# The mathematics of Ivo Rosenberg (Dedicated to the memory of Professor Ivo Rosenberg)

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***Abstract*—We survey some of the most well known results of Professor Ivo G. Rosenberg as well as present some new ones related to the study of maximal partial clones and their intersections.**

After a long struggle with Parkinson's disease, yet hard working and keeping a very kind spirit, Professor Ivo G. Rosenberg died on January 18, 2018. He was born in Brno (Moravia, Czech Republic), on December 13, 1934. One year before obtaining the doctoral degree, he published his Completeness Criterion [49] in 1965. In July 1966, Professor Rosenberg received his doctoral degree on his very same wedding day. Together with his wife Vlasta (known as Loty to all family friends), he moved to Saskatoon, Canada, in September 1968. After the birth of their two children, Madeleine (known as Misha) and Marc, the family moved to Montréal in September 1971. Ivo held the position of researcher with the rank of Full Professor from 1971 to 1983 at the "Centre de recherches mathématiques de l'Université de Montréal", and later he was appointed as a full Professor at the "Département de mathématiques et de statistique" from 1983 to August 2011. Ivo retired in August 2011.

Professor Ivo G. Rosenberg was an eminent scholar, brilliant mathematician, and one of the leading experts in universal algebra and discrete mathematics. His huge impact and contributions to these areas of mathematics are very hard to assess due to the great extension and ramifications of his works. According to [38], Ivo Rosenberg wrote more than 200 papers and several books, and the number of his co-authors exceeds 55, among which are the three authors of this paper.

Professor Rosenberg contributed to more than 50 conference papers in the various International Symposia on Multiple-Valued Logic. Some of his contributions to these symposia are listed and discussed in [38]. He was extremely generous towards his colleagues and provided great hospitality both personally and through his position at the University of Montréal. He wrote numerous research reports and doctoral thesis reports.

For his outstanding research and contributions to mathematics, Professor Rosenberg received two honorary doctorates: the first from the Technische Universität Wien (Vienna Polytechnic) in 2006 and the second from the Universität Rostock (Rostock University) in 2013.

Ivo Rosenberg organized several scientific meetings in Montréal and elsewhere, notably many sessions of the well known "Séminaire de Mathématiques Supérieures de l'Université de Montréal" like, for example, the "24e session: Algèbre universelle et relations" (Montréal, July 23–August 10, 1984), and the "30e session "Algèbres et ordres" (Montréal, July 29–August 9, 1991) as well as the "premier colloque montréalais sur la combinatoire et l'informatique" (Montréal, April 27–May 2, 1987).

Together with Maurice Pouzet, he organized the conference "Aspects of Discrete Mathematics and Computer Science" (Lyon, June 24–27, 1987) within the first "Entretiens du Centre Jacques-Cartier". The second "Entretiens", called "Discrete Mathematics and Computer Science" (Montréal, October 12–14, 1988), took place in Montréal the year after. It was organized by Ivo Rosenberg together with A. Achache, M. Pouzet (Lyon) and G. Hahn (Montréal). Still within his collaboration with French colleagues, Ivo organized a French-Quebec project "Ordered sets and their applications" from 1984 to 1986.

On the other hand, Ivo Rosenberg organized several colloquia such as **(1)** *Algebraic, Extremal and Metric Combinatorics*, (1986) with M. Deza and P. Frankl (Proceedings published in London Mathematical Society Lecture Notes Series, **131**), **(2)** *Algebras and Orders* (1991) with Gert Sabidussi (Proceedings published in Nato Advanced Study Institute Series, Math. and Phys. Sci., **389**), and **(3)** *Structural Theory of Automata, Semigroups and Universal Algebra*, (2003), with V. B. Kudryavtsev (Proceedings published in Nato Science Series, Math. Phys. and Chem. **207**).

Several of his students became prominent mathematicians and computer scientists, to mention a few: Anne Fearnley, Jens-Uwe Grabowski, Lucien Haddad, Simone Hazan, Sebastian Kerkhoff, Gérard Kientega, Benoît Larose, Maxime Lauzon, Florence Magnifo, Qinghe Sun, Bogdan Szczepara and Calvin Wuntcha. The main goal of this survey is not to give an exhaustive account of Professor Rosenberg's contributions, but rather to give a very brief survey of some of his most influential works in mathematics. For further information see, e.g., [24], [28], [38].

## I. CLONES

### A. Completeness in Multiple-Valued Logic

Professor Ivo G. Rosenberg has been extremely active in research and became one of the world's leading mathematicians in universal algebra and discrete mathematics. He became known in the 70's for his general completeness criterion in multiple-valued logic. It was first presented in 1965 [49] and the full proof was published in [50]. Rosenberg's completeness criterion is one of the most fundamental results in multiple valued logic and universal algebra, and its impact is impossible to estimate given the ever increasing number of research papers that followed it.

Let $\mathbf{k} := \{0, 1, \ldots, k-1\}$, $\mathrm{Op}(\mathbf{k})$ be the set of all operations on $\mathbf{k}$ and let $\Pi_k$ be the set of all projections on $\mathbf{k}$. For $X \subseteq \mathrm{Op}(\mathbf{k})$, we denote by $\mathrm{Clone}(X)$ the set of all operations on $\mathbf{k}$ that can be realized by composing functions from $X \cup \Pi_k$. A set $X \subseteq \mathrm{Op}(\mathbf{k})$ is a *clone* if $\mathrm{Clone}(X) = X$ and $X$ is said to be *complete* if $\mathrm{Clone}(X) = \mathrm{Op}(\mathbf{k})$. Thus $X$ is complete if and only if any function $f : \mathbf{k}^n \to \mathbf{k}$ can be realized by composing functions from $X \cup \Pi_k$. For a relation $\rho$ on $\mathbf{k}$, let $\mathrm{Pol}\,(\rho)$ be the set of all functions $f \in \mathrm{Op}(\mathbf{k})$ preserving $\rho$.

