Fixed-time estimation of parameters for non-persistent excitation

Jian Wang, Denis Efimov, Stanislav Aranovskiy, Alexey Bobtsov

▶ **To cite this version:**

Jian Wang, Denis Efimov, Stanislav Aranovskiy, Alexey Bobtsov. Fixed-time estimation of parameters for non-persistent excitation. European Journal of Control, Elsevier, 2020, 55, pp.24-32. 10.1016/j.ejcon.2019.07.005 . hal-02196637

HAL Id: hal-02196637

https://hal.inria.fr/hal-02196637

Submitted on 29 Jul 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Fixed-time estimation of parameters for non-persistent excitation

J. Wang, D. Efimov, S. Aranovskiy, A.A. Bobtsov

Abstract

The problem of estimation in the linear regression model is studied under the hypothesis that the regressor may be excited on a limited initial interval of time only. Then the estimation solution is searched on a finite interval of time also based on the framework of finite-time or fixed-time converging dynamical systems. The robustness issue is analyzed and a short-time input-to-state stability property is introduced for fixed-time converging time-varying systems with a sufficient condition, which is formulated with the use of a Lyapunov function. Several estimation algorithms are proposed and compared with existing solutions. The performance of the estimators is demonstrated in numerical experiments.

I. Introduction

One of the basic and the most popular problems in the theory of identification and estimation is the parameter estimation in a linear regression model:

$$y(t) \quad = \quad \omega^{\top}(t)\theta + w(t),\ t \in \mathbb{R},$$

where $\theta \in \mathbb{R}^n$ is the vector of unknown constant parameters that is necessary to find, $\omega : \mathbb{R} \to \mathbb{R}^n$ is the regressor function (usually assumed to be bounded and known), $y(t) \in \mathbb{R}$ is the signal available for measurements with a measurement noise $w : \mathbb{R} \to \mathbb{R}$ (here $\mathbb{R}$ denotes the set of real numbers). There are plenty of methods to solve this problem that need a complete statistics on the process (in other words these tools are mainly oriented on estimation off-line): the linear least squares, the maximum-likelihood estimation, the Bayesian linear regression, the principal component regression [1], [2], to mention a few. In the theory of adaptive control and identification there exist also many methods for adaptive and on-line estimation [3], [4], [5], [6], and applicability of many of them is based on the condition of persistence of excitation [7], [8], thus these approaches are also implicitly based on the asymptotic statistics. Recently, several concepts have been proposed to relax the requirement on the excitation [9], [10], [11], with improved estimation algorithms [12], [13], [14], [15], which require only an interval estimation of the regressor $\omega(t)$.

Considering convergence on a finite interval, the amplitude of the initial error becomes of great importance, since if this deviation is not bounded, then it is complicated to ensure global convergence of the estimates to their ideal values in a limited time. A notion that overcomes this drawback has been proposed recently, and it is called fixed-time or predefined-time stability or convergence [16], [17], [18], [19]. The objective of the present work is to design several fixed-time convergent algorithms solving the parameter estimation problem in a linear regression model, which are independent in the initial guesses for the values of parameters and regressor excitation[1], and whose robustness against the measurement noise is assessed using the input-to-state stability theory. To this end, a new notion of fixed-time input-to-state stability on a short interval of time is proposed together with a Lyapunov function analysis. The introduced approach is based on the dynamic regressor extension and mixing (DREM) method [21], [15], which allow the vector estimation problem to be decoupled on a series of scalar ones. The obtained solutions are compared with ones from [12], [13], [22].

The outline of this note is as follows. Some preliminary results are introduced in section II. The problem statement is given in Section III. The estimation algorithms are designed in Section IV, where also the convergence and robustness conditions are established. Simple illustrating examples are considered in Section V.

Jian Wang is with Hangzhou Dianzi University, Hangzhou, China.
Denis Efimov is with Inria, Univ. Lille, CNRS, UMR 9189 - CRIStAL, F-59000 Lille, France.
Stanislav Aranovskiy is with CentaleSupélec –IETR, Avenue de la Boulaie, 35576 Cesson-Sévigné, France.
Alexey A. Bobtsov is with ITMO University, 49 av. Kronverkskiy, 197101 Saint Petersburg, Russia.
This work was partially supported by 111 project No. D17019 (China) and by the Government of Russian Federation (Grant 08-08).

[1] In our previous study [20] only a finite-time convergence has been guaranteed, therefore, the synthesized algorithms use an upper bound on the initial error.

Notation

- $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, where $\mathbb{R}$ is the set of real number.
- $|x|$ denotes the absolute value for $x \in \mathbb{R}$ or a vector norm for $x \in \mathbb{R}^n$, and the corresponding induced matrix norm for a matrix $A \in \mathbb{R}^{n\times n}$ is denoted by $\|A\|$.
- For a Lebesgue measurable and essentially bounded function $x : \mathbb{R} \to \mathbb{R}^n$ denote $\|x\|_\infty = \operatorname{ess\,sup}_{t\in\mathbb{R}} |x(t)|$, and define by $\mathcal{L}_\infty(\mathbb{R}, \mathbb{R}^n)$ the set of all such functions with finite norms $\|\cdot\|_\infty$; if
$$\int_{-\infty}^{+\infty} |x(t)|^2 dt < +\infty$$
then this class of functions is denoted by $\mathcal{L}_2(\mathbb{R}, \mathbb{R}^n)$.
- A continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}$ if $\alpha(0) = 0$ and the function is strictly increasing, a function $\alpha \in \mathcal{K}$ belongs to the class $\mathcal{K}_\infty$ if it is increasing to infinity. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$ for each fixed $t \in \mathbb{R}_+$ and $\beta(s, \cdot)$ is decreasing and $\lim_{t\to+\infty} \beta(s, t) = 0$ for each fixed $s \in \mathbb{R}_+$; a function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{GKL}$ if $\beta(s, 0) \in \mathcal{K}$, $\beta(s, \cdot)$ is decreasing and for each $s \in \mathbb{R}_+$ there is $T_s \in \mathbb{R}_+$ such that $\beta(s, t) = 0$ for all $t \geq T_s$.
- The identity matrix of dimension $n \times n$ is denoted as I_n.
- A sequence of integers $1, 2, ..., n$ is denoted by $\overline{1, n}$.
- Define $\mathbf{e} = \exp(1)$.
- Define the Lambert function Lambert : $\mathbb{R} \to \mathbb{R}$, also called the omega function or product logarithm, as the branches of the inverse relation of the function $f(z) = z\mathbf{e}^z$ for $z \in \mathbb{R}$.
- Denote $\lceil s\rfloor^\alpha = |s|^\alpha \operatorname{sign}(s)$ for any $s \in \mathbb{R}$ and $\alpha \in \mathbb{R}_+$.

II. Preliminaries

Consider a time-dependent differential equation:

$$dx(t)/dt = f(t, x(t), d(t)),\ t \geq t_0,\ t_0 \in \mathbb{R}, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $d(t) \in \mathbb{R}^m$ is the vector of external disturbances and $d \in \mathcal{L}_\infty(\mathbb{R}, \mathbb{R}^m)$; $f : \mathbb{R}^{n+m+1} \to \mathbb{R}^n$ is a continuous function with respect to x, d and piecewise continuous with respect to t, $f(t, 0, 0) = 0$ for all $t \in \mathbb{R}$. A solution of the system (1) for an initial condition $x_0 \in \mathbb{R}^n$ at time instant $t_0 \in \mathbb{R}$ and some $d \in \mathcal{L}_\infty(\mathbb{R}, \mathbb{R}^m)$ is denoted as $X(t, t_0, x_0, d)$, and we assume that f ensures definiteness and uniqueness of solutions $X(t, t_0, x_0, d)$ in forward time at least on some finite time interval $[t_0, t_0 + T)$, where $T > 0$ may be dependent on the initial condition x_0, the input d and the initial time t_0.

A. Stability definitions

Let Ω, Ξ be open neighborhoods of the origin in $\mathbb{R}^n$, $0 \in \Omega \subset \Xi$, then following [23], [14] introduce several stability notions:

Definition 1. At the steady state $x = 0$ the system (1) with $d = 0$ is said to be

(a) *short-time stable* with respect to (Ω, Ξ, T^0, T_f) if for any $x_0 \in \Omega$ and $t_0 \in [-T^0, T^0]$, $X(t, t_0, x_0, 0) \in \Xi$ for all $t \in [t_0, t_0+T_f]$;

(b) *short-finite-time stable* with respect to (Ω, Ξ, T^0, T_f) if it is short-time stable with respect to (Ω, Ξ, T^0, T_f) and finite-time converging from Ω with the convergence time $T^{t_0,x_0} \leq t_0 + T_f$ for all $x_0 \in \Omega$ and $t_0 \in [-T^0, T^0]$;

(c) *globally short-finite-time stable* for $T^0 > 0$ if for any bounded set $\Omega \subset \mathbb{R}^n$ containing the origin there exist a bounded set $\Xi \subset \mathbb{R}^n$, $\Omega \subset \Xi$ and $T_f > 0$ such that the system is short-finite-time stable with respect to (Ω, Ξ, T^0, T_f);

(d) *short-fixed-time stable* for $T^0 > 0$ and $T_f > 0$, if for any bounded set $\Omega \subset \mathbb{R}^n$ containing the origin there exists a bounded set $\Xi \subset \mathbb{R}^n$, $\Omega \subset \Xi$ such that the system is short-finite-time stable with respect to (Ω, Ξ, T^0, T_f).

