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Abstract In this paper we consider the following setting: suppose that we are given a manifold in $\mathbb{R}^d$ with positive reach. Moreover assume that we have an embedded simplical complex $\mathscr{A}$ without boundary, whose vertex set lies on the manifold, is sufficiently dense and such that all simplices in $\mathscr{A}$ have sufficient quality. We prove that if, locally, interiors of the projection of the simplices onto the tangent space do not intersect, then $\mathscr{A}$ is a triangulation of the manifold, that is they are homeomorphic.

1 Introduction

Triangulations have played a central role in Computational Geometry since its foundation, Delaunay triangulations being the ones that have been studied most frequently

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[1, 3, 10, 16]. One of the main applications of Delaunay triangulations was to find triangulations of surfaces or more generally manifolds embedded in Euclidean space. Sometimes a distinction is made between meshing, where one assumes that the manifold is known, and reconstruction or learning, where one cannot query the manifold as needed but can only use a given sample.

Although the Computational Geometry community has mainly focused on Delaunay triangulations (until recently), the classical mathematics literature did not constrain itself to this [7, 17]. This paper, together with its companion [4], places itself in this broader scope.

In the Delaunay setting, the closed ball property [12] is often used to prove that a simplicial complex is homeomorphic to the manifold in question, see for example [1]. Edelsbrunner and Shah [12] defined the restricted Delaunay complex of a subset M of Euclidean space as the nerve of the Voronoi diagram on M when the ambient Euclidean metric is used. They showed that if M is a compact manifold, then the restricted Delaunay complex is homeomorphic to M when the Voronoi diagram satisfies the closed ball property: Voronoi faces are closed topological balls of the appropriate dimension. The closed ball property is purely topological and finding sampling conditions that ensure that the closed ball property holds is not easy [8, 10].

A first attempt to depart from the use of the closed ball property and to define conditions similar in spirit to those reported in this paper can be found in [2]. In [4] we explored triangulation conditions in a very general setting, that does not require the manifold to be embedded, and for general maps. The conditions were chosen such that they are more directly applicable compared to the closed ball property.

The triangulation criteria of [4] encompass tangential Delaunay complexes [2], and the intrinsic triangulations explored in [11]. The search for a universal framework incorporating [2, 11] was the main motivation for [4].

The new conditions we introduce in this paper are of the same vein as those results. There are however also noticeable differences:

- The setting of [4] is more abstract. In this paper we restrict ourselves to submanifolds of Eulcidean space. This affords a more precise analysis and thus better constants.
- The conditions in [4] were formulated in terms of *vertex sanity*. Vertex sanity says that if a vertex is mapped by a ‘nice’ coordinate map into the star of some (other) vertex, then the vertex is in fact a vertex of this star. This is quite different from the conditions that we formulate here.
- In [4] the simplicial complex $\mathscr{A}$ was assumed to be a (PL) manifold. Which is a much stronger condition than the one we examine here.

The conditions in this paper are very natural and seem to be generally applicable and complementary to the results of [4]. In particular, they apply to a recent triangulation algorithm due to one of the authors of the present paper [15].

In this algorithm, the triangulation is found as the support of a simplicial cycle over $\mathbb{Z}/2\mathbb{Z}$. As such, it is pure and any $(m-1)$-simplex has an even number of (and therefore at least 2) m-dimensional cofaces. It is therefore what is called below a *simplicial complex without boundary*. It is proved [15] that, under precise sampling conditions, the support of this minimal cycle meets both topological and local geometrical conditions required by Theorem 4 below.

The conditions of Theorem 4 can be informally stated as follows:

Simplicial complexes without boundary We make a topological assumption on $\mathscr{A}$, namely that $\mathscr{A}$ is m-dimensional and each $(m-1)$-simplex in $\mathscr{A}$ has at least two m-dimensional cofaces. We call a complex satisfying these conditions a *simplicial complex without boundary* (Definition 18). We stress that this is a rather weak topological assumption compared to the condition of being a *combinatorial m-manifold*, which requires the link of any k-simplex to be homeomorphic to the $(m-k-1)$-sphere $\mathbb{S}^{m-k-1}$.

Local geometric conditions We assume that the simplex diameters are small with respect to $\text{rch}(M)$ and that the ratio between the smallest height and the diameter of each simplex is lower bounded by some constant (see Theorem 4 (a) for a precise statement). Moreover, for any $p \in \mathscr{P}$, two m-simplices lying in a small neighborhood of p have disjoint interior projection on the plane tangent to M at p (Theorem 4, condition (b)).

Intuitively, once the interiors of the (local) projection of m-simplices on a local tangent plane of M are disjoint, the conditions for homeomorphism seem not far. In fact we will see that $\mathscr{A}$ is *ambient isotopic* to M.

Even though we have specific applications in mind, we formulate the statements in a setting that is as general as possible, albeit in the embedded setting. This is in the hope that it will be used in a wide range of applications in Manifold Meshing and Learning. We restrict ourselves to connected manifolds since the extension to non-connected manifolds consists merely of applying the result to each connected component.

1.1 Notation

Notation 1 (Simplex quality) The *thickness* of a m-simplex σ, denoted $t(\sigma)$, is given by $\frac{a}{mL}$, where $a = a(\sigma)$ is the smallest altitude of σ and $L = L(\sigma)$ is the length of the longest edge. The altitude of a vertex in a simplex is the distance from the vertex to the affine hull of the opposite face. Observe that $t \leq 1/m$ and might be in fact $O(m^{-3/2})$ [9]. We set $t(\sigma) = 1$ if σ has dimension 0.

Notation 2 (Simplicial complexes) We consider a simplicial complex $\mathscr{A}$ whose vertex set $\mathscr{A}^0$ is identified with a finite set $\mathscr{P} \subset \mathbb{R}^N$. The carrier of $\mathscr{A}$ (i.e., the underlying topological space), is denoted $|\mathscr{A}|$, and we have a natural piecewise linear map $\iota : |\mathscr{A}| \to \mathbb{R}^N$, but we do not assume a priori that ι is an embedding, i.e., we cannot assume that $|\mathscr{A}| \subset \mathbb{R}^N$. Nonetheless, we identify the simplices in $\mathscr{A}$ with their image under ι, and when there is no ambiguity, $\sigma \in \mathscr{A}$ may refer to a geometric simplex $\iota(\sigma) \subset \mathbb{R}^N$ as well as the abstract simplex $\sigma \in \mathscr{A}$. Similarly, we write $x \in |\mathscr{A}|$ as a shorthand for $x \in \iota(|\mathscr{A}|)$.