As a closure system, the set of all clones constitutes a lattice under $C \wedge C' = C \cap C'$ and $C \vee C' = \mathrm{Clone}(C \cup C')$, and where $\mathrm{Op}(\mathbf{k})$ and $\Pi_k$ are its greatest and least elements. It is also well known that this lattice is atomic and co-atomic, and its atoms and co-atoms are referred to as *minimal* and *maximal* clones, respectively.

For the purpose of presenting Rosenberg's classification of all maximal clones over $\mathbf{k}$, we recall some families of relations on $\mathbf{k}$. For $1 \leq h \leq k$, let

$$\iota_k^h := \{(a_1, \ldots, a_h) \in \mathbf{k} \mid a_i = a_j \text{ for some } 1 \leq i < j \leq h\}.$$

Let $\rho$ be an $h$-ary relation on $\mathbf{k}$ and denote by $S_h$ the set of all permutations on $\{0, \ldots, h-1\}$. For $\pi \in S_h$ set

$$\rho^{(\pi)} := \{(x_{\pi(1)}, \ldots, x_{\pi(h)}) \mid (x_1, \ldots, x_h) \in \rho\}.$$

The $h$-ary relation $\rho$ is said to be

1) *totally symmetric* (or simply *symmetric* in the case $h = 2$) if $\rho^{(\pi)} = \rho$ for every $\pi \in S_h$,
2) *totally reflexive* (in case $h = 2$ *reflexive*) if $\iota_k^h \subseteq \rho$,
3) *prime affine* if $h = 4$, $\mathbf{k} = p^m$ where $p$ is a prime number, $m \geq 1$, $\mathbf{p} := \{0, \ldots, p-1\}$ and we can define an elementary abelian $p$-group $(\mathbf{k}, +)$ on $\mathbf{k}$ so that
$$\rho := \{(\vec{a}, \vec{b}, \vec{c}, \vec{d}) \in \mathbf{k}^4 \mid \vec{a} + \vec{b} = \vec{c} + \vec{d}\}.$$
4) *central*, if $\rho \neq \mathbf{k}^h$, $\rho$ is totally symmetric, totally reflexive and $\{c\} \times \mathbf{k}^{h-1} \subseteq \rho$ for some $c \in \mathbf{k}$ (notice that for $h = 1$ each $\emptyset \neq \rho \subset \mathbf{k}$ is central and for $h \geq 2$ such $c$ is called a *central element* of $\rho$),
5) *elementary*, if $k = h^m$, $h \geq 3$, $m \geq 1$ and
$$(a_1, \quad a_2 \quad, \ldots, a_h) \in \rho \iff$$
$$\forall \quad i = 0, \ldots, m-1, (a_1^{[i]}, a_2^{[i]}, \ldots, a_h^{[i]}) \in \iota_h^h,$$
where $a^{[i]}$ ($a \in \{0, 1, \ldots, h^m - 1\}$) denotes the $i$-th digit in the $h$-adic expansion
$$a = a^{[m-1]} \cdot h^{m-1} + a^{[m-2]} \cdot h^{m-2} + \cdots + a^{[1]} \cdot h + a^{[0]},$$
6) a *homomorphic inverse image of an $h$-ary relation $\rho'$ on* $\mathbf{k}'$, if there is a surjective mapping $q : \mathbf{k} \to \mathbf{k}'$ such that
$$(a_1, \ldots, a_h) \in \rho \iff (q(a_1), \ldots, q(a_h)) \in \rho'$$
for all $a_1, \ldots, a_h \in \mathbf{k}$,
7) *$h$-universal*, if $\rho$ is a homomorphic inverse image of an $h$-ary elementary relation.

Furthermore, we denote by

- $\mathcal{C}_k$ the set of all central relations on $\mathbf{k}$;
- $\mathcal{C}_k^h$ the set of all $h$-ary central relations on $\mathbf{k}$;
- $\mathcal{U}_k$ the set of all non-trivial equivalence relations on $\mathbf{k}$;
- $P_{k,p}$ the set of all fixed-point-free permutations on $\mathbf{k}$ consisting of cycles of the same prime length $p$ ;
- $\mathcal{S}_{k,p} := \{s^0 \mid s \in P_{k,p}\}$, where $s^0 := \{(x, s(x)) \mid x \in \mathbf{k}\}$ is the *graph* of $s$;
- $\mathcal{S}_k := \bigcup\{\mathcal{S}_{k,p} \mid p \text{ is a prime divisor of } k\}$;
- $\mathcal{M}_k$ the set of all order relations on $\mathbf{k}$ with a least and a greatest element;
- $\mathcal{M}_k^\star$ the set of all lattice orders on $\mathbf{k}$;
- $\mathcal{L}_k$ the set of all prime affine relations on $\mathbf{k}$;
- $\mathcal{B}_k$ the set of all $h$-universal relations, $3 \leq h \leq k-1$.

Endowed with this terminology and notation, we can now state Rosenberg's celebrated classification of maximal clones on $\mathbf{k}$.

**Theorem 1.** ([49], [50]) *Let $k \geq 2$. Every proper clone on $\mathbf{k}$ is contained in a maximal one. Moreover a clone $M$ is a maximal clone over $\mathbf{k}$ if and only if $M = \mathrm{Pol}\,\rho$ for some relation $\rho \in \mathcal{C}_k \cup \mathcal{M}_k \cup \mathcal{S}_k \cup \mathcal{U}_k \cup \mathcal{L}_k \cup \mathcal{B}_k$.*

Since every clone $C \neq \mathrm{Op}(\mathbf{k})$ is contained in a maximal clone on $\mathbf{k}$, we have the following completness criterion.