The notions of Definition 1 can be equivalently formulated using the functions from the classes $\mathcal{K}$ and $\mathcal{GKL}$, *e.g.* the system (1) with $d = 0$ is globally short-fixed-time stable for $T^0 > 0$ and $T_f > 0$ at the origin if there exists $\beta \in \mathcal{GKL}$ such that for all $x_0 \in \mathbb{R}^n$ and $t_0 \in [-T^0, T^0]$:

$$|X(t, t_0, x_0, 0)| \leq \beta(|x_0|, t - t_0) \quad \forall t \in [t_0, t_0 + T_f]$$

and $\beta(|x_0|, T_f) = 0$. In addition, if (1) is short-fixed-time stable, then it is also globally short-finite-time stable. In [24], [25], [26], [27] the short-time stability is considered for a fixed initial time instant t_0 only. The notion of short-fixed-time stability has not been defined in [23], [14].

Remark 1. In the literature, short-time stability [26] is frequently called stability over a finite interval of time [24], [25], [27], but following [23], we prefer here the former notion to avoid a confusion with finite-time stability from [28], [29], since both concepts of stability are used in the paper.

B. Robust stability definitions

Consider the following definition of robust stability for (1) with $d \neq 0$:

Definition 2. [14] The system (1) is said to be

(a) *short-finite-time ISS* with respect to (Ω, T^0, T_f, D) if there exist $\beta \in \mathcal{GKL}$ and $\gamma \in \mathcal{K}$ such that for all $x_0 \in \Omega$, all $d \in \mathcal{L}_\infty(\mathbb{R}, \mathbb{R}^m)$ with $\|d\|_\infty < D$ and $t_0 \in [-T^0, T^0]$:

$$|X(t, t_0, x_0, d)| \leq \beta(|x_0|, t - t_0) + \gamma(\|d\|_\infty) \quad \forall t \in [t_0, t_0 + T_f]$$

and $\beta(|x_0|, T_f) = 0$;

(b) *globally short-finite-time ISS* for $T^0 > 0$ if there exist $\beta \in \mathcal{GKL}$ and $\gamma \in \mathcal{K}$ such that for any bounded set $\Omega \subset \mathbb{R}^n$ containing the origin there is $T_f > 0$ such that for all $x_0 \in \Omega$, all $d \in \mathcal{L}_\infty(\mathbb{R}, \mathbb{R}^m)$ and $t_0 \in [-T^0, T^0]$:

$$|X(t, t_0, x_0, d)| \leq \beta(|x_0|, t - t_0) + \gamma(\|d\|_\infty) \quad \forall t \in [t_0, t_0 + T_f]$$

and $\beta(|x_0|, T_f) = 0$ (the system is short-finite-time ISS with respect to $(\Omega, T^0, T_f, +\infty)$);

(c) *short-fixed-time ISS* for $T^0 > 0$ and $T_f > 0$, if there exist $\beta \in \mathcal{GKL}$ and $\gamma \in \mathcal{K}$ such that for all $x_0 \in \mathbb{R}^n$, all $d \in \mathcal{L}_\infty(\mathbb{R}, \mathbb{R}^m)$ and $t_0 \in [-T^0, T^0]$:

$$|X(t, t_0, x_0, d)| \leq \beta(|x_0|, t - t_0) + \gamma(\|d\|_\infty) \quad \forall t \in [t_0, t_0 + T_f]$$

and $\beta(|x_0|, T_f) = 0$ (for any bounded set $\Omega \subset \mathbb{R}^n$ containing the origin the system is short-finite-time ISS with respect to $(\Omega, T^0, T_f, +\infty)$).

As we can conclude from Definition 2, if the system (1) is short-finite-time ISS, then for $d = 0$ there exists $\Xi \subset \mathbb{R}^n$, $\Omega \subset \Xi$ such that the system is short-finite-time stable with respect to (Ω, Ξ, T^0, T_f). Again, if (1) is short-fixed-time ISS, then it is also globally short-finite-time ISS. The difference of global short-finite-time or short-fixed-time ISS and a conventional (finite-time or fixed-time) ISS [30], [31] is that in the former case the stability property is considered on a finite interval of time $[t_0, t_0 + T_f]$ only.

Theorem 1. *Let the constants $T^0 \geq 0$ and $T_f > 0$ be given. Let the system in* (1) *possess a Lyapunov function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_+$ such that for all $x \in \mathbb{R}^n$, $d \in \mathbb{R}^m$ and $t \in [-T^0, T^0 + T_f]$:*

$$\begin{gathered}\alpha_1(|x|) \leq V(t, x) \leq \alpha_2(|x|) \quad \alpha_1, \alpha_2 \in \mathcal{K}_\infty; \\ \dot{V}(t, x) \leq -u(t)\left(V^{1-\eta}(t, x) + V^{1+\eta}(t, x)\right) + \kappa(|d|)\end{gathered} \tag{2}$$

for $\kappa \in \mathcal{K}$, $\eta \in (0, 1)$ and a function $u : \mathbb{R} \to \mathbb{R}_+$ satisfying

$$\int_t^{t+\ell} u(s)ds \geq \upsilon > 0, \ \ell > 0$$

for all $t \in [-T^0, T^0 + T_f]$. Then the system (1) *is short-fixed-time ISS for T^0 and T_f with*

$$\begin{gathered}\beta(s, t) = \alpha_1^{-1}(2^{\eta^{-1}} \min\{\max\{0, \sqrt[\eta]{\alpha_2^\eta(s) - \eta\frac{\upsilon}{4\ell}(t - \ell)}\}, \\ \frac{1}{\sqrt[\eta]{\alpha_2^{-\eta}(s) + \eta\frac{\upsilon}{4\ell}(t - \ell)}}\}), \\ \gamma(s) = \alpha_1^{-1}[2^{\eta^{-1}}(\ell\kappa(s) + \max\{(\frac{4\ell}{\upsilon}\kappa(s))^{\frac{1}{1-\eta}}, (\frac{4\ell}{\upsilon}\kappa(s))^{\frac{1}{1+\eta}}\})]\end{gathered}$$

provided that

$$T_f \geq 2(1 + \frac{4}{\eta\upsilon})\ell.$$

Proof. First of all note that for any $t\in(0,T_f]$ such that $t=\nu\ell+\mu$ with an integer $\nu\geq 1$ and $\mu\in[0,\ell)$ we have:

$$\begin{aligned}\int_{t_0}^{t_0+t}u(t)dt&=\sum_{i=1}^{\nu}\int_{t_0+(i-1)\ell}^{t_0+i\ell}u(t)dt+\int_{t_0+\nu\ell}^{t_0+t}u(t)dt\\&\geq\sum_{i=1}^{\nu}\int_{t_0+(i-1)\ell}^{t_0+i\ell}u(t)dt\geq\nu\upsilon=\nu\upsilon\frac{t}{t}\\&=\upsilon\frac{\nu}{\nu\ell+\mu}t\geq\upsilon\frac{\nu}{\nu+1}\frac{t}{\ell}\geq\frac{\upsilon}{2\ell}t.\end{aligned}$$

Denote $V(t)=V(t,X(t,t_0,x_0,d))$ and define two sets of instants of time:

$$\mathcal{T}_1=\{t\in[t_0,t_0+T_f):\frac{4\ell}{\upsilon}\kappa(\|d\|_\infty)<V^{1-\eta}(t)+V^{1+\eta}(t)\},$$
$$\mathcal{T}_2=\{t\in[t_0,t_0+T_f):\frac{4\ell}{\upsilon}\kappa(\|d\|_\infty)\geq V^{1-\eta}(t)+V^{1+\eta}(t)\},$$

then obviously $[t_0,t_0+T_f)=\mathcal{T}_1\cup\mathcal{T}_2$. Next, consider $t\in[t_1,t_2)\subset\mathcal{T}_1$, then

$$\begin{aligned}\dot V(t)&\leq&-\left(u(t)-\frac{\upsilon}{4\ell}\right)\left(V^{1-\eta}(t)+V^{1+\eta}(t)\right)\\&\leq&-\left(u(t)-\frac{\upsilon}{4\ell}\right)\max\{V^{1-\eta}(t),V^{1+\eta}(t)\}\end{aligned}$$

and integrating this inequality we obtain:

$$\begin{aligned}V^{\eta}(t)&\leq&\min\{\max\{0,V^{\eta}(t_1)-\eta\int_{t_1}^{t}\left(u(s)-\frac{\upsilon}{4\ell}\right)ds\},\\&&\frac{1}{V^{-\eta}(t_1)+\eta\int_{t_1}^{t}\left(u(s)-\frac{\upsilon}{4\ell}\right)ds}\}\end{aligned}$$

for all $t\in[t_1,t_2)$. Thus,

$$\begin{aligned}V^{\eta}(t)&\leq&\min\{\max\{0,V^{\eta}(t_1)-\eta\frac{\upsilon}{4\ell}(t-t_1)\},\\&&\frac{1}{V^{-\eta}(t_1)+\eta\frac{\upsilon}{4\ell}(t-t_1)}\}\end{aligned}$$

for all $t\in[t_1+\ell,t_2)$. From the inequality

$$\dot V(t)\leq\kappa(|d|)$$

for all $t\in[t_1,t_1+\ell)$ we get:

$$V(t)\leq V(t_1)+\kappa(\|d\|_\infty)(t-t_1)$$

or

$$V^{\eta}(t)\leq V^{\eta}(t_1)+\ell^{\eta}\kappa^{\eta}(\|d\|_\infty).$$

Therefore,

$$\begin{aligned}V^{\eta}(t)&\leq&\min\{\max\{0,V^{\eta}(t_1)-\eta\frac{\upsilon}{4\ell}(t-t_1-\ell)\},\\&&\frac{1}{V^{-\eta}(t_1)+\eta\frac{\upsilon}{4\ell}(t-t_1-\ell)}\}+\ell^{\eta}\kappa^{\eta}(\|d\|_\infty)\end{aligned}$$