Notation 3 (Projection maps) We denote by pr_{T_pM} the projection on the tangent plane T_pM to M at $p \in \mathscr{P}$ and by $\mathrm{pr}_M|_{|\mathscr{A}|}$ the composition $\mathrm{pr}_M \circ \iota$.

1.2 Main result

Theorem 4 (Triangulation of submanifolds) *Let $M \subset \mathbb{R}^N$ be a connected C^2 m-dimensional submanifold of $\mathbb{R}^N$ with reach* $\mathrm{rch}(M) > 0$*, and $\mathscr{P} \subset M$ a finite set of points. Suppose that $\mathscr{A}$ is an m-dimensional simplicial complex without boundary whose vertex set $\mathscr{A}^0$ is identified with $\mathscr{P}$. Let $L, t > 0$ be such that for any m-simplex $\sigma \in \mathscr{A}$ one has:*

$$t \leq t(\sigma) \quad \textit{and} \quad L(\sigma) \leq L.$$

If:

(a) *All simplices are small with respect to the reach:*

$$\frac{L}{\mathrm{rch}(M)} \leq \min\left(\frac{1}{8}, t\,\sin\pi/8\right).$$

(b) *The projection of m-simplices on local tangent planes have disjoint interiors:*

$$\forall p \in \mathscr{P},\ \forall \sigma_1, \sigma_2 \in \mathscr{A} \textit{ with } |\sigma_1|, |\sigma_2| \subset B(p, 2.8L),$$
$$\sigma_1 \neq \sigma_2 \Rightarrow \mathrm{pr}_{T_pM}(|\sigma_1|)^\circ \cap \mathrm{pr}_{T_pM}(|\sigma_2|)^\circ = \emptyset.$$

Then:

(1) *The inclusion $\iota : |\mathscr{A}| \to \mathbb{R}^N$ is an embedding, and we can identify $\iota(|\mathscr{A}|)$ with $|\mathscr{A}|$.*
(2) *The closest-point projection map* $\mathrm{pr}_M|_{|\mathscr{A}|} : |\mathscr{A}| \to M$ *is a homeomorphism, so M is compact, and there is an ambient isotopy bringing $|\mathscr{A}|$ to M.*

Remark 1 As noticed in Notation 1, t decreases with m and the bound on $L/\mathrm{rch}(M)$ in Condition (a) decreases at least as fast as $O(1/m)$ or perhaps $O(m^{-3/2})$ as the dimension m of the manifold increases.

Remark 2 Condition (a) of the theorem could be improved with minor, only quantitative, changes in the proof. In the bound

$$\frac{L}{\mathrm{rch}(M)} \leq \min\left(\frac{1}{8}, t\,\sin\pi/8\right)$$

there is in fact a tradeoff between the constant bound (i.e., $\frac{1}{8}$) and the bound depending on t (i.e., $t\,\sin\pi/8$). The latter one could be replaced by any value strictly below $t\sin\pi/4$ by decreasing enough the constant bound, which is certainly better for large m, since, as seen in the previous remark, t may decreases as fast as $O(m^{-3/2})$.

2 Definitions and submanifold geometry

In this section, we make the following assumptions.

Hypothesis 5 (Geometric assumptions) $M \subset \mathbb{R}^N$ is a connected C^2 m-dimensional submanifold of $\mathbb{R}^N$ with positive reach, $\mathrm{rch}(M) > 0$, and $\mathscr{P} \subset M$ is a finite set of points. $\mathscr{A}$ is an m-dimensional simplicial complex whose vertex set $\mathscr{A}^0$ is identified with $\mathscr{P}$. Let $L, t > 0$ be such that for any m-simplex $\sigma \in \mathscr{A}$, one has

$$t \leq t(\sigma) \quad \text{and} \quad L(\sigma) \leq L.$$

We now recall the following five results. Lemma 6 is proved in [5, Corollary 8]. Lemma 7 is a variant of a result of Whitney [17, Section IV.15] proved in [3, Lemma 8.11]. Lemma 8 is proved in [5, Corollary 12], Lemma 9 is proved in [13, Theorem 4.8(7)] and Lemma 10 is proved in [3, Lemma 7.9]. In the following, $B(x,\rho)$ and $B^\circ(x,\rho)$ rspectively denote the closed and open ball with center x and radius ρ.

Lemma 6 (Tangent Balls) *For any $p \in M$, any open ball $B^\circ(c,r)$ that is tangent to M at p and whose radius r satisfies $r \leq \mathrm{rch}(M)$ does not intersect M.*

Lemma 7 (Simplex-tangent space angle bounds) *Under Hypothesis 5, if $\sigma \in \mathscr{A}$ and p is a vertex of σ, then*

$$\sin \angle(\sigma, T_pM) \leq \frac{L}{t\,\mathrm{rch}(M)}.$$

Lemma 8 (Variation of tangent space) *Under Hypothesis 5, if $p, q \in M$, then*

$$\sin \frac{\angle(T_pM, T_qM)}{2} \leq \frac{\|p-q\|}{2\,\mathrm{rch}(M)}.$$

Lemma 9 (Distance to tangent space) *Under Hypothesis 5, if $p, q \in M$, then*

$$d(q, T_pM) \leq \frac{\|p-q\|^2}{2\,\mathrm{rch}(M)}.$$

Lemma 10 (Hausdorff distance between M and $|\mathscr{A}|$) *Under Hypothesis 5, if $x \in |\mathscr{A}|$, then*

$$\|\mathrm{pr}_M(x) - x\| < \frac{2L^2}{\mathrm{rch}(M)}.$$

Apart from the results we just recalled we'll also use:

Lemma 11 (Hausdorff distance between a simplex and its vertices) *For any compact set $C \in \mathbb{R}^d$ with diameter L (the largest distance between any two points in C) one has*

$$\forall x \in \mathrm{hull}(C), \exists p \in C, \|x-p\| \leq L\sqrt{\frac{d}{2(d+1)}} < \frac{L}{\sqrt{2}}.$$

where $\mathrm{hull}(C)$ *denotes the convex hull of C.*

In particular, for any simplex σ with diameter L, if $x \in \sigma$ then there is a vertex p of σ such that

$$\|x-p\| \leq L\sqrt{\frac{d}{2(d+1)}} < \frac{L}{\sqrt{2}}.$$

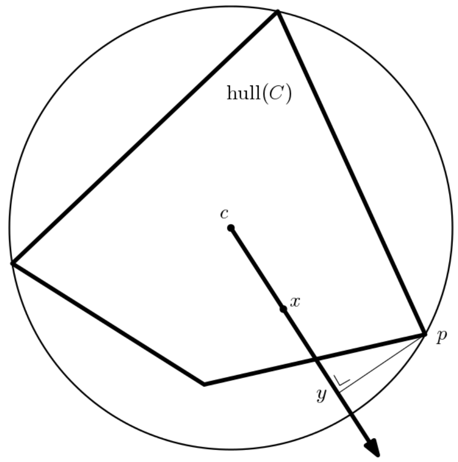

Fig. 1 The notation for Lemma 11.

Proof Jung's Theorem [14] says that in the d-dimensional Euclidean space, the radius r of the smallest ball enclosing a convex polyhedron is at most $L\sqrt{\frac{d}{2(d+1)}}$. The compact set C can be approximated arbitrary well by a finite set $\widetilde{C}$ in Hausdorff distance and Jung's Theorem applies to the convex polyhedron $\mathrm{hull}(\widetilde{C})$.

Since the map that sends a compact set to its convex hull: $C \mapsto \mathrm{hull}(C)$ is 1-Lipschitz for the Hausdorff distance, one can extend Jung's Theorem by continuity to get the same bound for the smallest ball enclosing $\mathrm{hull}(C)$. Note that $L\sqrt{\frac{d}{2(d+1)}} \leq \frac{L}{\sqrt{2}}$. Let $B(c,r)$ be the smallest ball enclosing C. The center c of that ball belongs to $\mathrm{hull}(C)$ since, otherwise, we would decrease the distance of c to any point in C by projecting c to its closest point in $\mathrm{hull}(C)$. We have that any $p \in C$ is at distance at most r from c. Moreover, if $x \in \mathrm{hull}(C) \setminus \{c\}$, then there is $p \in C$ extremal in the direction $\overrightarrow{cx}$. Let us write y for the orthogonal projection of p on the line cx, then the segment cy contains x and the distance to p cannot increase when going from c to y. Therefore $d(x,p) \leq d(c,p) \leq r$. □

3 Pseudo-manifolds, Whitney's lemma, and simplicial complexes without boundary

Definition 12 (Pure simplicial complex) An m-dimensional simplicial complex is *pure* if any simplex has *at least one* coface of dimension m.

Definition 13 (Pseudo-manifold) An m-dimensional simplicial complex is called a *pseudo-manifold* if it is pure and if any $(m-1)$-simplex has *exactly two* m-dimensional cofaces.

Definition 14 (Pseudo-manifold with boundary) An m-dimensional simplicial complex is a *pseudo-manifold with boundary* if it is pure and if any $(m-1)$-simplex has *at most two* m-dimensional cofaces. The *boundary* $\partial\mathscr{B}$ of an m-dimensional pseudo-manifold simplicial complex $\mathscr{B}$ with boundary, is the $(m-1)$-simplicial complex made of the closure of all $(m-1)$-simplices with exactly one m-dimensional coface, that is the simplices and their faces.

Remark 3 Usual definitions of pseudo-manifolds require moreover the complex to be *strongly connected*, which means that its dual graph, i.e. the graph with one vertex for each m-simplex and one edge for each pair of m-simplices sharing an $(m-1)$-simplex, is connected. In our context, this property is not required.

Our main result (Theorem 4) does not require any global orientability for the manifold M and simplicial complex without boundary $\mathscr{A}$. It applies to non-orientable manifolds as well. However, in the proof of the theorem, we will need to orient locally a pseudo-submanifold of $\mathscr{A}$ with boundary.

Definition 15 (Oriented pseudo-manifold) An m-dimensional pseudo-manifold with boundary $\mathscr{B}$ is said to be *oriented* if each m-simplex is given an orientation such that, if Γ is the m-chain over $\mathbb{Z}$ (or $\mathbb{R}$) with coefficient 1 on each m-simplex of $\mathscr{B}$ then the support of $\partial\Gamma$ is precisely $\partial\mathscr{B}$.

Definition 16 (Simplexwise positive map) Let $\mathscr{B}$ be an oriented m-dimensional pseudo-manifold with boundary. A piecewise linear map $F : \mathscr{B} \to \mathbb{R}^m$ is said to be *simplexwise positive* if the image $F(\sigma) = [F(v_0),\dots,F(v_m)]$ of each oriented m-simplex $\sigma = [v_0,\dots,v_m] \in \mathscr{B}$ is a non-degenerate m-simplex embedded in $\mathbb{R}^m$ and is positively oriented.

Having introduced these definitions, we are ready to state an adapted version of a topological result of Whitney [17, Appendix II, Section 15] that will be the main tool for the proof of Theorem 4.

In Lemma 17 , the simplexwise positive map is piecewise linear instead of being merely assumed piecewise smooth as in Whitney's original statement. Indeed, this makes the proof simpler and suffices in our context. Also our version states that the map is not only one-to-one but in fact an homeomorphism: this is an easy consequence of Whitney's original proof as well. The proof given in [17, Appendix II, Section 15] skiping some small steps, is not very easy to follow. Because we want to remedy this and we altered Whitney's original statement we give a detailed exposition of Whitney argument in Section A.