**Theorem 2.** *A set of functions $X \subseteq \mathrm{Op}(\mathbf{k})$ is complete if and only if for every relation $\rho$ described in Theorem 1, $X$ contains a function $f$ not preserving $\rho$.*

Several variant completeness criteria have been obtained from Theorem 1. For example, we say that $X \subseteq \mathrm{Op}(\mathbf{k})$ is *complete with constants* if the set $X \cup \{c_a \mid a \in \mathbf{k}\}$ is complete, where $c_a : \mathbf{k} \to \mathbf{k}$ is the unary constant map with value $\{a\}$. Here are two examples of sets that are complete with constants.

A *simple group* is a non-trivial group with only two normal (trivial) subgroups. A *simple ring* is a non-zero ring (i.e., there are $x, y$ such that $xy$ is not the zero element) with no non-trivial double-sided ideals. We have

**Corollary 3.** (1) *A finite group is complete with constants if and only if it is a simple non-abelian group.*

(2) *A finite ring is complete with constants if and only if it is a simple ring.*

### B. Minimal clones

Recall that a clone $C$ is said to be minimal if $C$ is an atom in the lattice of all clones, or, equivalently, if $C$ is generated by any of its non-trivial functions. A non-trivial function of smallest arity in a minimal clone is called a *minimal function*. The following is one of the deepest results in the literature concerning minimal clones. It is due to Ivo Rosenberg and was first published in [53].

**Theorem 4.** *Let $f$ be a minimal function of arity $n$ over $\mathbf{k}$. Then $f$ satisfies one of the following conditions:*

1) *$n = 1$ and $f$ satisfies $f^2 = f$ or $f^p(x) = x$ for some prime number $p$,*
2) *$n = 2$ and $f$ is an idempotent function, i.e., $f(x, x) = x$,*
3) *$n = 3$ and $f$ is a majority function, i.e., $f(x, x, y) = f(x, y, x) = f(y, x, x) = x$,*
4) *$n = 3$ and $f(x, y, z) = x + y + z$, where $(\mathbf{k}, +)$ is an elementary 2-group, or*
5) *$n > 2$ and $f$ is an $n$-ary semiprojection, i.e., there is an $i \in \{1, \ldots, n\}$ such that $f(x_1, \ldots, x_n) = x_i$ whenever $|\{x_1, \ldots, x_n\}| < n$.*

A significant amount of research has been devoted to the study of minimal clones because, among other things, of their use in the study of strongly rigid relations. This is discussed in Section 3. Further fruitful applications of minimal clones and their descriptions can be found in the theory of constraint satisfaction problems (CSPs, see, e.g. [31]) and the theory of essential arguments of functions [3], [5], [10], [54], [59].

## II. Partial clones

Professor Rosenberg made significant contributions in many other areas of multiple-valued logic, like for example to the theory of partial clones. We will focus here on one of his main contributions in this domain, the completeness criterion for finite partial algebras and some related results.

### A. Completeness Criterion for Finite Partial Algebras.

Denote by $\mathrm{Par}(\mathbf{k})$ the set of all partial functions on $\mathbf{k}$ and for a relation $\rho$ over $\mathbf{k}$, we denote by $\mathrm{pPol}\,\rho$ the set of all partial functions $f \in \mathrm{Par}(\mathbf{k})$ that preserve $\rho$. We need to introduce some terminology to state the Completeness Criterion for Finite Partial Algebras.

A relation $\rho$ on $\mathbf{k}$ *extremal* if $M := \mathrm{pPol}\,\rho$ is a maximal partial clone and if $\rho$ is of minimal size among all relations that determine the same maximal partial clone $M$. Let $E_h$ denote the set of all equivalence relations on the set $\mathbf{h}$, and let $\omega_h$ be the smallest element in $E_h$, i.e., $\omega_h := \{(x, x) \mid x \in \mathbf{h}\}$. Then for $\varepsilon \in E_h$, put

$$\Delta_\varepsilon := \{(x_0, \ldots, x_{h-1}) \in \mathbf{k}^h \mid (i, j) \in \varepsilon \Rightarrow x_i = x_j\}.$$

We often denote $\Delta_\varepsilon$ by $\Delta_{X_1,\ldots,X_n}$, where $X_1, \ldots, X_n$ are the nonsingleton equivalence classes of $\varepsilon$. Moreover let

$$\Gamma_k^h := \bigcup_{0 \le i < j \le h-1} \Delta_{\{i,j\}}.$$

Notice that $\Gamma_k^h$ is the set of all tuples $(a_0, \ldots, a_{h-1}) \in \mathbf{k}^h$ such that $a_i = a_j$ for some $0 \le i < j \le h - 1$.

An $h$-ary relation $\rho$ is said to be

(a) *diagonal* if there exists $\varepsilon \in E_h$ such that $\rho = \Delta_\varepsilon$,
(b) *areflexive* if $\rho \cap \Delta_\varepsilon = \emptyset$ for each $\varepsilon \in E_h$, $\varepsilon \neq \omega_h$, i.e., for all $(x_0, \ldots, x_{h-1}) \in \rho$, $x_i \neq x_j$ holds for all $0 \le i < j \le h - 1$,
(c) *quasi-diagonal* if $\rho = \sigma \cup \Delta_\varepsilon$ where $\sigma$ is a non-empty areflexive relation, $\varepsilon \in E_h \setminus \{\omega_h\}$, and in addition, $\rho \neq \mathbf{k}^2$ if $h = 2$,
(d) *totally reflexive* if $\Gamma_k^h \subseteq \rho$.

