Uniform k-step recovery with CMF dictionaries

Clément Elvira, Rémi Gribonval, Cédric Herzet, Charles Soussen

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Clément Elvira, Rémi Gribonval and Cédric Herzet
Univ Rennes, Inria, CNRS, IRISA
F-35000 Rennes, France
Email: firstname.lastname@inria.fr

Charles Soussen
L2S, CentraleSupélec-CNRS-Université
Paris-Saclay, 91192 Gif-sur-Yvette, France
Email: firstname.lastname@centralesupelec.fr

***Abstract*—We present new theoretical results on sparse recovery guarantees for a greedy algorithm, orthogonal matching pursuit (OMP), in the context of continuous parametric dictionaries, *i.e.*, made up of an infinite uncountable number of atoms. We build up a family of dictionaries for which k-step recovery is possible.**

I. Introduction

In the context of parametric dictionaries, the *sparse representation* problem aims at decomposing some observation $\boldsymbol{y} \in \mathcal{H}$ as

$$\boldsymbol{y} = \sum_{\ell=1}^{k} c_\ell \boldsymbol{a}(\theta_\ell) \quad \text{with} \quad c_\ell, \theta_\ell \in \mathbb{R} \times \Theta,\ \forall \ell \in [\![1, k]\!] \tag{1}$$

where $\mathcal{H}, \Theta, \boldsymbol{a}$ are the Hilbert space of observations, the parameter sets and an atom, respectively. The set $\mathcal{A} = \{\boldsymbol{a}(\theta) \mid \theta \in \Theta\}$ of all atoms is called the *dictionary*. If $\boldsymbol{y}$ writes as in (1), we say that $\boldsymbol{y}$ is supported in $\mathcal{S}^\star = \{\theta_\ell^\star\}_{\ell=1}^k$. Traditional approaches address the problem in the so-called discrete setting, *i.e.*, using a finite grid on the parameter set Θ. Among the most popular, one may mention convex minimization, *e.g.*, Basis Pursuit (BP) [1] or Lasso [2], nonconvex continuous optimization [3, 4] and greedy algorithms such as OMP [5]. The latter is a procedure that iteratively increments the support and evaluates the value of the nonzero coefficients by solving a least-square problem. A question of broad interest with OMP is the k step recovery of sparse supports. Given a finite support $\mathcal{S}^\star \subsetneq \Theta$, if $\boldsymbol{y}$ satisfies (1) one desires to known if OMP is able to recover $\mathcal{S}^\star$ in exactly k steps. If such a property holds for all choices of $\boldsymbol{y}$ supported in $\mathcal{S}^\star$, we say that OMP uniformly recovers $\mathcal{S}^\star$.

In the last few years, several works have tackled the problem of sparse representation in continuous dictionaries, *i.e.*, where $\mathcal{A}$ is made up of an uncountable number of atoms [6–8]. However, most of the contributions were made on the side of convex optimization where the problem is expressed over the space of Radon measures. In this work, we study the recovery properties of OMP in the context of continuous dictionaries. In particular, we rephrase k step recovery as a property of the kernel induced by the inner product between atoms.

II. Main results

Prior to stating our main results on k-step recovery in continuous dictionaries, we define the class of kernels that will be considered:

Definition 1 ([9], Def. 2.1)**.** *A function $\varphi : \mathbb{R}_+ \longmapsto \mathbb{R}$ is said to be a Completely Monotone Function (CMF) on $[0, +\infty[$ if and only if: it is right continuous at 0, it is infinitely differentiable on $]0, +\infty[$ and its derivatives obey*

$$(-1)^n \varphi^{(n)}(x) > 0 \quad \forall x, n \in \mathbb{R}_+^* \times \mathbb{N}. \tag{2}$$

Definition 2 (CMF dictionaries)**.** *A dictionary $\mathcal{A} = \{\boldsymbol{a}(\theta) \mid \theta \in \Theta\}$ is a CMF dictionary in $\mathbb{R}^D$ if $\Theta \subset \mathbb{R}^D$ and there exists a CMF φ verifying $\varphi(0) = 1$, and $0 < p \leq 1$ such that*

$$\langle \boldsymbol{a}(\theta), \boldsymbol{a}(\theta') \rangle = \varphi\left(\|\theta - \theta'\|_p^p\right) \quad \forall \theta, \theta' \in \mathbb{R}^D. \tag{3}$$