Lemma 17 (After Whitney) *Assume that the following conditions are satisfied:*

(C1) *$\mathscr{C}$ is an oriented finite m-pseudo-manifold with boundary and $F: |\mathscr{C}| \to \mathbb{R}^m$ is a simplexwise positive map.*
(C2) *$\mathbf{R} \subset \mathbb{R}^m$ is a connected open set such that $\mathbf{R} \cap F(|\partial\mathscr{C}|) = \emptyset$.*
(C3) *There is $y \in \mathbf{R} \setminus F(|\mathscr{C}^{m-1}|)$ such that $F^{-1}(y)$ is a single point.*

Then the restriction of F to $F^{-1}(\mathbf{R})$ is an homeomorphism between $F^{-1}(\mathbf{R})$ and $\mathbf{R}$.

Theorem 4 actually holds for simplicial complexes without boundary (as defined above) which forms a larger class than the class of pseudo-manifolds.

Definition 18 (Simplicial complex without boundary) An m-dimensional simplicial complex is called a *simplicial complex without boundary* if it is pure and if any $(m-1)$-simplex has *at least two* m-dimensional cofaces.

Any pseudo-manifold (without boundary) is a simplicial complex without boundary but the converse is not true in general.

4 Proof of Theorem 4

The proof of Theorem 4 makes use of this classical observation:

Theorem 19 (Triangulation of manifolds) *Let H be a continuous mapping from a (non-empty) m-dimensional finite simplicial complex $\mathscr{A}$, to a connected m-manifold without boundary M.*

If H is injective and the underlying space $|\mathscr{A}|$ of $\mathscr{A}$ is a manifold without boundary, then H is a homeomorphism.

Proof By the invariance of domain theorem [6], we have that H is open. Being injective, continuous and open, H is a homeomorphism on its image. Since $\mathscr{A}$ is finite, it is compact and $H(|\mathscr{A}|)$ is the image of an open and compact set by an open and continuous map and is therefore open and compact. Since M is connected its only open and closed non-empty subset is M itself, therefore $H(|\mathscr{A}|) = M$. □

4.1 Overview of the proof

We establish Theorem 4 by means of three primary observations. First, in Section 4.2 we show that the conditions of the theorem imply that $\mathscr{A}$ is manifold (Lemma 20). This observation, together with the results that are obtained in demonstrating it, make it a relatively easy exercise in Section 4.3 to demonstrate that $\mathrm{pr}_M|_{|\mathscr{A}|}$ is injective, and therefore, by Theorem 19, it is a homeomorphism (Proposition 25). Finally, in Section 4.4 we show that $|\mathscr{A}|$ and M are ambient isotopic.

Constants and definitions

From Lemma 10, we have

$$\|\mathrm{pr}_M(x) - x\| \leq \frac{2L^2}{\mathrm{rch}(M)}$$

which, together with Condition (a) of the theorem, gives

$$\|\mathrm{pr}_M(x) - x\| \leq \eta := \frac{L}{4}. \tag{1}$$

For $p \in \mathscr{P}$ and $\rho > 0$, we define $\mathscr{A}_{p,\rho}$ as the subcomplex of $\mathscr{A}$ consisting of all m-simplices entirely included in $B(p,\rho)$ together with all their faces:

$$\mathscr{A}_{p,\rho} = \overline{\{\sigma \in \mathscr{A} \mid \dim(\sigma) = m, \iota(\sigma) \subset B(p,\rho)\}}$$

The proof relies on the properties of the continuous piecewise linear function

$$F_p : |\mathscr{A}_{p,2.8L}| \to T_pM$$

defined as the restriction of pr_{T_pM} to $\iota(|\mathscr{A}_{p,2.8L}|)$. We will focus in particular on the restriction of F_p to

$$W_p = F_p^{-1}(\mathbf{R}_p),$$

where

$$\mathbf{R}_p = T_pM \cap B^\circ\left(p, \frac{L}{\sqrt{2}} + 2\eta)\right) = T_pM \cap B^\circ\left(p, L\left(\frac{1}{\sqrt{2}} + \frac{1}{2}\right)\right).$$

Remark 4 The size of the set $\mathbf{R}_p$ is primarily motivated for convenience in establishing the injectivity of $\mathrm{pr}_M|_{|\mathscr{A}|}$ (see (13)). The set $\mathscr{A}_{p,2.8L}$ is chosen to be large enough to ensure that condition (C2) of Whitney's lemma is satisfied for this choice (see (11)).

4.2 $\mathscr{A}$ is manifold

Lemma 20 (Manifold complex) *If the conditions of Theorem 4 are met then $\mathscr{A}$ is an m-manifold complex with $\{(W_p, F_p)\}_{p\in\mathscr{P}}$ an atlas for $|\mathscr{A}|$.*

Overview of the proof

We first prove that the map F_p and the set $\mathbf{R}_p$ meet the conditions of Whitney's Lemma. Using a separate step for each of the three conditions (C1)–(C3), they are shown to be satisfied which gives a homeomorphism beween W_p and $\mathbf{R}_p$, and thus that $\mathscr{A}$ is a manifold.

Step 1: (C1) is satisfied

Claim The m-simplices of $\mathscr{A}_{p,2.8L}$ can be oriented in such a way that:

(1) $\mathscr{A}_{p,2.8L}$ is an oriented pseudo-manifold with boundary,
(2) F_p is simplexwise positive.

Proof of Claim 4.2 In order to prove this claim, we need the angle between a simplex $\sigma \in \mathscr{A}_{p,2.8L}$ and the tangent space T_pM to be strictly less $\pi/2$. If $\sigma \in \mathscr{A}_{p,2.8L}$, consider a vertex q of σ.