Let $\sigma$ be an $h$-ary relation on $\mathbf{k}$ and suppose that there is a subgroup $G$ of $S_h$ such that $\sigma = \sigma^{(\pi)}$ for all $\pi \in G$ and $\sigma \cap \sigma^{(\alpha)} = \emptyset$ for all $\alpha \in S_h \setminus G$. Then $G$ is called the *group of symmetries* of the relation $\sigma$. An $h$-ary relation $\rho$ is *totally symmetric* if $S_h$ is its group of symmetries, i.e., if

$$(x_0, \ldots, x_{h-1}) \in \rho \Leftrightarrow (x_{\pi(0)}, \ldots, x_{\pi(h-1)}) \in \rho,\ \forall \pi \in S_h.$$

The following quaternary relations on $\mathbf{k}$ play an important role in the study of maximal partial clones (see Theorem 5). Let

$$\begin{aligned} R_1 &:= \Delta_{\{0,1\},\{2,3\}} \cup \Delta_{\{0,2\},\{1,3\}} \cup \Delta_{\{0,3\},\{1,2\}}, \\ R_2 &:= \Delta_{\{0,1\},\{2,3\}} \cup \Delta_{\{0,3\},\{1,2\}}, \\ R_3 &:= \Delta_{\{0,1\},\{2,3\}} \cup \Delta_{\{0,2\},\{1,3\}}, \text{ and} \\ R_4 &:= \Delta_{\{0,2\},\{1,3\}} \cup \Delta_{\{0,3\},\{1,2\}}. \end{aligned}$$

Observe that $(x_0, x_1, x_2, x_3) \in R_2$ if and only if $[x_0 = x_1$ and $x_2 = x_3]$ or $[x_0 = x_3$ and $x_1 = x_2]$.

Now let $\sigma$ be an areflexive $h$-ary relation and let $F \subset E_h$. Put $G_\sigma := \{\pi \in S_h \mid \sigma \cap \sigma^{(\pi)} \neq \emptyset\}$ and suppose that the $h$-ary relation $\rho$ is of the form

$$\rho = \sigma \cup (\bigcup_{\varepsilon \in F} (\Delta_\varepsilon)).$$

Then the *model* of $\rho$ is the $h$-ary relation

$$M(\rho) := \{(\pi(0), \ldots, \pi(h-1)) \mid \pi \in G_\sigma\} \cup \left( \bigcup_{\varepsilon \in F} \{(x_0, \ldots, x_{h-1}) \in \mathbf{h}^h \mid (i, j) \in \varepsilon \Rightarrow x_i = x_j\} \right)$$

on the set $\mathbf{h}$.

Furthermore, suppose that $h$, $F$ and $\sigma$ satisfy one of the following five conditions:

1) $h \ge 2$, $F = \emptyset$ and $\sigma \neq \emptyset$, i.e., $\rho$ is a nonempty $h$-ary areflexive relation;
2) $h \ge 2$, $F = \{\varepsilon\}$ where $\varepsilon \neq \omega_h$, $\sigma \neq \emptyset$ and $\sigma \cup \Delta_\varepsilon \neq \mathbf{k}^2$, i.e., $\rho$ is a nontrivial quasi-diagonal $h$-ary relation;
3) $h = 4$ and $F = \{\{[0, 1], [2, 3]\}, \{[0, 3], [1, 2]\}, \{[0, 2], [1, 3]\}\}$, i.e., $\rho = \sigma \cup R_1$, where $\sigma$ is a (possibly empty) areflexive 4-ary relation;
4) $h = 4$ and $F = \{\{[\pi(0), \pi(1)], [\pi(2), \pi(3)]\}, \{[\pi(0), \pi(3)], [\pi(1), \pi(2)]\}\}$, where $\pi \in S_4$, i.e., $\rho = \sigma \cup R_i$, where $i = 2, 3, 4$ and $\sigma$ is a (possibly empty) areflexive 4-ary relation;

5) $h \neq 2$, $h \leq k$, $F = \bigcup_{i<j} \{i, j\}$ and $\rho \neq \mathbf{k}^h$, i.e., $\rho$ is a totally reflexive and totally symmetric nontrivial relation.

Then the $h$-ary relation $\rho$ is said to be *coherent* if

(A) the following conditions hold:

1) when either of conditions (1) or (2) are satisfied,

$$G_\sigma = \{\pi \in S_h \mid \sigma^{(\pi)} = \sigma\} \text{ and}$$
$$\pi(\varepsilon) := \{(\pi(x), \pi(y)) \mid (x, y) \in \varepsilon\} = \varepsilon, \ \forall\, \pi \in G_\sigma;$$

2) when condition (3) is satisfied,

$$G_\sigma = \{\pi \in S_4 \mid \sigma^{(\pi)} = \sigma\} = \{\pi \in S_4 \mid \pi(F) = F\} = S_4;$$

3) when condition (4) is satisfied,

$$G_\sigma = \{\pi \in S_4 \mid \sigma^{(\pi)} = \sigma\} = \{\pi \in S_4 \mid \pi(F) = F\};$$

4) when condition (5) is satisfied,

$$G_\sigma = \{\pi \in S_h \mid \sigma^{(\pi)} = \sigma\} = S_h; \text{ and}$$

(B) for every non-empty subrelation $\sigma' \subseteq \sigma$, there exists a relational homomorphism $\gamma : \mathbf{k} \to \mathbf{h}$ from $\sigma'$ to $M(\rho)$ such that $(\gamma(i_0), \ \ldots, \ \gamma(i_{h-1})) = (0, \ \ldots, \ h-1)$ for at least one $h$-tuple $(i_0, \ldots, i_{h-1}) \in \sigma'$.

The description of all maximal partial clones on a $k$-element set as given in [22] follows.

