(T\gamma_0)^n}\Bigg((n+1)L_n^{(0)}\left(-(T\gamma_0\theta)\gamma^{1/2}\right) + (n+1)L_{n-1}^{(1)}\left(-(T\gamma_0\theta)\gamma^{1/2}\right) - nL_n^{(1)}\left(-(T\gamma_0\theta)\gamma^{1/2}\right)\Bigg)e^{\theta\int_0^t \gamma(\tau)d\tau}$$

Applying the second recurrence relation (48), we finally obtain

$$F^{(n)}(t) = n!\frac{\gamma(t)\gamma^{n/2}(t)}{(T\gamma_0)^n}L_n^{(1)}\left(-(T\gamma_0\theta)\gamma^{1/2}(t)\right)e^{\theta\int_0^t \gamma(\tau)d\tau}$$

Therefore, replacing it into (33) and by virtue of (30), we obtain (39). This concludes the proof. •

Corollary 2. *[Espitia et al. (2019)] The system* (12)-(14),(22) *with* $\epsilon = 1$ *has a well-posed* C^∞ *solution on* $\mathcal{T}$, *given by*

$$k(x,y,t) = -y\gamma(t)e^{\frac{\sqrt{\gamma(t)}(x^2-y^2)}{4T\gamma_0\theta}}\frac{I_1\left(\sqrt{\gamma(t)(x^2-y^2)}\right)}{\sqrt{\gamma(t)(x^2-y^2)}} \tag{50}$$

3.2 Solution of the PDE kernel (17)-(19)

Proceeding similarly as before, the characterizations for the time-varying kernel of the inverse transformation are given as follows:

Theorem 2. *Let* $T > 0$ *be given and* γ *satisfy* (22). *The problem* (17)-(19) *has a well-posed solution on* $\mathcal{T}$, *given by*

$$\begin{aligned} l(x,y,t) = &-\frac{y}{2}\frac{\gamma^{1/(\epsilon+1)}(t)}{T\gamma_0\theta}\sum_{n=0}^{\infty}\frac{(\gamma^{1/(\epsilon+1)}(t)(x^2-y^2))^n}{4^n(T\gamma_0\theta)^n n!(n+1)!} \\ &\times(-1)^n B_{n+1}\Big((\epsilon+1)_0(T\gamma_0\theta)\gamma^{\epsilon/(\epsilon+1)}(t), \\ &-(\epsilon+1)_1(T\gamma_0\theta)\gamma^{\epsilon/(\epsilon+1)}(t), \\ &..., (-1)^{n+1}(\epsilon+1)_{n+1}(T\gamma_0\theta)\gamma^{\epsilon/(\epsilon+1)}(t)\Big) \end{aligned} \tag{51}$$

Corollary 3. *Let* $T > 0$ *be given and* γ *satisfy* (22). *If* $\epsilon = 1$, *then the problem* (17)-(19) *has a well-posed solution on* $\mathcal{T}$, *given by*

$$l(x,y,t) = -\frac{y}{2}\gamma(t)\sum_{n=0}^{\infty}\frac{(\gamma^{1/2}(t)(x^2-y^2))^n}{4^n(T\gamma_0\theta)^n(n+1)!} \times L_n^{(1)}\left((T\gamma_0\theta)\gamma^{1/2}(t)\right) \tag{52}$$

Corollary 4. *[Espitia et al. (2019)] The system* (17)-(19),(22) *with* $\epsilon = 1$ *has a well-posed* C^∞ *solution on* $\mathcal{T}$, *given by*

$$l(x,y,t) = -y\gamma(t)e^{\frac{\sqrt{\gamma(t)}(x^2-y^2)}{4T\gamma_0\theta}}\frac{J_1\left(\sqrt{\gamma(t)(x^2-y^2)}\right)}{\sqrt{\gamma(t)(x^2-y^2)}} \tag{53}$$

4. FIXED-TIME STABILIZATION

Lemma 1. *The target system* (8)-(10) *with* γ *satisfying* (22) *is fixed-time stable, i.e. for any* $w_0 \in L^2((0,1),\mathbb{R})$, *it holds*

$$\|w(t,\cdot)\|_{L^2((0,1),\mathbb{R})} \to 0 \quad as \quad t \to T \tag{54}$$

Proof. The proof essentially follows the same lines of Espitia et al. (2019) as we consider the Lyapunov function candidate, $V : L^2((0,1),\mathbb{R}) \to \mathbb{R}^+$, $V(w) = \frac{1}{2}\int_0^1 w^2(x)dx$ such that, for all $t \in [0,T)$, it holds that:

$$V(w(t,\cdot)) \leq \zeta(t)V(w_0) \tag{55}$$

where

$$\zeta(t) := e^{-2\frac{\theta\gamma_0 T}{\epsilon}\gamma(t)^{\epsilon/(1+\epsilon)}}e^{2\frac{\theta\gamma_0 T}{\epsilon}\gamma(0)^{\epsilon/(1+\epsilon)}+\lambda t}$$

(with $\epsilon > 0$) is monotonically decreasing with the properties $\zeta(0) = 1$ and $\zeta(T) = 0$. Therefore, it is straightforward to obtain $\|w(t,\cdot)\|_{L^2((0,1),\mathbb{R})} \to 0 \quad \text{as} \quad t \to T$. •

Since kernel k is continuous on $\mathcal{T}$, it holds, for each $t \in [0,T)$, that $\|\mathcal{K}(t)\|_\infty \leq M_k(t)$ where $\|\mathcal{K}(t)\|_\infty = \sup_{0\leq y\leq x\leq 1}|k(x,y,t)|$. Similarly, it holds, for each $t \in [0,T)$, that $\|\mathcal{L}(t)\|_\infty \leq M_l(t)$ where $\|\mathcal{L}(t)\|_\infty = \sup_{0\leq y\leq x\leq 1}|l(x,y,t)|$. In this paper, we state the following assumption on the characterization of $M_k(t)$ and $M_l(t)$.