for all $t\in[t_1,t_2)\subset\mathcal{T}_1$, then the estimate that is satisfied for all $t\in[t_0,t_0+T_f)=\mathcal{T}_1\cup\mathcal{T}_2$ is

$$\begin{aligned}V^{\eta}(t)&\leq&\min\{\max\{0,V^{\eta}(t_0)-\eta\frac{\upsilon}{4\ell}(t-t_0-\ell)\},\\&&\frac{1}{V^{-\eta}(t_0)+\eta\frac{\upsilon}{4\ell}(t-t_0-\ell)}\}+\ell^{\eta}\kappa^{\eta}(\|d\|_\infty)\\&&+\max\left\{\left(\frac{4\ell}{\upsilon}\kappa(\|d\|_\infty)\right)^{\frac{\eta}{1-\eta}},\left(\frac{4\ell}{\upsilon}\kappa(\|d\|_\infty)\right)^{\frac{\eta}{1+\eta}}\right\}\end{aligned}$$

and the system is ISS as needed for the given β and γ. The constraint on T_f follows from the estimate on the fixed time of convergence for the case $\|d\|_\infty=0$. Indeed, $V^{\eta}(t)\leq 1$ for all $t\geq t_0+T_1$ with $T_1=(1+\frac{4}{\eta\upsilon})\ell$ using the term with the power $1+\eta$, and starting from the unit value $V(t)=0$ for all $t\geq t_0+T_1+T_2$ with $T_2=(1+\frac{4}{\eta\upsilon})\ell$ using the term with the power $1-\eta$. Then $T_f\leq T_1+T_2$. $\square$

The case without the second term $V^{1+\eta}$ has been analyzed in [14]:

Theorem 2. *[14] Let the constants $\varrho>0$, $D>0$, $T^0,T_f\in\mathbb{R}_+$ and $\ell>0$ be given. Let the system in* (1) *possess a Lyapunov function $V:\mathbb{R}\times\Omega\to\mathbb{R}_+$, where $\Omega=\{x\in\mathbb{R}^n:|x|\leq\varrho\}$, such that for all $x\in\Omega$, all $|d|\leq D$ and all $t\in[-T^0,T^0+T_f]$:*

$$\alpha_1(|x|)\leq V(t,x)\leq\alpha_2(|x|)\quad \alpha_1,\alpha_2\in\mathcal{K}_\infty;$$
$$\dot V(t,x)\leq -u(t)V^{1-\eta}(t,x)+\kappa(|d|)\quad \kappa\in\mathcal{K},\ \eta\in(0,1)$$

for a function $u:\mathbb{R}\to\mathbb{R}_+$ satisfying

$$\int_t^{t+\ell}u(s)ds\geq\upsilon>0$$

for all $t\in[-T^0,T^0+T_f]$. Then the system (1) *is short-finite-time ISS with respect to (Ω',T^0,T_f,D) for*

$$\Omega'=\{x\in\Omega:\beta(|x|,0)+\gamma(D)\leq\varrho\},$$
$$\beta(s,t)=\alpha_1^{-1}\left(3^{\frac{1-\eta}{\eta}}4\left(\alpha_2^\eta(s)-\frac{\eta}{2^\eta}\frac{\upsilon}{4\ell}(t-\ell)\right)^{\frac{1}{\eta}}\right),$$
$$\gamma(s)=\alpha_1^{-1}\left(3^{\frac{1-\eta}{\eta}}4\left(\left(4\ell\upsilon^{-1}\kappa(s)\right)^{\frac{1}{1-\eta}}+\ell\kappa(s)\right)\right)$$

provided that

$$T_f\geq\ell[1+\frac{2^\eta}{\eta}\frac{4}{\upsilon}\sup_{x_0\in\Omega'}\alpha_2^\eta(|x_0|)].$$

In both theorems, the condition imposed on u is a version of PE for a finite interval of time. Finally, let us formulate a useful lemma:

Lemma 1. *[32] Let $x,y\in\mathbb{R}$ and $p>0$, then for any $\kappa_1\in(0,1)$ there exists $\kappa_2>0$ such that*

$$x\lceil x+y\rfloor^p\geq\kappa_1|x|^{p+1}-\kappa_2|y|^{p+1}.$$

In particular, $\kappa_2=\max\{1+\kappa_1,\frac{\kappa_1}{(1-\kappa_1^{1/p})^p}\}$.

C. Dynamic regressor extension and mixing method

Consider the estimation problem:

$$\begin{aligned} x(t) &= \omega^\top(t)\theta,\ t\in\mathbb{R}, \\ y(t) &= x(t)+w(t), \end{aligned} \tag{3}$$

where $x(t)\in\mathbb{R}$ is the model output, $\theta\in\mathbb{R}^n$ is the vector of unknown constant parameters that is necessary to estimate, $\omega:\mathbb{R}\to\mathbb{R}^n$ is the regressor function (usually assumed to be bounded and known), $y(t)\in\mathbb{R}$ is the signal available for measurements with a measurement noise $w:\mathbb{R}\to\mathbb{R}$. Introduce the following hypothesis:

Assumption 1. *Let $\omega\in\mathcal{L}_\infty(\mathbb{R},\mathbb{R}^n)$ and $w\in\mathcal{L}_\infty(\mathbb{R},\mathbb{R})$.*

As it has been proposed in [21], in order to overcome the limitations imposed by the condition that ω is PE and also to improve the transient performance, the DREM procedure transforms (3) to n new one-dimensional regression models, which allows the decoupled estimates of θ_i, $i=\overline{1,n}$ to be computed under a condition on the regressor ω that differs from the persistent excitation.

For this purpose $n-1$ linear operators $H_j:\mathcal{L}_\infty(\mathbb{R},\mathbb{R})\to\mathcal{L}_\infty(\mathbb{R},\mathbb{R})$ are introduced for $j=\overline{1,n-1}$ (for instance an operator H_j can be chosen as a stable linear time-invariant filter with the transfer function $W_j(s)=\frac{\alpha_j}{s+\beta_j}$, where $s\in\mathbb{C}$ is a complex variable and $\alpha_j\neq0$, $\beta_j>0$ are selected to filter the noise w in (3); or it can realize the delay operation with the transfer function $W_j(s)=e^{-\tau_j s}$ for $\tau_j>0$). Note that $y\in\mathcal{L}_\infty(\mathbb{R},\mathbb{R})$ under Assumption 1, then these operators are applied to the measured output $y(t)$ of (3), and using the superposition principles (the operators H_j are linear) we obtain:

$$\tilde y_j(t)=H_j(y(t))=\tilde\omega_j^\top(t)\theta+\tilde w_j(t),\ j=\overline{1,n-1},$$

where $\tilde{y}_j(t) \in \mathbb{R}$ is the j^{th} operator output, $\tilde{\omega}_j : \mathbb{R} \to \mathbb{R}^n$ is the j^{th} filtered regressor function and $\tilde{w}_j(t) : \mathbb{R} \to \mathbb{R}$ is the new j^{th} noise signal, which is composed by the transformation of the noise $w(t)$ by H_j and other exponentially converging components related to the initial conditions of the filters. By construction $\tilde{\omega}_j \in \mathcal{L}_\infty(\mathbb{R}, \mathbb{R}^n)$ and $\tilde{w}_j \in \mathcal{L}_\infty(\mathbb{R}, \mathbb{R})$ for all $j = \overline{1, n-1}$. Define new vector variables

$$\tilde{Y}(t) = [y(t)\ \tilde{y}_1(t) \ldots \tilde{y}_{n-1}(t)]^\top \in \mathbb{R}^n,$$
$$\tilde{W}(t) = [w(t)\ \tilde{w}_1(t) \ldots \tilde{w}_{n-1}(t)]^\top \in \mathbb{R}^n$$

and a time-varying matrix

$$M(t) = [\omega(t)\ \tilde{\omega}_1(t) \ldots \tilde{\omega}_{n-1}(t)]^\top \in \mathbb{R}^{n\times n},$$

then stacking the original equation (3) with the $n-1$ filtered regressor models we construct an extended regressor system:

$$\tilde{Y}(t) = M(t)\theta + \tilde{W}(t).$$

For any matrix $M(t) \in \mathbb{R}^{n\times n}$ the following equality is true [33]:

$$\text{adj}\,(M(t))\, M(t) = \det\,(M(t))\, I_n,$$

even if $M(t)$ is singular, where adj $(M(t))$ is the adjugate matrix of $M(t)$ and $\det\,(M(t))$ is its determinant. Recall that each element of the matrix adj $(M(t))$,

$$\text{adj}\,(M(t))_{k,s} = (-1)^{k+s}\mathbf{M}_{k,s}(t)$$

for all $k, s = \overline{1, n}$, where $\mathbf{M}_{k,s}(t)$ is the (k, s) minor of $M(t)$, *i.e.* it is the determinant of the $(n-1)\times(n-1)$ matrix that results from deleting the k^{th} row and the s^{th} column of $M(t)$. Define

$$Y(t) = \text{adj}\,(M(t))\,\tilde{Y}(t),\ W(t) = \text{adj}\,(M(t))\,\tilde{W}(t),$$
$$\phi(t) = \det\,(M(t))\,,$$

then multiplying from the left the extended regressor system by the adjugate matrix adj $(M(t))$ we get n scalar regressor models of the form:

$$Y_i(t) = \phi(t)\theta_i + W_i(t) \tag{4}$$

for $i = \overline{1, n}$. Again, by construction $Y \in \mathcal{L}_\infty(\mathbb{R}, \mathbb{R}^n)$, $W \in \mathcal{L}_\infty(\mathbb{R}, \mathbb{R}^n)$ and $\phi \in \mathcal{L}_\infty(\mathbb{R}, \mathbb{R})$. For the scalar linear regression model (4) the conventional gradient estimation algorithm takes the form:

$$\dot{\hat{\theta}}_i(t) = \gamma_i\phi(t)\left(Y_i(t) - \phi(t)\hat{\theta}_i(t)\right),\ \gamma_i > 0 \tag{5}$$

for all $i = \overline{1, n}$, where now the estimation processes for all components of θ are decoupled, and the adaptation gain γ_i can be adjusted separately for each element of θ. However, all these estimation algorithms are dependent on the same regressor $\phi(t)$ (determinant of $M(t)$).