**Theorem 5** ([18], [22], [26])**.** *Let $k \geq 2$. Every proper partial clone on $\mathbf{k}$ extends to a maximal one. If $M$ is a maximal partial clone on $\mathbf{k}$, then either*

$$C = \mathcal{O}_{\mathbf{k}} \cup \{f \in \mathcal{P}_{\mathbf{k}} \mid \mathrm{dom}\,(f) = \emptyset\}$$

*or $M$ is determined by an extremal relation on $\mathbf{k}$. Furthermore a relation $\rho$ is an extremal relation on $\mathbf{k}$ if and only if it is of one of the following types of relations:*

1) *an $h$-ary areflexive or quasi-diagonal relation which is coherent and $h \geq 2$,*
2) *an $h$-ary non-trivial totally reflexive and totally symmetric relation and $h \neq 2$,*
3) *one of the quaternary relations $R_1$ or $R_2$,*
4) *a quaternary coherent relation $\sigma \cup R_i$ where $i = 1, \ldots, 4$ and $\sigma \neq \emptyset$ is a quaternary areflexive relation.*

**Remark 6.** *The number of maximal partial clones on a finite set $\mathbf{k}$ greatly exceeds the number of maximal clones on $\mathbf{k}$. For example, Theorem 1 says that any order relation that is bounded defines a maximal clone, while Theorem 5 says that any non-trivial order relation defines a maximal partial clone on $\mathbf{k}$. Let $\mathcal{M}_k$ and $p\mathcal{M}_k$ be the families of all maximal and maximal partial, respectively, clones on $\mathbf{k}$. It is known (see, e.g., [16], [56]) that $|\mathcal{M}_2| = 5$ and $|p\mathcal{M}_2| = 8$, $|\mathcal{M}_3| = 18$ and $|p\mathcal{M}_3| = 58$, $|\mathcal{M}_4| = 82$ and $|p\mathcal{M}_4| = 1102$, $|\mathcal{M}_5| = 634$ and $|p\mathcal{M}_5| = 325,722$, $|\mathcal{M}_6| = 15,182$ and $|p\mathcal{M}_6| = 5,242,621,816$.*

Besides the completeness problem for partial algebras, Professor Rosenberg deeply contributed to the research on partial clones on a finite set. For space limitation, we will mention only one main contribution to the study of clones of partial Boolean functions. Let $\mathbf{k} = \{0, 1\}$, $\mathcal{I}(SM)$ be the interval of all partial clones containing the self-dual and monotone functions on $\{0, 1\}$. In the late 2000's it was widely believed that this interval must be finite. A breakthrough was obtained when the second author constructed an infinite set of partial clones contained in the interval $\mathcal{I}(SM)$ (see [29]) proving that the interval is actually *infinite*. Later on, the first two authors, together with Ivo Rosenberg, proved that this interval is not just infinite, but it has the cardinality of the continuum ([8]). Ivo Rosenberg came with the main construction of this paper and that construction, adapted to different situations, led to the complete solution, for the case of Boolean functions, to the following open problem in clone theory due to D. Lau: *For each total clone $C$ on $\mathbf{k}$, describe the interval of partial clones $\mathcal{I}(C) := \{D \mid D$ is a partial clone on $\mathbf{k}$ and $D \cap O_k = C\}$.*

This problem was considered by several authors and many partial results were available in the literature. It is shown in [9] that if $C$ is a clone on $\{0, 1\}$, then the interval of partial clones $\mathcal{I}(C)$ is finite for some finitely many clones listed in [9] and $\mathcal{I}(C)$ has the cardinality of the continuum otherwise.

## III. Rigidity and Projectivity

Many works of Professor Rosenberg deal with several notions of rigidity and projectivity. We briefly survey a few in this section.

A relational structure $R := (A, (\rho_i)_{i \in I})$ on a set $A$ is

- *rigid* if the identity is the only endomorphism of $R$;
- *strongly rigid* if the projections are the only maps of several variables which preserve $R$;
- *semirigid* if the only unary functions which preserve $\rho$ are the identity and all constant maps;
- *$n$-projective* if the only idempotent $n$-ary functions which preserve $R$ are the projections.

### A. Rigidity and strong rigidity

In 1965, Vopěnka, Pultr, and Hedrlín [58] proved that a rigid binary relation exists on any set. The existence of rigid relations was known for cardinality less that the first inaccessible cardinal and used by Pultr [47] to prove that the category $\mathcal{R}$ of binary relations is *universal* in the sense that every small category (the collection of objects is a set) is a full subcategory of $\mathcal{R}$. This result supposed that no inaccessible existed. With the above result this restriction was not needed. Rosenberg [52] observed that the relation constructed in [58] is not strongly rigid and he proved that strongly rigid binary relations exist on any set with at least three elements.

### B. Semirigidity

We recall that $R$ is semirigid if the functions that preserve $R$ are either the projections or the constant maps (Langer and Pöschel). Demetrovics, Miyakawa, Rosenberg, Simovici and Stojmenović [7] introduced the following notion: two orders

$\rho$ and $\tau$ on the same set $E$ are *orthogonal* if $(E, \tau, \rho)$ is semirigid. Nozaki, Miyakawa, Pogosyan and Rosenberg [41] investigated the existence of a linear order orthogonal to a given finite linear order. They observed that there is always one provided that the number of elements is not equal to three and proved:

**Theorem 7.** *The proportion $q(n)/n!$ of linear orders orthogonal to the natural order on $[n] := \{1, ..., n\}$ goes to $e^{-2} = 0.1353...$ when $n$ goes to infinity.*

Their counting argument was based on the fact that two linear orders on the same *finite* set are orthogonal if and only if they do not have a common nontrivial interval. The notion capturing the properties of intervals of a linear order was extended a long time ago to posets, graphs and binary structures and a decomposition theory has been developed (see, e.g., [13], [14], [15], [17]). One of the terms in use for this notion is *autonomous set*; structures with no nontrivial autonomous subset–the building blocks in the decomposition theory–are called *prime* (or *indecomposable*). With this terminology, the above fact can be expressed by saying that two linear orders $\rho$ and $\tau$ on the same finite set $V$ are orthogonal if and only if the binary structure $B := (V, \rho, \tau)$, that we call a *bichain*, is prime. This lead to results relating primality and orthogonality ([48], [61]).