Assumption 1. *There exist* c_k *and* $c_l > 0$ *depending on* γ_0 *such that*

$$M_k(t) = e^{c_k\gamma(t)^{\epsilon/(1+\epsilon)}} \tag{56}$$

and

$$M_l(t) = e^{c_l\gamma(t)^{\epsilon/(1+\epsilon)}} \tag{57}$$

Remark 2. For $\epsilon = 1$, $M_k(t)$ and $M_l(t)$ have been precisely characterized in Espitia et al. (2019). The proof to verify that (56)-(57) hold with $\epsilon \neq 1$ requires long technical developments and is not provided here for space limitation. ◦

Theorem 3. *Let* $\theta, T > 0$ *be fixed. Under Assumption 1, if* γ_0 *is chosen such that* $c_l - \frac{T\gamma_0\theta}{\epsilon} < 0$, *then the time-varying feedback controller*

$$U(t) = \int_0^1 k(1,y,t)u(t,y)dy \tag{58}$$

with $k(1,y,t)$ *as in* (29) *(at* $x = 1$*), stabilizes the system* (1)-(4) *in a prescribed* T, *i.e. for any initial condition* $u_0 \in L^2((0,1),\mathbb{R})$, *it holds*

$$\|u(t,\cdot)\|_{L^2((0,1),\mathbb{R})} \to 0 \quad as \quad t \to T \tag{59}$$

Moreover, $U(t)$ *remains bounded and* $|U(t)| \to 0$ *as* $t \to T$.

Proof. It follows the same reasoning as in Espitia et al. (2019). Thus, by Lemma 1 and Assumption 1 the bounded invertibility of the backstepping transformation can be verified as the estimate of the L^2- norm of the original system, which is given by

$$\|u(t,\cdot)\|_{L^2} \leq (1 + M_l(t))\sqrt{\zeta(t)}(1 + M_k(0))\|u_0\|_{L^2} \tag{60}$$

goes to zero in a fixed time T since the term $M_l(t)\sqrt{\zeta(t)} = e^{(c_l-\frac{\theta\gamma_0 T}{\epsilon})\gamma(t)^{\epsilon/(1+\epsilon)}}e^{\frac{\theta\gamma_0 T}{\epsilon}\gamma(0)^{\epsilon/(1+\epsilon)}+\lambda T}$ decreases to zero as t goes to T. The remaining part of the proof immediately follows. •

5. SIMULATIONS

Let us illustrate the fixed-time stability result with $\epsilon = 1$ by considering a scalar reaction-diffusion system with $\theta = 1$, $\lambda = 12$ and initial condition $u(0,x) = 1025x(1-x)$. Note that, in open loop (e.g. $U(t) = 0$), the system is unstable. The continuous boundary time-varying feedback is implemented by using the obtained closed-form solution for the kernel gain (50). Figure 1 shows the time evolution of L^2- norm of the closed-loop system plotted on a

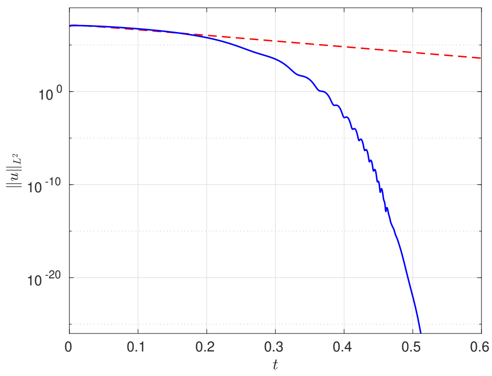

Fig. 1. Evolution of the L^2-norm of the closed-loop system (logarithmic scale) with time-varying feedback (blue line) for a prescribed time $T = 0.6$s and linear control feedback (red dashed line) for exponential stabilization.

logarithmic scale. It illustrates that with the time-varying feedback, the closed-loop system converges in a prescribe time $T = 0.6$s. It can be observed that the convergence to zero is faster than using the traditional linear control (e.g. in Smyshlyaev and Krstic (2004)) for exponential stabilization (red-dashed line).

6. CONCLUSION

In this work we have obtained more general characterizations of the time-varying kernels whose solutions are given in terms of power series representation involving the exponential complete Bell polynomials. By particularizing the characterization, one can recover kernel solutions in terms of the generalized Laguerre polynomials and the modified Bessel functions. The fixed-time stability property can be guaranteed for both the target system and the original one. For future works, we expect to study robustness analysis as well as feasibility issues. In addition, the main methodology for fixed-time stabilization of other classes of PDEs is going to be addressed.

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