We have from Lemma 7 that $\sin\angle(\sigma, T_qM) \le \frac{L}{t\,\mathrm{rch}(M)}$. From condition (a) of Theorem 4, one has $\frac{L}{t\,\mathrm{rch}(M)} \le \sin\pi/8$ and therefore

$$\angle(\sigma, T_qM) \le \pi/8. \tag{2}$$

Also, from Lemma 8, since $\|p-q\| \le 2.8L$ and using condition (a) of the theorem we have

$$\sin\frac{(\angle T_pM, T_qM)}{2} \le \frac{\|p-q\|}{2\,\mathrm{rch}(M)} \le \frac{2.8 \times \frac{1}{8}}{2} = 0.175 < \sin\pi/8.$$

It follows that

$$\angle(T_pM, T_qM) < \pi/4, \tag{3}$$

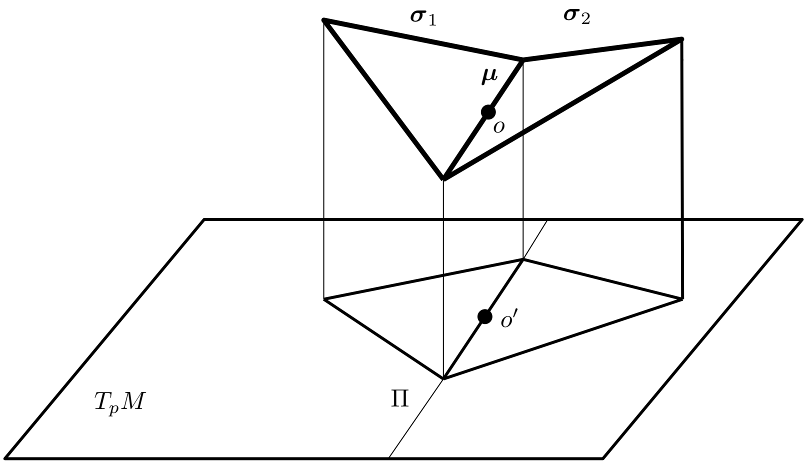

Fig. 2 The notation for the proof of Claim 4.2.

which, together with (2) gives

$$\angle(\sigma, T_pM) < 3\pi/8. \tag{4}$$

This last inequality shows that the restriction of pr_{T_pM} to each m-simplex in $\mathscr{A}_{p,2.8L}$ is injective.

For a given orientation of T_pM, each m-simplex $\sigma \in \mathscr{A}_{p,2.8L}$ is oriented in such a way that $F_p(\sigma)$ has positive orientation in T_pM.

Consider two distinct simplices $\sigma_1, \sigma_2 \in \mathscr{A}_{p,2.8L}$ sharing a common $(m-1)$-face $\mu = \sigma_1 \cap \sigma_2$. Since F_p is non-degenerate on σ_1 it is non-degenerate on μ and $F_p(\mu)$ spans a hyperplane Π in T_pM. Consider a point $o' = F_p(o)$ in the relative interior of $F_p(\mu)$ (see Figure 2). If V_1 is a neigboorhood of o in σ_1, then $F_p(V_1)$ covers a neighborhood of o' in one of the half-spaces bounded by Π. The same appears for a neighborhood V_2 of o in σ_2 and then condition (b) of the theorem enforces these two half-spaces to be distinct.

Following the same argument, we see that μ cannot have as a coface a third m-simplex σ_3 as there is no room for three pairwise disjoint open half-spaces in $\mathbb{R}^m$ to share a same bounding hyperplane Π. Thus $\mathscr{A}_{p,2.8L}$ is a pseudo-manifold with boundary.

Now consider $F_p(\sigma_1)$ and $F_p(\sigma_2)$ as simplicial chains with coefficients in $\mathbb{Z}$ and choose any orientation of $F_p(\mu)$. It follows from the previous observation that the signs of the coefficients of $F_p(\mu)$ in the respective expressions of $\partial F_p(\sigma_1)$ and $\partial F_p(\sigma_2)$ are opposite. It follows that the coefficient of μ in $\partial(\sigma_1 + \sigma_2)$ is zero.

Thus $\mathscr{A}_{p,2.8L}$ and F_p meet the respective conditions in Definitions 15 and 16, and the claim is proven. □

Notice that a consequence of Claim 4.2 is that $\mathscr{A}$ is a pseudo-manifold, not just a simplicial complex without boundary.

Step 2: (C2) is satisfied

In order to be able to apply Whitney's lemma, we need a second claim.

We remind ourselves of the notation $\mathbf{R}_p = T_pM \cap B^\circ\left(p, \frac{L}{\sqrt{2}} + 2\eta)\right)$ and F_p is the restriction of pr_{T_pM} to $\iota(|\mathscr{A}_{p,2.8L}|)$.

Claim 21 $\mathbf{R}_p \cap F_p(|\partial\mathscr{A}_{p,2.8L}|) = \emptyset$.

Proof of Claim 21 Let $x \in |\partial\mathscr{A}_{p,2.8L}|$. Since $\mathscr{A}$ is a pseudo-manifold without boundary, x must belong to a simplex in $\mathscr{A}_{p,2.8L}$ and also to a simplex in $\mathscr{A} \setminus \mathscr{A}_{p,2.8L}$. The later condition together with the definition of $\mathscr{A}_{p,2.8L}$ implies that

$$x \in B(p, 2.8L) \setminus B(p, 1.8L). \tag{5}$$

Equations (1) and (5) give:

$$\mathrm{pr}_M(x) \in B(p, 2.8L + \eta) \setminus B(p, 1.8L - \eta) = B(p, 3.05L) \setminus B(p, 1.55L).$$

We are now going to decompose $\mathrm{pr}_M(x) - p$ into vectors $\mathbf{u} \in T_pM$ and $\mathbf{v} \in N_pM$, where N_pM denotes the normal space at p. One can write (see Figure 3):

$$\mathrm{pr}_M(x) - p = \mathbf{u} + \mathbf{v}.$$

We now bound $\|\mathbf{u}\|$ and distinguish whether $\mathbf{v} = 0$ or not.

If $\mathbf{v} = 0$, i.e., if $\mathrm{pr}_M(x) \in T_pM$, then since $\mathrm{pr}_M(x) \notin B(p, 1.55L)$ one has $\|\mathbf{u}\| \geq 1.55L$.

Let us assume now that $\mathbf{v} \neq 0$.