The notion of primality and Theorem 7 reappeared in recent years under quite a different setting: a study of permutations motivated by the Stanley-Wilf conjecture, now settled by Marcus and Tardos [39]. This study, which developed in many papers, can be presented as follows: To a permutation $\sigma$ on $[n]$ associate first the linear order $\leq_\sigma$ defined by $x \leq_\sigma y$ if $\sigma(x) \leq \sigma(y)$ for the natural order on $[n]$; next associate the bichain $B_\sigma := ([n], (\leq, \leq_\sigma))$. On the set $\mathfrak{S} := \cup_{n \in \mathbb{N}} \mathfrak{S}_n$ of all permutations, set $\sigma \leq \tau$ if $B_\sigma$ is embeddable into $B_\tau$. Say that a subset $\mathfrak{C}$ of $\mathfrak{S}$ is *hereditary* if $\sigma \leq \tau$ and $\tau \in \mathfrak{C}$ imply $\sigma \in \mathfrak{C}$. The goal is to evaluate the growth rate of the function $\varphi_{\mathfrak{C}}$ which counts for each integer $n$ the number $\varphi_{\mathfrak{C}}(n)$ of permutations $\sigma$ on $[n]$ which belong to $\mathfrak{C}$ (the Stanley-Wilf conjecture asserted that $\varphi_{\mathfrak{C}}$ is bounded by an exponential if $\mathfrak{C} \neq \mathfrak{S}$). For this purpose, simple permutations were introduced. A permutation $\sigma$ is *simple* if $\leq_\sigma$ and the natural order $\leq$ on $[n]$ have no nontrivial interval in common. Arbitrary permutations being obtained by means of simple permutations, the enumeration of permutations belonging to a hereditary class of permutations can be then reduced to the enumeration of simple permutations belonging to that class. This fact was illustrated in many papers ([1], [32], see also [4] for a survey on simple permutations and [2], where the asymptotic result mentioned in Theorem 7 is rediscovered).

Research on classes of finite permutations leads to the study of infinite bichains and particularly the prime one. But, in the infinite, primality and orthogonality no longer coincide. In [55], it was proved that the chain of the rational numbers admits an orthogonal linear order of the same order type. Pairs of orthogonal countable well-ordered chains have been described recently [45].

Relational systems of equivalence relations are prototypes of semirigid systems, since, if a set $E$ has at least three elements, only the constant functions and the identity preserve all equivalence relations on $E$. From this it follows that if a set $\{\rho_i : i \in I\}$ generates by means of joins and meets the lattice of equivalences on $E$ then $M := (E, (\rho_i)_{i \in I})$ is semirigid. The converse does not hold. Indeed, according to Strietz [57], if $E$ is finite with at least four elements, four equivalences are needed to generate the lattice of equivalence relations on $E$ and Zádori [60] has described for every set $E$, whose size $|E|$ is finite and distinct from 2 and 4, a semirigid system made of three equivalence relations. A general method of constructing semirigid systems of three equivalence relations on sets of cardinality at most the continuum and distinct from 2 and 4 is developed in [12].

### C. Projectivity

Corominas [6] introduced the notion of 2-projectivity for ordered sets (posets). For posets, 2-projectivity is equivalent to $n$-projectivity [44]. Larose, then a student Professor Rosenberg, showed that for $P$ with at least three elements, projectivity is equivalent to the apparently weaker notion of quasi-projectivity[1] (projections beeing replaced by quasi-projections, i.e., maps $f$ such that $f(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\}$) [33], [34]. With Tardif [35] he obtained the same conclusion for graph (with no loops). More results of the same vein were obtained by Hazan [30], another student of Professor Rosenberg, and by Delhommé [11] for posets and reflexive graphs. In [44] it was shown that a relational structure $R$ on a set $A$ is 2-projective and not $n$-projective for some $n > 2$ if and only if its clone is the clone of maps which preserve the congruences of a 2-elementary group on a set of at least four elements. As a consequence, a proof of a form of Arrow's theorem on social choice is obtained.

## IV. Conclusion

In "Ivo G. Rosenberg, *A commemoration of his outstanding research contributions and friendship*", (IEEE Computer Society, 2012), several colleagues wrote wonderful tributes about Ivo Rosenberg as a mathematician and even more about his exceptional qualities. Ivo will always be remembered not only for his huge contributions to mathematics, but also for all his human values, for the ways he treated colleagues, visitors, young researchers and graduate students. He will be remembered for his generosity, his modesty and for his extreme kindness. As our late colleague Dietlinde Lau described him, Ivo Rosenberg was a *real gentleman of the old school.* May his soul rest in peace.

## References


[1] M.H. Albert, M.D. Atkinson. *Simple permutations and pattern restricted permutations. Discrete Mathematics*, **300** (2005) 1–15.

[2] M.H. Albert, M.D. Atkinson, M. Klazar. *The enumeration of simple permutations.* Journal of integer sequences, Vol. 6 (2003), Article 03.4.4.

[3] J. Berman, A. Kisielewicz, On the number of operations in a clone, *Proc. Amer. Math. Soc.* **122** (1994) 359–369.