Define the parameter estimation error as $e(t) = \theta - \hat{\theta}(t)$, then its dynamics admits the differential equation:

$$\dot{e}_i(t) = -\gamma_i\phi(t)\left(\phi(t)e_i(t) + W_i(t)\right),\ i = \overline{1, n} \tag{6}$$

and the following result can be proven for the DREM method:

Theorem 3. *[21], [14] Consider the linear regression system* (3) *under Assumption 1. Assume that for the selected operators* $H_j : \mathcal{L}_\infty(\mathbb{R}, \mathbb{R}) \to \mathcal{L}_\infty(\mathbb{R}, \mathbb{R})$, $j = \overline{1, n-1}$*:*

$$\int_{t_0}^{+\infty} \phi^2(t)dt = +\infty \tag{7}$$

for any $t_0 \in \mathbb{R}$*, then the estimation algorithm* (5) *has the following properties:*

(A) *If* $\|W\|_\infty = 0$*, then the system* (6) *is globally asymptotically stable at the origin iff* (7) *is valid.*

(B) *For all $W \in \mathcal{L}_2(\mathbb{R},\mathbb{R}^n)$ we have $e \in \mathcal{L}_\infty(\mathbb{R},\mathbb{R}^n)$, in addition, for all $t_0 \in \mathbb{R}$ and $e_i(t_0) \in \mathbb{R}$:*

$$\begin{aligned} |e_i(t)| \quad &\leq \quad \mathbf{e}^{-\gamma_i \int_{t_0}^{t} \phi^2(\tau)d\tau}|e_i(t_0)| \\ &+\sqrt{\frac{\gamma_i}{2}}\sqrt{1-\mathbf{e}^{-2\gamma_i \int_{t_0}^{t} \phi^2(\tau)d\tau}}\sqrt{\int_{t_0}^{t} W_i^2(s)ds} \end{aligned}$$

for all $t \geq t_0$ and $i = \overline{1,n}$.

Obviously, if the signal $\phi(t)$ is PE, then the error dynamics is ISS with respect to $W \in \mathcal{L}_\infty(\mathbb{R},\mathbb{R}^n)$ and an exponential convergence rate can be guaranteed [7], [8], [11].

III. Problem statement

Consider the static linear regression model (3) under Assumption 1, and assume that the DREM method has been applied in order reduce the initial problem of vector estimation to n scalar regressor models in the form (4). Note that $Y \in \mathcal{L}_\infty(\mathbb{R},\mathbb{R}^n)$, $W \in \mathcal{L}_\infty(\mathbb{R},\mathbb{R}^n)$ and $\phi \in \mathcal{L}_\infty(\mathbb{R},\mathbb{R})$ under Assumption 1 and due to properties of the DREM approach. Relaxing the condition (7), which is also imposed on an infinite interval of time, assume that $\omega(t)$ may not admit PE, and that $\phi(t)$ is a converging function of time (more precise properties will be formulated separately for the designed algorithms).

It is necessary to propose an algorithm generating an estimate $\hat{\theta}(t) \in \mathbb{R}^n$ of the vector of unknown parameters $\theta \in \mathbb{R}^n$, and for $\|W\|_\infty = 0$ providing the property of short-fixed-time stability (see Definition 1) of the estimation error $e(t) = \theta - \hat{\theta}(t)$ dynamics under assumptions 1 for some given T^0 and T_f. If $\|W\|_\infty \neq 0$ then short-fixed-time ISS for T^0 and T_f (see Definition 2) has to be guaranteed.

Since by applying DREM method the problem is decoupled on n independent ones, for brevity of notation, we will further omit the index i in (4) by assuming that $n = 1$:

$$Y(t) = \phi(t)\theta + W(t), \tag{8}$$

then $\theta \in \mathbb{R}$, $Y \in \mathcal{L}_\infty(\mathbb{R},\mathbb{R})$, $W \in \mathcal{L}_\infty(\mathbb{R},\mathbb{R})$ and $\phi \in \mathcal{L}_\infty(\mathbb{R},\mathbb{R})$.

IV. Design of estimation algorithms converging in short-finite-time

Two different solutions to the posed estimation problem are proposed in this section, whose difference consists in the requirements imposed on excitation of $\phi(t)$ and on the guaranteed robustness abilities with respect to $W(t)$.

A. Algorithm 1

Consider an adaptive estimation algorithm proposed in [34], [35]:

$$\begin{aligned} \dot{\hat{\theta}}(t) \quad &= \quad \phi(t)\{\gamma_1 \left\lceil Y(t) - \phi(t)\hat{\theta}(t) \right\rfloor^{1-\alpha} \\ &+\gamma_2 \left\lceil Y(t) - \phi(t)\hat{\theta}(t) \right\rfloor^{1+\alpha}\} \end{aligned} \tag{9}$$

for $\gamma_1 > 0$, $\gamma_2 > 0$ and $\alpha \in [0,1)$, with $\hat{\theta}(t_0) \in \mathbb{R}$, which admits the following properties:

Proposition 1. *Let Assumption 1 be satisfied, and for given $T^0 > 0$ and $T_f > 0$,*

$$\int_t^{t+\ell} \min\{|\phi(s)|^{2-\alpha}, |\phi(s)|^{2+\alpha}\}ds \geq \upsilon > 0 \tag{10}$$

for all $t \in [-T^0, T^0 + T_f]$ and some $\ell \in \left(0, \frac{T_f}{2}\right)$. Take

$$\min\{\gamma_1,\gamma_2\} > \frac{2^{2+\frac{\alpha}{2}}}{\alpha\upsilon\left(\frac{T_f}{2\ell} - 1\right)},$$

then the estimation error $e(t) = \theta - \hat{\theta}(t)$ dynamics of (9) is short-fixed-time ISS for T^0 and T_f.

Proof. The error dynamics for the estimation algorithm (9) can be written as follows:

$$\begin{aligned} \dot{e}(t) \quad &= \quad -\phi(t)\{\gamma_1 \lceil \phi(t)e(t) + W(t) \rfloor^{1-\alpha} \\ &+\gamma_2 \lceil \phi(t)e(t) + W(t) \rfloor^{1+\alpha}\}. \end{aligned}$$

Consider a Lyapunov function candidate $V(e)=0.5e^2$, whose derivative has an upper estimate for some $\kappa_{11},\kappa_{12}\in(0,1)$ and $\kappa_{21},\kappa_{22}>0$ coming from Lemma 1:

$$\begin{aligned}\dot{V}(t) &= -\gamma_1 e(t)\phi(t)\lceil\phi(t)e(t)+W(t)\rfloor^{1-\alpha}\\ &\quad -\gamma_2 e(t)\phi(t)\lceil\phi(t)e(t)+W(t)\rfloor^{1+\alpha}\\ &\le -\gamma_1\kappa_{11}|e(t)\phi(t)|^{2-\alpha}-\gamma_2\kappa_{12}|e(t)\phi(t)|^{2+\alpha}\\ &\quad +\gamma_1\kappa_{21}|W(t)|^{2-\alpha}+\gamma_2\kappa_{22}|W(t)|^{2+\alpha}\\ &\le -u(t)\left(V^{1-\frac{\alpha}{2}}(t)+V^{1+\frac{\alpha}{2}}(t)\right)+\sigma(|W(t)|)\end{aligned}$$

for any $e(t)\in\mathbb{R}$ and $W(t)\in\mathbb{R}$, where

$$u(t)=\min\{2^{1-\frac{\alpha}{2}}\gamma_1\kappa_{11}|\phi(t)|^{2-\alpha},2^{1+\frac{\alpha}{2}}\gamma_2\kappa_{12}|\phi(t)|^{2+\alpha}\},$$
$$\sigma(s)=\gamma_1\kappa_{21}s^{2-\alpha}+\gamma_2\kappa_{22}s^{2+\alpha}$$

is a function from class $\mathcal{K}_\infty$. Note that

$$u(t)\ge 2^{1-\frac{\alpha}{2}}\min\{\gamma_1\kappa_{11},\gamma_2\kappa_{12}\}\min\{|\phi(t)|^{2-\alpha},|\phi(t)|^{2+\alpha}\},$$

then under the imposed restrictions for ϕ, the system is short-fixed-time ISS for T^0 and T_f due to Theorem 1 provided that the constraint

$$T_f\ge 2(1+\frac{2^{2+\frac{\alpha}{2}}}{\alpha\min\{\gamma_1\kappa_{11},\gamma_2\kappa_{12}\}\upsilon})\ell$$

is satisfied. The imposed restriction on $\min\{\gamma_1,\gamma_2\}$ guarantees that there exist $\kappa_{11},\kappa_{12}\in(0,1)$ such that all trajectories converge to the origin faster than T_f if $\|W\|_\infty=0$. □

It is important to clarify the differences between the restrictions imposed on $\phi(t)$ in Theorem 3 and Proposition 1 for the case $\|W\|_\infty=0$. Note that (7) allows us to establish the fact of asymptotic convergence of the estimation error $e(t)$, but it does not permit to evaluate the rate of convergence. Of course, the condition (7) can be formulated on a finite interval of time:

$$\int_{t_0}^{t_0+T}\phi^2(t)dt=+\infty$$

that implicitly implies unboundedness of $\phi(t)$ contrarily to its admissible convergence in (10).