Result 1 ([10])**.** *Let $\mathcal{A}$ be a CMF dictionary in $\mathbb{R}$, $k \geq 1$ be an integer, $\{\theta_\ell^\star\}_{\ell=1}^k \subset \Theta$ and $\boldsymbol{y} \in \operatorname{span}(\{\boldsymbol{a}(\theta_\ell^\star)\}_{\ell=1}^k)$. Then OMP with input $\boldsymbol{y}$ recovers $\mathcal{S}^\star$ in k steps.*

Although the parameter space is a continuum, k-step recovery is always achieved whatever the support. Surprisingly, such a dictionary does not require separation between parameters, as it is usually the case for signed combination of atoms [7, 8]. In higher dimension, separation is needed again. To express the result we specialize to the family of Laplace dictionaries, which are particular cases of CMF dictionaries. It remains work in progress to determine whether the result holds for any CMF dictionary.

Definition 3 (Laplace dictionary)**.** *A dictionary $\mathcal{A} = \{\boldsymbol{a}(\theta) \mid \theta \in \Theta\}$ is a Laplace dictionary if it is a CMF dictionary with function $\varphi : x \to \mathrm{e}^{-\lambda x}$ for some $\lambda > 0$.*

We also define the following set augmenter operator

$$\Gamma(\mathcal{S}) = \prod_{d=1}^{D} \left\{ \theta[d] \mid \theta \in \mathcal{S} \right\} \tag{4}$$

where $\prod$ is the cartesian product. Figure 1 illustrates these definitions for $\mathcal{S}^\star = \{\theta_\ell^\star\}_{\ell=1}^3$.

Result 2 ([11])**.** *Let $\mathcal{A}$ be a Laplace dictionary in $\mathbb{R}^D$ and $\mathcal{S}^\star = \{\theta_\ell^\star\}_{\ell=1}^k$ be a set of k distinct parameters. Then OMP uniformly recovers $\mathcal{S}^\star$ in k steps if and only if*

$$\max_{\theta \in \Gamma(\mathcal{S}^\star) \setminus \mathcal{S}^\star} \left\| \mathbf{G}^{-1} \mathbf{g}_\theta \right\|_1 < 1 \tag{5-L-ERC}$$

In particular, if (5-L-ERC) does not hold, there exists (at least) one linear combination $\boldsymbol{y} = \sum_{\ell=1}^k c_\ell \boldsymbol{a}(\theta_\ell^\star)$ such that OMP with $\boldsymbol{y}$ as input selects some $\theta \in \Gamma(\mathcal{S}^\star) \setminus \mathcal{S}^\star$ at the first iteration.

One recognizes in (5-L-ERC) the flavor of classical Exact Recovery Conditions (ERC) [12]. The main difference with the discrete setting is that the condition involves only a (finite) subset of the dictionary.

III. Conclusion and perspectives

We show that OMP achieves k-step recovery for a broad family of continuous dictionaries built upon completely monotone functions. Our recovery results vary with the dimension of the parameter set. In dimension 1, uniform recovery is met for all k-sparse supports. So far, in higher dimension, an algebraic condition is necessary and uniform recovery holds only for Laplace dictionaries. In [10], we leverage the proof of these results to identify three conditions that are sufficient to allow for uniform k-step recovery.

In light of the existing links between Tropp's ERC and recovery guarantees for ℓ_1 minimization [13], future work will study whether these guarantees extend to sparse spike recovery with total variation norm minimization [7, 8].

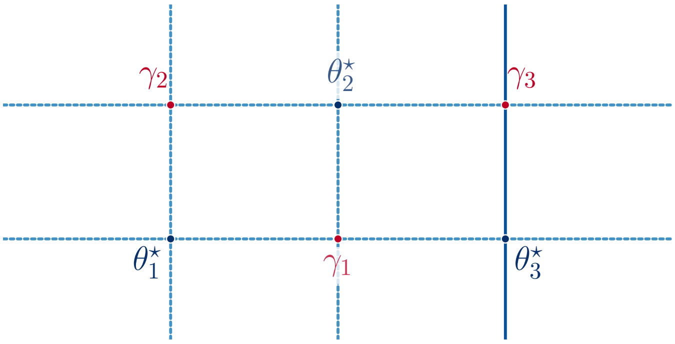

Figure 1. Illustration of the definition of the superset $\Gamma(\mathcal{S}^\star)$ with $D = 2$ and $k = 3$, see (4). Here, the support $\mathcal{S}^\star$ is composed of $\{\theta_\ell^\star\}_{\ell=1}^3$ and 3 additional points denoted $\{\gamma_\ell\}_{\ell=1}^3$.

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