[1] Also called *quasitriviality* or *conservativeness* in the literature.

[4] R. Brignall. A survey of simple permutations. Permutation patterns. 41-65, London Math. Soc. Lecture Note Ser., 376, Cambridge Univ. Press, Cambridge, 2010.

[5] K. N. Čimev, On some properties of functions, in: B. Csákány, I. Rosenberg (eds.) *Finite Algebra and Multiple-Valued Logic,* Abstracts of lectures of the colloquium on finite algebra and multiple-valued logic (Szeged, 1979), North-Holland, 1981, pp. 38–40.

[6] E. Corominas. *Sur les ensembles ordonnés projectifs et la propriété du point fixe*. C. R. Acad. Sci. Paris Sér. I Math. 311 (1990), no. 4, 199–204

[7] J. Demetrovics, M. Miyakawa, I. G. Rosenberg, D. A. Simovici, I. Stojmenović. *Intersections of isotone clones on a finite set.* Proc. 20th Internat. Symp. Multiple-valued Logic. Charlotte, NC, 1990. 248–253.

[8] M. Couceiro, L. Haddad, I.G. Rosenberg. Partial clones containing all Boolean monotone self-dual partial functions. *J. of Mult.-Valued Logic and Soft Computing.*, Vol 27, pp 183 – 192 (2016).

[9] M. Couceiro, L. Haddad, K. Schölzel, T. Waldhauser. A Solution to a Problem of D. Lau: Complete Classification of Intervals in the Lattice of Partial Boolean Clones, *J. of Mult.-Valued Logic and Soft Computing*, **28** (1) pp 47–58 (2017).

[10] M. Couceiro, E. Lehtonen, T. Waldhauser. A survey on the arity gap, *Journal of Multiple-Valued Logic and Soft Computing* **24**:1–4 (2015) 223–249

[11] C. Delhommé. *Propriétés de projection des graphes sans triangle ni carré*. European J. Combin. 17 (1996), no. 1, 15–22.

[12] C. Delhommé, M. Miyakawa, M. Pouzet, I. G. Rosenberg, H. Tatsumi. Semirigid Systems of Three Equivalence Relations. *Multiple-Valued Logic and Soft Computing* **28**:4–5 (2017) 511–535.

[13] A. Ehrenfeucht, T. Harju, G. Rozenberg. *The theory of 2-structures: a framework for decomposition and transformation*, World Scientific, 1999.

[14] R. Fraïssé. *On a decomposition of relations which generalizes the sum of ordering relations.* Bull. Amer. Math. Soc., 59:389, 1953.

[15] R. Fraïssé. *L'intervalle en théorie des relations; ses généralisations; filtre intervallaire et clôture d'une relation. (French) [The interval in relation theory; its generalizations; interval filter and closure of a relation].* Orders: description and roles. (L'Arbresle, 1982), 313–341, North-Holland Math. Stud., 99, North-Holland, Amsterdam, 1984.

[16] R. V. Freivald. Completness criteria for functions of the algebra of logic and many-valued logics. *Dokl. Akad. Nauk. SSSR*, 167, **6** (1966) 1249–1250.

[17] T. Gallai. *Transitiv orientbare graphen*. Acta Math. Acad. Sci. Hungar. 18 (1967) 25-66 (English translation by F. Maffray and M. Preissmann in J.J. Ramirez-Alfonsin, B. Reed (Eds), Perfect graphs, Wiley 2001, 25–66.

[18] L. Haddad, I.G. Rosenberg, Critère général de complétude pour les algèbres partielles finies, *C. R. Acad. Sci.* **304**, (1987), Ser 1, No 17, 507–509.

[19] L. Haddad, I.G. Rosenberg, Maximal, partial clones determined by areflexive relations. *Discrete Appl. Math.* 24 (1989) 133–143.

[20] L. Haddad, I. G. Rosenberg. Finite Clones Containing all Permutations. *Can. Journ. Math.* 46 (1990) 34–41.

[21] L. Haddad, I. G. Rosenberg, Partial Sheffer Operations. *European Journal of Combinatorics* 12 (1991) 9–21.

[22] L. Haddad, I. G. Rosenberg, Completeness theory for finite partial algebras *Algebra Universalis* **29** (1992) 378–401.

[23] L. Haddad, I. G. Rosenberg. Partial Clones Containing all Permutations. *Bull. Austral. Math. Soc.* **52** (1995) 263–278.

[24] L. Haddad. Foreword: Ivo G. Rosenberg's 65th Birthday, *Multiple-Valued Logic, An International Journal* **5**:3 (2000).

[25] L. Haddad I.G. Rosenberg, H. Machida. Maximal and Minimal Partial Clones, *Journal of Automata, Languages and Combinatorics* **7**:1 (2002) 83–93.

[26] L. Haddad, D. Lau, I. G. Rosenberg. Intervals of partial clones containing Maximal Clones. *Journal of Automata, Languages and Combinatorics* **11**:4 (2006) 399–421.

[27] L. Haddad, I.G. Rosenberg, H. Machida. Monoidal Intervals of Partial Clones, *Proc.* 37th *IEEE International Symposium on Multiple-Valued Logic*, Oslo, Norway, May 2007.

[28] L. Haddad, M. Pouzet. Some Aspects of the Mathematics of Ivo Rosenberg. In Ivo G. Rosenberg, *A commemoration of his outstanding research contributions and friendship, IEEE Computer Society*, 2012.

[29] L. Haddad. Infinite chains of partial clones containing all self-dual monotonic partial functions. *J. of Mult.-Valued Logic and Soft Computing*, **18**:2 (2011) 139–152.

[30] S. Hazan. On triangle-free projective graphs. Algebra Universalis **35**:2 (1996) 185–196.