B. Algorithm 2

And, finally, let us introduce a version of the algorithm (9), which extends the nonlinear paradigm of the former by time-varying powers:

$$\begin{aligned}\dot{\hat{\theta}}(t)&=\text{sign}(\phi(t))\{\gamma_1\left\lceil Y(t)-\phi(t)\hat{\theta}(t)\right\rfloor^{\beta(t)}\\ &\quad+\gamma_2\left\lceil Y(t)-\phi(t)\hat{\theta}(t)\right\rfloor^{\varsigma+\beta(t)}\}\\ &\gamma_1>0,\ \gamma_2>0,\ \varsigma>1,\ \beta(t)=\frac{|\phi(t)|}{1+|\phi(t)|},\end{aligned}\tag{11}$$

where $\hat{\theta}(t_0)\in\mathbb{R}$. The idea of this design is that the power $\beta(t)$ is approaching zero together with the regressor $\phi(t)$, then the contribution of the regressor in the adaptation rate is proportional to $|\phi(t)|^{\beta(t)}$, which is strictly separated with zero even for a convergent regressor, the detailed proof of this fact is given below.

Proposition 2. *Let Assumption 1 be satisfied, and $\vartheta\in\mathcal{L}_\infty(\mathbb{R},\mathbb{R}_+)$ where $\vartheta(t)=\frac{W(t)}{\phi(t)}$. Then the estimation error $e(t)=\theta-\hat{\theta}(t)$ dynamics of (11) is short-finite-time ISS for any $T^0\ge 0$ with the input ϑ.*

Let additionally for given $T^0>0$ and $T_f>0$,

$$\int_t^{t+\ell}|\phi(s)|^\varsigma ds\ge\upsilon>0\tag{12}$$

for all $t\in[-T^0,T^0+T_f]$ and some $\ell\in(0,T_f)$, and

$$\min\{\gamma_1,\gamma_2\}>\sqrt{2}\frac{1+\phi_{\max}+\frac{4\ell}{(\varsigma-1)\upsilon}}{(T_f-\ell)g(x_{\min})},$$

where $\phi_{\max} = \max_{t\in[-T^0,T^0+T_f]} |\phi(t)|$, $g(x) = x^{\frac{x}{1+x}}$ *and* $x_{\min} = \text{Lambert}(\mathrm{e}^{-1})$, *then the estimation error* $e(t) = \theta - \hat{\theta}(t)$ *dynamics of* (11) *is short-fixed-time ISS for* T^0 *and* T_f *with the input* ϑ.

Proof. The error dynamics for the estimation algorithm (11) can be written as follows:

$$\dot{e}(t) = -\text{sign}(\phi(t))\{\gamma_1 \lceil \phi(t)e(t) + W(t)\rfloor^{\beta(t)} \\ +\gamma_2 \lceil \phi(t)e(t) + W(t)\rfloor^{\varsigma+\beta(t)}\}.$$

Consider a Lyapunov function $V(e) = 0.5e^2$ and observe that

$$r^{a(t)}(t) \geq \begin{cases} r^{a_{\min}}(t) & r(t) \geq 1 \\ r^{a_{\max}}(t) & r(t) < 1 \end{cases} \geq \min\{1, r_{\min}^{a_{\max}}\},$$
$$r^{a(t)}(t) \leq \begin{cases} r^{a_{\max}}(t) & r(t) \geq 1 \\ r^{a_{\min}}(t) & r(t) < 1 \end{cases} \leq \max\{1, r_{\max}^{a_{\max}}\}$$

for any $r : \mathbb{R} \to \mathbb{R}_+$ and $a : \mathbb{R} \to \mathbb{R}_+$ such that $r_{\min} = \inf_{t\in\mathbb{R}} r(t)$, $r_{\max} = \sup_{t\in\mathbb{R}} r(t)$, $a_{\min} = \inf_{t\in\mathbb{R}} a(t)$ and $a_{\max} = \sup_{t\in\mathbb{R}} a(t)$ for some $r_{\min}, r_{\max}, a_{\min}, a_{\max} \in \mathbb{R}_+$, then the time derivative of V for the estimation error dynamics admits an upper estimate:

$$\dot{V}(t) = -\text{sign}(\phi(t))e(t)\{\gamma_1 \lceil \phi(t)e(t) + W(t)\rfloor^{\beta(t)} \\ +\gamma_2 \lceil \phi(t)e(t) + W(t)\rfloor^{\varsigma+\beta(t)}\} \\ = -\gamma_1|\phi(t)|^{\beta(t)}e(t) \lceil e(t) + \vartheta(t)\rfloor^{\beta(t)} \\ -\gamma_2|\phi(t)|^{\varsigma+\beta(t)}e(t) \lceil e(t) + \vartheta(t)\rfloor^{\varsigma+\beta(t)} \\ \leq -\gamma_1\kappa_{11}|\phi(t)|^{\beta(t)}|e(t)|^{1+\beta(t)} - \gamma_2\kappa_{12}|\phi(t)|^{\varsigma+\beta(t)}|e(t)|^{1+\varsigma+\beta(t)} \\ +\gamma_1\kappa_{21}|\phi(t)|^{\beta(t)}|\vartheta(t)|^{1+\beta(t)} + \gamma_2\kappa_{22}|\phi(t)|^{\varsigma+\beta(t)}|\vartheta(t)|^{1+\varsigma+\beta(t)} \\ \leq -\sqrt{2}\min\{\gamma_1\kappa_{11}, \gamma_2\kappa_{12}\}g(|\phi(t)|)[\xi_1(V(t)) \\ +|\phi(t)|^{\varsigma}\xi_2(V(t))] + \sigma(\|\vartheta\|_\infty),$$

where

$$\xi_1(s) = \begin{cases} s^{0.5} & s \geq 1 \\ s^{\frac{0.5+\phi_{\max}}{1+\phi_{\max}}} & s < 1 \end{cases}, \ \xi_2(s) = \begin{cases} s^{\frac{1+\varsigma}{2}} & s \geq 1 \\ s^{\frac{1+\varsigma+\frac{\phi_{\max}}{1+\phi_{\max}}}{2}} & s < 1 \end{cases},$$
$$\sigma(s) = [\gamma_1\kappa_{21}\max\{1, \phi_{\max}^{\frac{\phi_{\max}}{1+\phi_{\max}}}\} \\ +\gamma_2\kappa_{22}\max\{1, \phi_{\max}^{\varsigma+\frac{\phi_{\max}}{1+\phi_{\max}}}\}s^{\varsigma}] \\ \times \begin{cases} s^{\frac{1+2\phi_{\max}}{1+\phi_{\max}}} & s \geq 1 \\ s & s < 1 \end{cases}$$

and $\kappa_{11}, \kappa_{12} \in (0,1)$, $\kappa_{21}, \kappa_{22} > 0$ are from Lemma 1 (since $\phi(t)$ is upper bounded in amplitude due to Assumption 1 such constants exist). Note that $g(0) = g(1) = 1$ and

$$\frac{\partial g(x)}{\partial x} = \left(1 + \ln(x)\frac{1}{1+x}\right)\frac{g(x)}{1+x},$$
$$\frac{\partial^2 g(x)}{\partial x^2} = \frac{x^2\ln^2(x) + x(x+1)^2}{xg(x)(x+1)^4},$$

then $\frac{\partial^2 g(x)}{\partial x^2} \geq 0$ for $x \geq 0$, and the function $g(x)$ has the only minimum $x_{\min} \in [0,1]$ that satisfies the equality:

$$1 + \ln(x_{\min})\frac{1}{1+x_{\min}} = 0.$$

Solving this equation we obtain $x_{\min} = \text{Lambert}(\mathrm{e}^{-1})$, then $g(x_{\min}) \simeq 0.757$. Therefore, the estimate for the Lyapunov function can be represented as follows:

$$\dot{V}(t) \leq -\varpi[\xi_1(V(t)) + |\phi(t)|^{\varsigma}\xi_2(V(t))] + \sigma(\|\vartheta\|_\infty)$$
$$\leq -\varpi\min\{1, |\phi(t)|^{\varsigma}\}\xi(V(t)) + \sigma(\|\vartheta\|_\infty),$$

where $\varpi = \sqrt{2}\min\{\gamma_1\kappa_{11}, \gamma_2\kappa_{12}\}g(x_{\min})$ and

$$\xi(s) = \begin{cases} s^{0.5} + s^{\frac{1+\varsigma}{2}} & s \geq 1 \\ s^{\frac{0.5+\phi_{\max}}{1+\phi_{\max}}} + s^{\frac{1+\varsigma+\frac{\phi_{\max}}{1+\phi_{\max}}}{2}} & s < 1 \end{cases}.$$

By repeating the arguments of theorems 1 and 2, the short-fixed-time and the short-finite-time ISS can be established, respectively (in the latter case there is no restriction on excitation of $\phi(t)$ and even the conventional results [36] can be used).