The open ball B° centered at $c = p + \frac{\mathrm{rch}(M)}{\|\mathbf{v}\|}\mathbf{v}$ with radius $\mathrm{rch}(M)$ is tangent to M at p. Since $\mathrm{pr}_M(x) \in B(p, 3.05L) \setminus B(p, 1.55L)$ and since, from Lemma 6, B° has no intersection with M, one gets the following pair of equations:

$$(1.55L)^2 < \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \leq (3.05L)^2 \tag{6}$$

$$(\mathrm{rch}(M) - \|\mathbf{v}\|)^2 + \|\mathbf{u}\|^2 \geq \mathrm{rch}(M)^2. \tag{7}$$

We first use Equation (7) and the fact that $\mathrm{rch}(M) \geq 8L$ (by condition (a) of Theorem 4) to get

$$16L\|\mathbf{v}\| - \|\mathbf{v}\|^2 \leq \|\mathbf{u}\|^2.$$

Combining with Equation 6, we obtain

$$\max\{16L\|\mathbf{v}\| - \|\mathbf{v}\|^2, (1.55L)^2 - \|\mathbf{v}\|^2\} \leq \|\mathbf{u}\|^2.$$

Note that thanks to (6), $\|\mathbf{v}\| < 3.05L$. The minimum of $\max\{16L\|\mathbf{v}\| - \|\mathbf{v}\|^2, (1.55L)^2 - \|\mathbf{v}\|^2\}$ is attained when both terms are equal which happens when $\|\mathbf{v}\| = (1.55^2/16)L = 0.15015625L$. This gives us in particular

$$\|\mathbf{u}\| > 1.54L. \tag{8}$$

Hence we have in both cases $\mathbf{v} = 0$ and $\mathbf{v} \neq 0$

$$\|\mathbf{u}\| = \|\mathrm{pr}_{T_pM}(\mathrm{pr}_M(x)) - p\| > 1.54L. \tag{9}$$

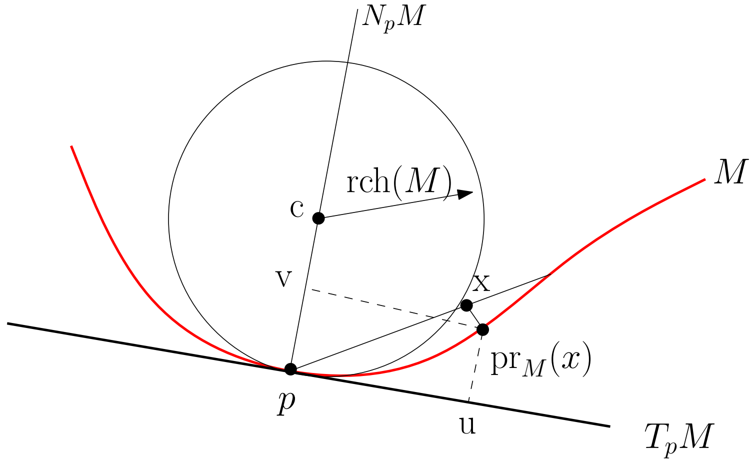

Fig. 3 The notation for the proof of Claim 21.

Moreover, since $\|x - \mathrm{pr}_M(x)\| < \eta$, Equation (1) yields

$$\|\mathrm{pr}_{T_pM}(x) - \mathrm{pr}_{T_pM}(\mathrm{pr}_M(x))\| < \eta = 0.25L, \tag{10}$$

since projection reduces length. By the triangle inequality, equations (9) and (10) give us

$$\|\mathrm{pr}_{T_pM}(x) - p\| > 1.54L - 0.25L = 1.29L. \tag{11}$$

Therefore, since $\mathbf{R}_p$ is defined as $\mathbf{R}_p = T_pM \cap B^\circ\left(p, \left(\frac{1}{\sqrt{2}} + 0.5\right)L\right)$
we have proved that $x \in \partial\mathscr{A}_{p,2.8L} \Rightarrow \mathrm{pr}_{T_pM}(x) \notin \mathbf{R}_p$. It follows that

$$\mathbf{R}_p \cap F_p(|\partial\mathscr{A}_{p,2.8L}|) = \emptyset,$$

which ends the proof of the claim. □

Step 3: (C3) is satisfied

We now denote by $\mathscr{C}^{m-1} = \mathscr{A}^{m-1}_{p,2.8L}$ the $(m-1)$-skeleton of $\mathscr{A}_{p,2.8L}$, i.e., the simplicial complex made of simplices of $\mathscr{A}_{p,2.8L}$ of dimension at most $m-1$. Since $F_p(|\mathscr{C}^{m-1}|)$ is a finite union of $(m-1)$-dimensional simplices, it cannot cover the projection of an m-simplex. Therefore condition (b) of the theorem shows that there is a $y \in \mathbf{R}_p \setminus F_p(|\mathscr{C}^{m-1}|)$ such that $F_p^{-1}(y)$ is a single point.

All Conditions (C1)–(C3) being satisfied, Whitney's lemma applies, proving the following proposition:

Proposition 22 *The restriction of F_p to $F_p^{-1}(\mathbf{R}_p)$ is an homeomorphism from $F_p^{-1}(\mathbf{R}_p)$ to $\mathbf{R}_p$.*

Step 4 : proof of lemma 20

We start with an easy lemma.

Lemma 23

$$\mathscr{A} \cap B(p,L) \subset F_p^{-1}(\mathbf{R}_p) \subset \mathscr{A} \cap B(p,2.8L).$$

Proof Recall that F_p is the restriction of pr_{T_pM} to $|\mathscr{A}_{p,2.8L}|$. By definition of $\mathscr{A}_{p,2.8L}$, and since the diameter of any simplex is upper bounded by L, one has:

$$|\mathscr{A}| \cap B(p,1.8L) \subset \mathscr{A}_{p,2.8L} \subset |\mathscr{A}| \cap B(p,2.8L) \tag{12}$$

The second inclusion in the statement of the lemma follows from the second inclusion in (12) and the definition of F_p. If $x \in |\mathscr{A}| \cap B(p,L)$ then $x \in \mathscr{A}_{p,2.8L}$. Since $\mathrm{pr}_{T_pM}(B(p,L)) \subset B(p,L)$ one has

$$\mathrm{pr}_{T_pM}(\mathscr{A} \cap B(p,L)) \subset T_pM \cap B(p,L) \subset \mathbf{R}_p.$$

This shows that $\mathscr{A} \cap B(p,L) \subset F_p^{-1}(\mathbf{R}_p)$. □

Claim 24 *For any $p \in \mathscr{P} = \mathscr{A}^0$, the restriction of F_p to $W_p = F_p^{-1}(\mathbf{R}_p)$ yields a homeomorphism onto $\mathbf{R}_p$. Thus $\mathscr{A}$ is a manifold, and $\{(W_p,F_p)\}_{p\in\mathscr{P}}$ is an atlas for $|\mathscr{A}|$.*

Proof From Whitney's lemma, F_p defines a homeomorphism from W_p to $\mathbf{R}_p$.