[31] P. Jeavons, D. Cohen. An Algebriac Characterization of Tractable Constraints. Computing and combinatorics 633–642, *Lecture Notes in Comput. Sci.* **959** (1995) 633–642, Springer, Berlin.

[32] M. Klazar. *Overview of general results in combinatorial enumeration. ArXiv: 0803.4292V1* [math.co], Mar 2008.

[33] B. Larose. Finite projective ordered sets. *Order* **8**:1 (1991) 33–40.

[34] B. Larose. A property of projective ordered sets. *European J. Combin.* **13**:5 (1992) 371–378.

[35] B. Larose, C. Tardif. Strongly rigid graphs and projectivity. Ivo G. Rosenberg's 65th birthday, Part 2. *Mult.-Valued Log.* **7**:5–6 (2001) 339–361.

[36] V. Lashkia, M. Miyakawa, A. Nozaki, G. Pogosyan, I.G. Rosenberg. Semirigid sets of diamond orders. *Discrete Math.* **156**:1-3 (1996) 277–283, (1996).

[37] D. Lau. *Function Algebras on Finite Sets*, a basic course on Multiple-Valued Logic and Clone Theory, 670 pages, Springer Monograph in Mathematics, 2006.

[38] H. Machida, T. Hikita, Honouring Ivo. G. Rosenberg: His contribution to ISMVL. *Proc.* 42nd *IEEE International Symposium on Multiple-Valued Logic*, Victoria, Canada, 322–330, May (2012).

[39] A. Marcus, G. Tardös. *Excluded permutation matrices and the Stanley-Wilf conjecture*, *J. Combin. Theory*, Ser. A 107 (2004), 153–160.

[40] M. Miyakawa, M. Pouzet, I.G. Rosenberg, H. Tatsumi. Semirigid equivalence relations on a finite set. *J. Mult.-Valued Logic Soft Comput.* **15**:4 (2009) 395–407.

[41] A. Nozaki, M. Miyakawa, G. Pogosyan and I. G. Rosenberg. The number of orthogonal permutations. *Europ. J. Combinatorics* **16** (1995) 71–85.

[42] D. Oudrar, M. Pouzet. *Profile and hereditary classes of relational structures*, Proceedings ISOR'11, International Symposium on Operational Research, Algiers , Algeria , May 30-June 2, 2011, H.Ait Haddadene, I.Bouchemakh, M.Boudar, S.Bouroubi (Eds)LAID3.

[43] M. Pouzet and I.G. Rosenberg. General metrics and contracting operations. *Discrete Math.* **130**:1–3 (1994) 103–169.

[44] M. Pouzet, I.G. Rosenberg and M.G. Stone. A projection property. *Algebra Universalis* **36**:2 (1996) 159–184.

[45] C. Laflamme, M. Pouzet, N. Sauer and I. Zaguia, Pairs of countable orthogonal ordinals. *Discreet Math.* **335** (2014) 35 – 44.

[46] R. W. Quackenbush, I. Rival and I.G. Rosenberg. Clones, order varieties, near unanimity functions and holes. *Order* **7**:3 (1990) 239– 247.

[47] A. Pultr. Concerning universal categories. *Comment. Math. Univ. Carolinae* **5** (1964) 227–239.

[48] I. Rival, N. Zaguia. Perpendicular orders. *Discrete Math.* **137**:1–3 (1995) 303–313.

[49] I.G. Rosenberg, La structure des fonctions de plusieurs variables sur un ensemble fini, *C.R. Acad. Sci. Paris*, Sér A-B, **260** (1965) 3817–3819.

[50] I. G. Rosenberg, Über die funktionale Vollständigkeit in den mehrwertigen Logiken. *Rozpravy Československé Akad. Věd. Řada Mat. Přírod. Věd* **80** (1970) 3–93.

[51] I. G. Rosenberg, Composition of functions on finite sets, completeness and relations, a short survey. In D. Rine (ed.) *Multiple-valued Logic and Computer Science*, 2nd edition, North-Holland, Amsterdam (1984) 150 – 192.

[52] I. G. Rosenberg. Strongly rigid relations. *Rocky Mountain J. Math.* **3**:4 (1973) 631–639.

[53] I. G. Rosenberg. Minimal clones I: the five types. *Coll. Math. Soc. János Bolyai*, **43**, North-Holland, Amsterdam (1986) 405–427.

[54] A. Salomaa, On essential variables of functions, especially in the algebra of logic, *Ann. Acad. Sci. Fenn. Ser. A I. Math.* **339** (1963) 3–11.

[55] N. Sauer and I. Zaguia. The order on the rationals has an orthogonal order with the same order type. *Order* **28** (2011) 377–385.

[56] K. Schölzel, Number of maximal partial clones. *Proc.* 40th *IEEE International Symposium on Multiple-Valued Logic*, Barcelona, Spain, 288–293 (2010).

[57] H. Strietz. Über Erzeugendenmengen endlicher Partitionenverbände, *Studia Scien. Math.* **12** (1977) 1–17.

[58] P. Vopěnka P, A. Pultr and Z. Hedrlín. A rigid relation exists on any set. *Comment. Math. Univ. Carolinae* **6** (1965) 149–155.

[59] R. Willard, Essential arities of term operations in finite algebras, *Discrete Math.* **149** (1996) 239–259.

[60] L. Zádori. Generation of finite partition lattices. *Lectures in Universal Algebra* (Proc. Colloq., Szeged, 1983), Colloq. Math. Soc. János Bolyai 43, North-Holland, Amsterdam (1986) 573–586.

[61] I. Zaguia. Prime two-dimensional orders and perpendicular total orders. *Europ. J. of Combinatorics* **19** (1998) 639–649.