In order to check the restrictions on T_f for the case $\|W\|_\infty = 0$, assume that $V(t_0) > 1$, denote by $T_1, T_2 > 0$ the instants of time such that $V(t_0 + T_1) = 1$ and $V(t_0 + T_1 + T_2) = 0$, and use the estimate

$$\dot{V}(t) \leq -\varpi\max\{V^{\frac{0.5+\phi_{\max}}{1+\phi_{\max}}}(t), |\phi(t)|^{\varsigma}V^{\frac{1+\varsigma}{2}}(t)\},$$

then since $\frac{0.5+\phi_{\max}}{1+\phi_{\max}} \in (0,1)$ and $\frac{1+\varsigma}{2} > 1$ we obtain

$$V^{\frac{\varsigma-1}{2}}(t) \leq \frac{1}{V^{-\frac{\varsigma-1}{2}}(t_0) + \frac{\varsigma-1}{2}\frac{\varpi\upsilon}{4\ell}(t - t_0 - \ell)}$$

for all $t \in [t_0, t_0 + T_1]$, and

$$V^{\frac{0.5}{1+\phi_{\max}}}(t) \leq 1 - \frac{0.5}{1+\phi_{\max}}\varpi(t - t_0 - T_1)$$

for all $t \in [t_0 + T_1, t_0 + T_1 + T_2]$. Hence, the upper bounds on T_1 and T_2 follow (due to the properties of $\beta(t)$ the time T_2 is independent in the excitation):

$$T_1 \leq \ell[1 + \frac{8}{(\varsigma-1)\varpi\upsilon}],\ T_2 \leq 2\frac{1+\phi_{\max}}{\varpi},$$

which give by resolving the inequality $T_1 + T_2 \leq T_f$ the required restriction on γ_1 and γ_2. □

Thus, the idea of the algorithm (11) consists in the utilization of a nonlinearity such that the function $g(|\phi(t)|)$ becomes separated with zero overcoming the absence of excitation in the system. The price for that is the robustness with respect to a noise W with a well-defined ratio $\frac{W(t)}{\phi(t)}$.

Remark 2. One of the most important features of estimation algorithms, after estimation error convergence in the ideal case, is their robustness with respect to measurement noises. In our case, since the regressor $\phi(t)$ may converge to zero, the appearance of $W(t) \neq 0$ additionally limit the time of convergence, since it is reasonable to use the output $Y(t)$ for estimation with $t \in [t_0, t_0 + T]$ only while

$$|Y(t)| > |W(t)| + \varepsilon$$

for some $\varepsilon > 0$. If $|Y(t)| \leq |W(t)| + \varepsilon$ (or $|Y(t)|$ is almost equal to $|W(t)|$ for a sufficiently small ε), then the measured output mainly contains the measurement noise, and it is ambiguous to ask an algorithm to estimate θ due to a bad ratio between the signal and the noise. In this sense the requirement, that the signal $\frac{W(t)}{\phi(t)}$ is well-defined, is not much restrictive (roughly speaking it just assumes that the ratio between the useful signal and the noise lies in reasonable limits).

Remark 3. If instead of (12), for any $t_0 \in [-T^0, T^0 + T_f]$ there exist $\ell_0 > 0$ and $\upsilon_0 > 0$ such that

$$\int_{t_0}^{t_0+\ell_0} |\phi(s)|^{\varsigma} ds \geq \upsilon_0,$$

i.e. the regressor is always excited on an initial interval of time $[t_0, t_0 + \ell_0]$ only, then using the same arguments as in the proof of Proposition 2 it is possible to show that there exists $T_f > 0$ for which the estimation error dynamics of (11) is short-fixed-time ISS for T^0 and T_f with the input ϑ.

If the regressor $\phi(t)$ is just asymptotically converging without crossing zero, then the algorithms (9) and (11) can be applied for any finite $T > 0$ and $t_0 \in \mathbb{R}$.

C. Comparison with other approaches

Two approaches will be considered in the next section for a numeric comparison.

1) Concurrent learning approach: The concurrent learning (CL) approach [12], [13] is based on combination of a conventional gradient algorithm with a static update law, whose application in our case leads to the following estimator:

$$\begin{aligned} \dot{\hat{\theta}}(t) &= \gamma_1 \phi(t)(Y(t) - \phi(t)\hat{\theta}(t)) \\ &+ \gamma_2 \sum_{r=0}^{p} \phi(t_r)(Y(t_r) - \phi(t_r)\hat{\theta}(t)), \\ &\gamma_1 > 0,\ \gamma_2 > 0, \end{aligned} \tag{13}$$

where $t_r \in [t_0, t_0 + T_f]$ for all $r = 0, \ldots, p$ and $p \geq 0$ is the number of stored points (in the above sum and in this subsection it is assumed that if $t_r \leq t$ for $r = 0, \ldots, h$ with $h < p$ only, then the sum on the first h items is taken). The part proportional to γ_1 is the conventional linear algorithm, the part proportional to γ_2 represents an additional static feedback, which allows the requirement on persistence of excitation to be relaxed by keeping in memory p successful measurements. Indeed, considering the dynamics of the estimation error $e(t) = \theta - \hat{\theta}(t)$ for (13) we obtain:

$$\begin{aligned} \dot{e}(t) &= -\left(\gamma_1 \phi^2(t) + \gamma_2 \sum_{r=0}^{p} \phi^2(t_r)\right) e(t) \\ &\quad -\gamma_1 \phi(t) W(t) - \gamma_2 \sum_{r=0}^{p} \phi(t_r) W(t_r). \end{aligned}$$

Thus, independently on the excitation properties of $\phi(t)$ on the interval $t \in [-T^0, T^0 + T_f]$, if the instants t_r, $r = 0, \ldots, p$ have been properly selected such that $\sum_{r=0}^{p} \phi^2(t_r) > 0$ (which is always realizable since $\phi(t)$ is assumed to be available for measurement), then the asymptotic convergence of $e(t)$ with an exponential rate to the origin in the absence of W and ISS for $\|W\|_\infty \neq 0$ are obviously guaranteed. In such a case the term proportional to γ_2 introduces a kind of σ modification [3], and $\gamma_1 = 0$ can be imposed.

In our simple scalar context the algorithm (13) can be further modified. Indeed, we do not need to wait for convergence of (13) since for the case with $\gamma_1 = 0$ the final estimate of (13) can be calculated directly:

$$\lim_{t \to +\infty} \hat{\theta}(t) = \theta + \frac{\sum_{r=0}^{p} \phi(t_r) W(t_r)}{\sum_{r=0}^{p} \phi^2(t_r)}.$$

Consequently, introducing the noise-free hypothesis:

$$Y(t_r) = \phi(t_r)\hat{\theta}(t),\ \phi(t_r) \neq 0 \quad \forall r = 0, \ldots, p,$$

the following simple fixed-time converging algorithm can be proposed:

$$\begin{aligned} \hat{\theta}(t) &= \frac{\sum_{r=0}^{p} \phi(t_r) Y(t_r)}{\sum_{r=0}^{p} \phi^2(t_r)} \\ &= \theta + \frac{\sum_{r=0}^{p} \phi(t_r) W(t_r)}{\sum_{r=0}^{p} \phi^2(t_r)}, \end{aligned} \tag{14}$$

whose estimation error $e(t) = \theta - \hat{\theta}(t)$ admits an upper bound for all $t \in [t_0, t_0 + T_f]$:

$$|e(t)| \leq \|W\|_\infty \max_{1 \leq h \leq p} \frac{\sum_{r=0}^{h} |\phi(t_r)|}{\sum_{r=0}^{h} \phi^2(t_r)}.$$

Obviously, in the noise-free scenario the ideal value is recovered by (14) at $t = t_0$ if $\phi(t_0) \neq 0$. In the presence of noise, application of (14) at $t = t_0$ eliminates dependence on the amplitude of initial error $|e(t_0)| = |\theta - \hat{\theta}(t_0)|$. The algorithm (14) needs minimal computational power, and the instants t_r can be optimized in order to ensure a better identification of θ.

The drawback of the CL approach is that the result of its application is very sensitive to selection of the instants t_r, $r = 0, \ldots, p$ and the values of the noise $W(t_r)$, which are unknown.