It follows that $W_p = F_p^{-1}(\mathbf{R}_p)$ is an open m-manifold that contains the star of p, since $F_p^{-1}(\mathbf{R}_p) \supset \mathscr{A} \cap B(p,L)$ (Lemma 23). Therefore, since any $y \in \mathscr{A}$ belongs to the star of some vertex p we have that $|\mathscr{A}|$ is a manifold and $\{(W_p,F_p)\}_{p\in\mathscr{P}}$ is an atlas. □

This completes the proof of Lemma 20.

Remark 5 In order to establish that $\mathscr{A}$ is manifold, we need only consider sets U_p that are large enough to ensure that $\{(U_p,F_p)\}_{p\in\mathscr{P}}$ is an atlas. The sets W_p are larger than we need; by Lemma 11, it would be sufficient to take $U_p = B\big(p, \frac{1}{\sqrt{2}}L\big) \cap \iota(|\mathscr{A}|)$. Notice, that since pr_{T_pM} does not increase distances, $F_p(U_p)$ is contained in the set $T_pM \cap B\big(p, \frac{1}{\sqrt{2}}L\big)$. We used larger sets for convenience in demonstrating below that $\mathrm{pr}_M|_{|\mathscr{A}|}$ is injective (Proposition 25).

Also, the existence of the manifold M is not essential in the demonstration that $\mathscr{A}$ is manifold; it suffices to have a collection of hyperplanes $\{T_p\}_{p\in\mathscr{P}}$ such that T_p makes a sufficiently small angle with all m-simplices that lie sufficently close to the vertex p.

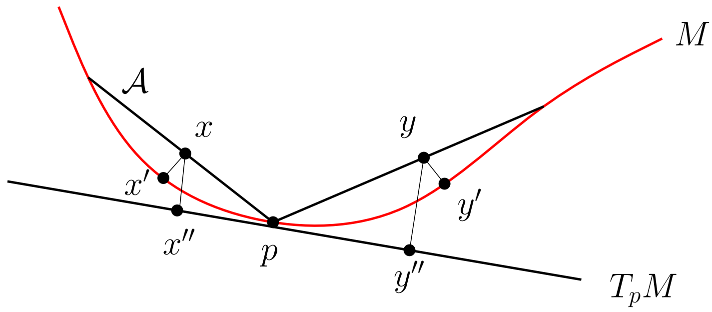

Fig. 4 For the definition of x', x'', y', y''.

4.3 $\mathrm{pr}_M|_{|\mathscr{A}|}$ is a homeomorphism

Proposition 25 ($\mathrm{pr}_M|_{|\mathscr{A}|}$ is a homeomorphism) *Let $M \subset \mathbb{R}^N$ be a connected C^2 m-dimensional submanifold of $\mathbb{R}^N$ with reach $\mathrm{rch}(M) > 0$, and $\mathscr{P} \subset M$ a finite set of points. Suppose that $\mathscr{A}$ is a m-dimensional manifold simplicial complex whose vertex set, $\mathscr{A}^0$, is identified with $\mathscr{P}$. Let $L, t > 0$ be such that for any m-simplex $\sigma \in \mathscr{A}$ one has:*

$$t \le t(\sigma) \quad \text{and} \quad L(\sigma) \le L.$$

If

(a) *Simplices are small with respect to the reach:*

$$\frac{L}{\mathrm{rch}(M)} \le \min\left(\frac{1}{8}, t \sin \pi/8\right),$$

(b) $\{(W_p, F_p)\}_{p \in \mathscr{P}}$ *is an atlas for* $|\mathscr{A}|$,

then:

(1) *The closest-point projection map $\mathrm{pr}_M|_{|\mathscr{A}|}$ is a homeomorphism $|\mathscr{A}| \to M$.*
(2) *The inclusion $\iota : |\mathscr{A}| \to \mathbb{R}^N$ is an embedding.*

Proof Let $x, y \in \mathscr{A}$. For convenience, we will write $x' = \mathrm{pr}_M(x)$, $x'' = \mathrm{pr}_{T_pM}(x)$ and similarly for y, y' and y'' (see Figure 4).

Assume that $\|x - y\| \ge 2\eta$. From Lemma 10, we have $\|x' - x\| < \eta$ and $\|y' - y\| < \eta$. The triangular inequality then shows that $\|x' - y'\| > 0$ and therefore $x' \ne y'$.

Assume now that $x, y \in \mathscr{A}$ with $x \ne y$ and $\|x - y\| < 2\eta$, and let us prove that $x' \ne y'$. From Lemma 11, we know that there is $p \in \mathscr{P}$ such that $\|x - p\| < L/\sqrt{2}$ and therefore one has

$$x, y \subset B\left(p, \frac{L}{\sqrt{2}} + 2\eta\right). \tag{13}$$

It follows that $x'', y'' \subset T_pM \cap B\left(p, \frac{L}{\sqrt{2}} + 2\eta\right)$, and with the notations of Whitney's Lemma applied in a neighborhood of p, this translates to $F_p(x), F_p(y) \in \mathbf{R}_p$ and $x, y \in F_p^{-1}(\mathbf{R}_p)$.

Since F_p is a homeomorphism from $F_p^{-1}(\mathbf{R}_p)$ to $\mathbf{R}_p$ (Claim 24), it has a continuous inverse $F_p^{-1}:\mathbf{R}_p \to F_p^{-1}(\mathbf{R}_p)$ and the set $F_p^{-1}(\mathbf{R}_p)$ can then be seen as the graph of the continuous map ϕ:

$$\phi : \mathbf{R}_p \to N_pM, \quad z \mapsto F_p^{-1}(z) - z,$$

where N_pM denotes the normal space at p.