2) Linear fixed-time converging observers: A very elegant estimation solution has been proposed in [22], following it the linear regression problem can be rewritten as follows:

$$\dot{\theta} = 0 = \gamma_i \phi(t)[Y(t) - \phi(t)\theta - W(t)],$$

where $\gamma_i \in \mathbb{R}$ with $i = 1, 2$ are design parameters defined later. Then, estimators can be introduced as usual linear observers (in the form of (5)):

$$\dot{\tilde{\theta}}_i(t) = \gamma_i \phi(t)[Y(t) - \phi(t)\tilde{\theta}_i(t)],$$

where $\tilde{\theta}_i(t) \in \mathbb{R}$ are the conventional estimates of θ for $i = 1, 2$. Next, for any delay $\tau > 0$ we obtain:

$$\begin{aligned} \theta - \tilde{\theta}_i(t) \quad &= \quad \mathbf{e}^{-\gamma_i \int_{t-\tau}^{t} \phi^2(s)ds}(\theta - \tilde{\theta}_i(t-\tau)) \\ &\quad -\gamma_i \int_{t-\tau}^{t} \mathbf{e}^{-\gamma_i \int_{\sigma}^{t} \phi^2(s)ds} \phi(\sigma) W(\sigma) d\sigma \end{aligned}$$

for $i = 1, 2$, and subtracting these equations we get:

$$\begin{aligned} &\left(\mathbf{e}^{-\gamma_1 \int_{t-\tau}^{t} \phi^2(s)ds} - \mathbf{e}^{-\gamma_2 \int_{t-\tau}^{t} \phi^2(s)ds}\right) \theta = \tilde{\theta}_2(t) - \tilde{\theta}_1(t) \\ &+\mathbf{e}^{-\gamma_1 \int_{t-\tau}^{t} \phi^2(s)ds} \tilde{\theta}_1(t-\tau) - \mathbf{e}^{-\gamma_2 \int_{t-\tau}^{t} \phi^2(s)ds} \tilde{\theta}_2(t-\tau) \\ &+ \int_{t-\tau}^{t} \left(\gamma_1 \mathbf{e}^{-\gamma_1 \int_{\sigma}^{t} \phi^2(s)ds} - \gamma_2 \mathbf{e}^{-\gamma_2 \int_{\sigma}^{t} \phi^2(s)ds}\right) \phi(\sigma) W(\sigma) d\sigma, \end{aligned}$$

from which the following fixed-time estimator can be designed:

$$\begin{aligned} \hat{\theta}(t) = &\left(\mathbf{e}^{-\gamma_1 \int_{t-\tau}^{t} \phi^2(s)ds} - \mathbf{e}^{-\gamma_2 \int_{t-\tau}^{t} \phi^2(s)ds}\right)^{-1} [\tilde{\theta}_2(t) - \tilde{\theta}_1(t) \\ &+\mathbf{e}^{-\gamma_1 \int_{t-\tau}^{t} \phi^2(s)ds} \tilde{\theta}_1(t-\tau) - \mathbf{e}^{-\gamma_2 \int_{t-\tau}^{t} \phi^2(s)ds} \tilde{\theta}_2(t-\tau)], \end{aligned} \tag{15}$$

under the hypothesis that the gains γ_1, γ_2 and the delay τ are selected such that

$$\mathbf{e}^{-\gamma_1 \int_{t-\tau}^{t} \phi^2(s)ds} \neq \mathbf{e}^{-\gamma_2 \int_{t-\tau}^{t} \phi^2(s)ds}$$

for all $t \in [t_0 + \tau, t_0 + T_f]$ (see [22] for the details). Obviously, the estimation error $e(t) = \theta - \hat{\theta}(t)$ for (15) satisfies:

$$|e(t)| \leq K_{t_0} \|W\|_\infty \quad \forall t \in [t_0 + \tau, t_0 + T_f],$$

where

$$\begin{aligned} K_{t_0} = &\sup_{t \in [t_0+\tau, t_0+T_f]} \left|\mathbf{e}^{-\gamma_1 \int_{t-\tau}^{t} \phi^2(s)ds} - \mathbf{e}^{-\gamma_2 \int_{t-\tau}^{t} \phi^2(s)ds}\right|^{-1} \\ &\times \int_{t-\tau}^{t} \left|\gamma_1 \mathbf{e}^{-\gamma_1 \int_{\sigma}^{t} \phi^2(s)ds} - \gamma_2 \mathbf{e}^{-\gamma_2 \int_{\sigma}^{t} \phi^2(s)ds}\right| |\phi(\sigma)| d\sigma, \end{aligned}$$

and the upper bound on the error is undefined for $t \in [t_0, t_0 + \tau)$. Therefore, an issue of this approach is its robustness. Indeed, if the regressor signal $\phi(t)$ is approaching zero, then the gain $\left|\mathbf{e}^{-\gamma_1 \int_{t-\tau}^{t} \phi^2(s)ds} - \mathbf{e}^{-\gamma_2 \int_{t-\tau}^{t} \phi^2(s)ds}\right|^{-1}$ becomes large, while the integral may be continuously growing, which leads to the growth of K_{t_0}. Since here the estimation is performed on a bounded interval of time, then some of the gains γ_1 and γ_2 can be selected to be slightly negative, to improve the accuracy of estimation and noise sensitivity.

V. Example

Consider a measured scalar signal $y(t)$ that consists of two terms. The first term is a harmonic oscillation, and the second term is a decaying process:

$$y(t) = \theta_1 \sin(t) + \theta_2 \frac{1}{t+1} + w(t),$$

where $w(t)$ is a measurement noise, and θ_1, θ_2 are unknown constant parameters. Since the second term decays, the parameter θ_1 can be estimated asymptotically with a standard gradient estimator; however, estimation of θ_2 is more challenging. The signal $y(t)$ can be rewritten as (3), where the regressor $\omega(t) = [\sin(t) \ \frac{1}{t+1}]^\top$, obviously, is not persistently exciting. Thus, standard gradient or least-square approaches do not guarantee asymptotic estimation. This section illustrates how θ_2 can be estimated by the means of the DREM procedure and the proposed fixed-time parameter estimation algorithms.

First, the DREM procedure is applied to get a scalar linear regression equation for θ_2. To this end, choose the delay operator $H(u(t)) = u(t - \tau_d)$, where $\tau_d > 0$ is the tuning parameter, and define

$$\tilde{\omega}_1(t) = H(\omega(t)) = \omega(t - \tau_d), \ \tilde{y}_1(t) = H(y(t)) = y(t - \tau_d).$$

Table I
THE MEAN VALUES (MV) AND ROOT-MEAN-SQUARED DEVIATIONS (RMS) OF ESTIMATION ERRORS FOR UNIFORMLY DISTRIBUTED MEASUREMENT NOISE.

	Algorithm 1	Algorithm 2	Algorithm 3	Algorithm 4
MV$\cdot 10^2$	0.1	4.0	6.1	3.8
RMS$\cdot 10^2$	0.7	20.6	0.1	23.5

Then the procedure described in subsection II-C yields

$$Y_2(t) = \phi(t)\theta_2 + W_2,$$

where Y_2, W_2, and ϕ are computed as it has been explained above, and

$$\phi(t) = \frac{\sin(t)}{1+t-\tau_d} - \frac{\sin(t-\tau_d)}{1+t}.$$

It is worth noting that

$$\int_{\tau_d}^{\infty} \phi^2(s)ds = C \leq \infty,$$

and the requirement (7) of Theorem 3 is not satisfied; thus, the algorithm (5) cannot be applied. However, since $\phi(t)$ decays only asymptotically, there exist ℓ and υ such that (10) and (12) are satisfied on any finite interval of time.

For simulations, the algorithms (9), (11), (14) and (15) are tuned as follows. For algorithms (9) and (11) the parameters $\gamma_1 = \gamma_2 = 10$, $\alpha = 0.25$, and $\varsigma = 1.25$ are chosen. For the algorithm (14), a point $\phi(t_r)$ is stored if it is sufficiently large and sufficient amount of time has passed:

$$t_r = \arg \inf_{t \geq t_{r-1}+0.1} \{|\phi(t)| \geq 0.1\}.$$

For the algorithm (15) the values $\tau = 0.5$, $\gamma_1 = 5$ and $\gamma_2 = 15$ are chosen; note that for this algorithm it is required that $\gamma_1 \neq \gamma_2$. To avoid singularities in (15), no division is performed when the absolute value of the denominator is less than 10^{-4}. All algorithms are initialized with $\hat{\theta}_2(0) = 0$, while the true value is $\theta_2 = -2$. For all algorithms no estimation is performed for $t \in [0, \tau_d]$, $\tau_d = 1$ since $\phi(t)$ is identically zero on this interval due to the DREM procedure.

Simulation results for the algorithms (9), (11), (14) and (15) for the noise-free case $w(t) \equiv 0$ are given in Fig. 1. In the left part the plot with transients for the nominal values of γ_1 and γ_2 is given, while in the right part, in order to illustrate the ability for acceleration of convergence, the estimates are shown for the values of γ_1 and γ_2 multiplied by 4 and $\tau_d = 0.5$ (note that in our case, decreasing the value of τ_d leads to a weaker excitation of $\phi(t)$, see the expression above). Simulation results for the same algorithms for the case when $w(t)$ is a uniformly distributed noise, $|w(t)| \leq 0.2$, are given in Fig. 2. The mean values and root-mean-squared deviations computed over the interval $t \in [5, 15]$ are given in Tab. I.

As expected, all algorithms provide finite-time parameter estimation. In the noise-free scenario, the algebraic algorithms (14) and (15) provide one-step estimation. Since the algorithm (14) does not gather new measurements as $\phi(t)$ decays, it has the smallest steady-state error deviations but suffers from the largest mean error. The algorithms (11) and (15) show high noise sensitivity, while the algorithm (9) has both small mean error and small error oscillations, where the trade-off is the largest transient time for similar tuning parameters.

All these results confirm the theoretical findings of this note.

VI. Conclusions

The problem of estimation in the linear regression model has been considered on a bounded interval of time. Several estimators are designed, which are based on the framework of finite-time or fixed-time converging dynamical systems. In order to analyze the robustness of these estimation algorithms, a short-time fixed-time input-to-state stability property has been introduced for time-varying systems and its sufficient Lyapunov condition is given. The synthesized estimation algorithms are compared with existing solutions in numerical experiments.

References

[1] X. Yan, *Linear Regression Analysis: Theory and Computing*. Singapore: World Scientific, 2009.

[2] A. Rencher and W. Christensen, *Methods of Multivariate Analysis*, 3rd ed., ser. Wiley Series in Probability and Statistics. New Jersey: John Wiley & Sons, 2012, vol. 709.

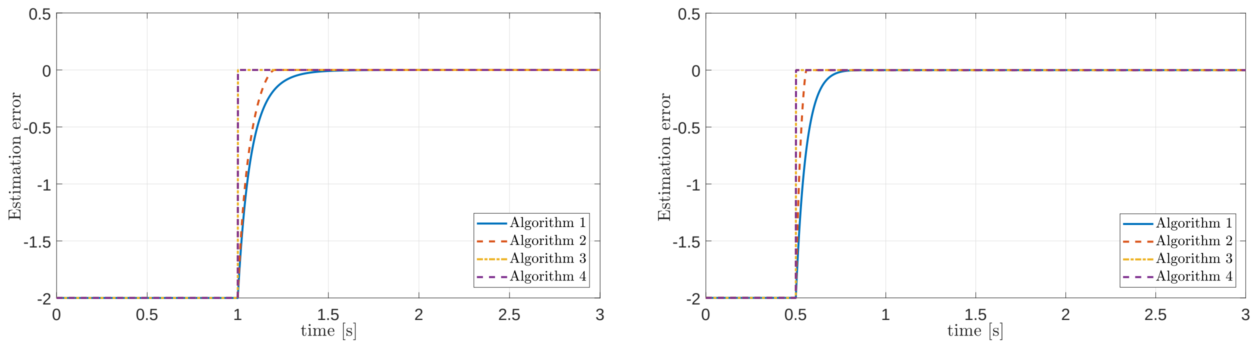

Figure 1. Simulation results for the algorithms (9), (11), (14) and (15), estimation error $\theta_2 - \hat{\theta}_2(t)$ vs. time, seconds. The case of no measurement noise (to the left with the nominal values of γ_1 and γ_2, to the right with four times higher values)

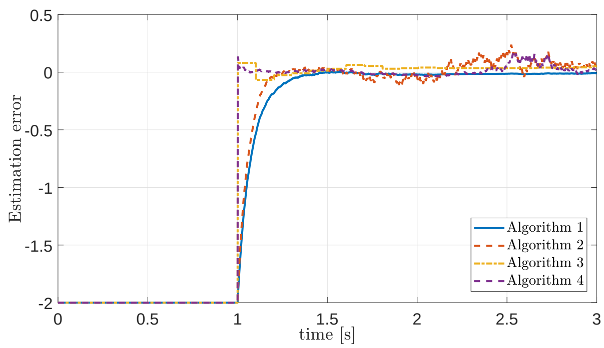

Figure 2. Simulation results for the algorithms (9), (11), (14) and (15), estimation error $\theta_2 - \hat{\theta}_2(t)$ vs. time, seconds. The case of uniformly distributed measurement noise

[3] P. Ioannou and P. Kokotovic, *Adaptive Systems with Reduced Models*. Secaucus, NJ: Springer Verlag, 1983.

[4] L. Ljung, *System Identification: Theory for the User*. Upper Saddle River, NJ: Prentice-Hall, 1987.

[5] S. Sastry and M. Bodson, *Adaptive Control: Stability, Convergence and Robustness*. London: Prentice-Hall, 1989.

[6] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Upper Saddle River, New Jersey: Prentice Hall, 2002.

[7] K. Narendra and A. Annaswamy, "Persistent excitation in adaptive systems," *International Journal of Control*, vol. 45, no. 1, pp. 127–160, 1987.

[8] D. Efimov and A. Fradkov, "Design of impulsive adaptive observers for improvement of persistency of excitation," *Int. J. Adaptive Control and Signal Processing*, vol. 29, no. 6, pp. 765–782, 2015.

[9] L. Praly, "Convergence of the gradient algorithm for linear regression models in the continuous and discrete time cases," December 26 2016, int. Rep. MINES ParisTech, Centre Automatique et Systèmes. [Online]. Available: https://hal.archives-ouvertes.fr/hal-01423048

[10] N. Barabanov and R. Ortega, "On global asymptotic stability of $\dot{x} = -\phi(t)\phi^\top(t)x$ with $\phi(t)$ bounded and not persistently exciting," *Systems & Control Letters*, vol. 109, pp. 24–27, 2017.

[11] D. Efimov, N. Barabanov, and R. Ortega, "Robust stability under relaxed persistent excitation conditions," *International Journal of Adaptive Control and Signal Processing*, 2019.

[12] G. Chowdhary, T. Yucelen, M. Mühlegg, and E. N. Johnson, "Concurrent learning adaptive control of linear systems with exponentially convergent bounds," *International Journal of Adaptive Control and Signal Processing*, vol. 27, no. 4, pp. 280–301, 2012.

[13] R. Kamalapurkar, P. Walters, and W. E. Dixon, "Model-based reinforcement learning for approximate optimal regulation," *Automatica*, vol. 64, pp. 94–104, 2016.

[14] J. Wang, D. Efimov, and A. Bobtsov, "On robust parameter estimation in finite-time without persistence of excitation," *IEEE Transactions on Automatic Control*, 2018, submitted.

[15] A. Belov, S. Aranovskiy, R. Ortega, N. Barabanov, and A. Bobtsov, "Enhanced parameter convergence for linear systems identification: The DREM approach," in *Proc. European Control Conference (ECC)*, Limassol, 2018.

[16] V. Andrieu, L. Praly, and A. Astolfi, "Homogeneous Approximation, Recursive Observer Design, and Output Feedback," *SIAM J. Control Optimization*, vol. 47, no. 4, pp. 1814–1850, 2008.

[17] E. Cruz-Zavala, J. Moreno, and L. Fridman, "Uniform robust exact differentiator," *IEEE Transactions on Automatic Control*, vol. 56, no. 11, pp. 2727–2733, 2011.

[18] A. Polyakov, "Nonlinear feedback design for fixed-time stabilization of linear control systems," *IEEE Transactions on Automatic Control*, vol. 57, no. 8, pp. 2106–2110, 2012.

[19] Y. Song, Y. Wang, J. Holloway, and M. Krstic, "Time-varying feedback for regulation of normal-form nonlinear systems in prescribed finite time," *Automatica*, vol. 83, pp. 243–251, 2017.

[20] J. Wang, D. Efimov, and A. Bobtsov, “Finite-time parameter estimation without persistence of excitation,” in *Proc. European Control Conference*, Naples, 2019.

[21] S. Aranovskiy, A. Bobtsov, R. Ortega, and A. Pyrkin, “Performance enhancement of parameter estimators via dynamic regressor extension and mixing,” *IEEE Transactions on Automatic Control*, vol. 62, no. 7, pp. 3546–3550, 2017.

[22] R. Engel and G. Kreisselmeier, “A continuous-time observer which converges in finite time,” *IEEE Transactions on Automatic Control*, vol. 47, no. 7, pp. 1202–1204, 2002.

[23] H. Ríos, D. Efimov, A. Polyakov, and W. Perruquetti, “Homogeneous time-varying systems: Robustness analysis,” *IEEE Transactions on Automatic Control*, vol. 61, no. 12, pp. 4075–4080, 2016.

[24] G. Kamenkov, “On stability of motion over a finite interval of time,” *Journal of Applied Math. and Mechanics (PMM)*, vol. 17, pp. 529–540, 1953.

[25] A. Lebedev, “The problem of stability in a finite interval of time,” *Journal of Applied Math. and Mechanics (PMM)*, vol. 18, pp. 75–94, 1954.

[26] P. Dorato, “Short-time stability in linear time-varying systems,” Ph.D. dissertation, Polytechnic Institute of Brooklyn, New York, 1961.

[27] L. Weiss and E. Infante, “On the stability of systems defined over a finite time interval,” *Proc. of the National Academy of Sciences*, vol. 54, pp. 440–448, 1965.

[28] E. Roxin, “On finite stability in control systems,” *Rendiconti del Circolo Matematico di Palermo*, vol. 15, pp. 273–283, 1966.

[29] S. P. Bhat and D. S. Bernstein, “Geometric homogeneity with applications to finite-time stability,” *Mathematics of Control, Signals and Systems*, vol. 17, pp. 101–127, 2005.

[30] Y. Hong, “H_∞ control, stabilization, and input-output stability of nonlinear systems with homogeneous properties,” *Automatica*, vol. 37, no. 7, pp. 819–829, 2001.

[31] E. Bernuau, A. Polyakov, D. Efimov, and W. Perruquetti, “Verification of ISS, iISS and IOSS properties applying weighted homogeneity,” *Systems & Control Letters*, vol. 62, pp. 1159–1167, 2013.

[32] J. G. Rueda-Escobedo, D. Ushirobira, R. Efimov, and J. A. Moreno, “Gramian-based uniform convergent observer for stable LTV systems with delayed measurements,” *Int. J. Control*, 2019. [Online]. Available: https://hal.inria.fr/hal-01889193

[33] F. Gantmacher, *The Theory of Matrices*. New York: Chelsea, 1960.

[34] H. Ríos, D. Efimov, J. A. Moreno, W. Perruquetti, and J. G. Rueda-Escobedo, “Time-varying parameter identification algorithms: Finite and fixed-time convergence,” *IEEE Transactions on Automatic Control*, vol. 62, no. 7, pp. 3671–3678, 2017.

[35] H. Ríos, D. Efimov, and W. Perruquetti, “An adaptive sliding-mode observer for a class of uncertain nonlinear systems,” *International Journal of Adaptive Control and Signal Processing*, vol. 32, no. 3, pp. 511–527, 2018.

[36] E. Sontag and Y. Wang, “On characterizations of the input-to-state stability property,” *Systems & Control Letters*, vol. 24, pp. 351–359, 1995.