Let σ be a simplex in $\mathscr{A}$ with a non-empty intersection with $F_p^{-1}(\mathbf{R}_p)$. Since $F_p^{-1}(\mathbf{R}_p) \subset \mathscr{A}_{p,2.8L}$ (Lemma 23), we have that $\sigma \in \mathscr{A}_{p,2.8L}$. By Equation (4),

$$\angle(\sigma, T_pM) < 3\pi/8.$$

Since the graph of ϕ restricted to $\mathbf{R}_p$ is made of a (finite) number of simplices whose angles with T_pM are less than $3\pi/8$, ϕ is Lipschitz with constant $\tan(3\pi/8)$. It follows that, with x and y as in Equation (13),

$$\|\phi(y'') - \phi(x'')\| \leq \tan(3\pi/8)\ \|y'' - x''\|,$$

and then

$$\angle(T_pM, y - x) < 3\pi/8. \tag{14}$$

On the other hand, since $x \in B\left(p, \frac{L}{\sqrt{2}} + 2\eta\right)$ and $\|x - x'\| < \eta$ we have

$$x' \in B\left(p, \frac{L}{\sqrt{2}} + 3\eta\right), \tag{15}$$

and, since $\eta = 0.5$,

$$x' \in B(p, 1.46L). \tag{16}$$

Equation (15) together with Lemma 8 and $L/\operatorname{rch}(M) \leq 1/8$ (Condition (a) of Theorem 4) gives

$$\sin\frac{\angle(T_pM, T_{x'}M)}{2} \leq \frac{1.46L}{2\operatorname{rch}(M)} = 0.925 < \sin(\pi/16).$$

It follows that

$$\angle(T_pM, T_{x'}M) < \pi/8.$$

Therefore, if we assume for a contradiction that $x' = y'$, we have that $x - y$ is orthogonal to $T_{x'}M$, that is $\angle(T_{x'}, y - x) = \pi/2$ and one gets:

$$\angle(T_pM, y - x) > \pi/2 - \pi/8 = 3\pi/8$$

a contradiction with Equation (14). This concludes the proof that $\mathrm{pr}_M|_{|\mathscr{A}|}$ is injective.

We now can apply Theorem 19 which completes the proof of the first claim of the proposition.

Since $\mathrm{pr}_M|_{|\mathscr{A}|}$ is defined as $\mathrm{pr}_M \circ \iota$, the fact that it is a homeomorphism implies the second claim of the proposition: ι is an embedding.

Lemma 26 below applies and we get the required ambient isotopy. □

4.4 Ambient isotopy

We now show that the homeomorphism $\mathrm{pr}_M|_{|\mathscr{A}|}$ induces an ambient isotopy, for $|\mathscr{A}|$ a distance at most the reach from M.

Proposition 26 *Let M be a m-dimensional submanifold of Euclidean space $\mathbb{R}^N$ with reach $\mathrm{rch}(M) > 0$ and $|\mathscr{A}|$ a subset of $\mathbb{R}^N$ such that:*

(1) $\sup_{x\in|\mathscr{A}|} \inf_{y\in M} d(x,y) < \mathrm{rch}(M)$
(2) the restriction of pr_M, the closest map projection on M to $|\mathscr{A}|$ is an homeomorphism.

Then $|\mathscr{A}|$ and M are ambient isotopic.

Observes that by definition of $\mathrm{rch}(M)$ condition (1) ensures that the definition of pr_M in condition (2) is unambiguous.
Proof In the conditions of the lemma, there are real numbers a and b such that:

$$\sup_{x\in|\mathscr{A}|} d(x, \mathrm{pr}_M(x)) < a < b < \mathrm{rch}(M)$$

We define a map $\Psi : [0,1]\times\mathbb{R}^N \to \mathbb{R}^N$ as follows. Denote by $M^{\oplus b}$ the b-offset of M.

For $x \in M^{\oplus b}$ we know that $x \mapsto \mathrm{pr}_M(x) \in M$ and $x \mapsto \alpha(x) = \mathrm{pr}_M|^{-1}_{|\mathscr{A}|}(\mathrm{pr}_M(x)) \in |\mathscr{A}|$ are continuous. For $x \in M^{\oplus b}$, x and $\alpha(x)$ are in the ball B centered at $\mathrm{pr}_M(x)$ with radius b in the normal space to M at $\mathrm{pr}_M(x)$. One has $x \in |\mathscr{A}| \iff x = \alpha(x)$ and if $x \neq \alpha(x)$ one defines $\beta(x)$ to be the intersection point of the boundary of B with the half-line starting at $\alpha(x)$ and going through x.

Define also the real valued function:

$$x \mapsto \lambda(x) = \frac{\|x-\alpha(x)\|}{\|\beta(x)-\alpha(x)\|}$$

Observes that if $x \neq \alpha(x)$ one has by the definition of λ that $x = (1-\lambda(x))\alpha(x) + \lambda(x)\beta(x)$. Also, since $|\lambda(x)| \leq \frac{\|x-\alpha(x)\|}{b-a}$ we can check that the map:

$$x \mapsto \begin{cases} (1-\lambda(x))\mathrm{pr}_M(x) + \lambda(x)\beta(x) & \text{if } x \neq \alpha(x) \\ \mathrm{pr}_M(x) & \text{if } x = \alpha(x) \end{cases}$$

is continuous.

We can now give an explicit expression for the ambient isotopy Ψ:

$$\Psi(t,x) = \begin{cases} x & \text{if } x \notin M^{\oplus b} \\ (1-t)x + t\,\mathrm{pr}_M(x) & \text{if } x = \alpha(x) \\ (1-t)x + t\big((1-\lambda(x))\mathrm{pr}_M(x) & \\ \qquad\qquad + \lambda(x)\beta(x)\big) & \text{if } x \in M^{\oplus b} \setminus \{\alpha(x)\} \end{cases}$$

It is a simple exercise to check that Ψ is continuous both in t and x. □

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A Proof of the (adapted) Whitney lemma

In this appendix we prove our variation of Whitney's lemma.
Proof of Lemma 17 The proof holds in five successive claims.

Claim 27 *The restriction of F to the star* $\mathrm{star}(\sigma^{m-1})$ *of some $(m-1)$-simplex σ^{m-1} of $\mathscr{C}$ which is not in $\partial\mathscr{C}$ is injective and open, then $F|_{\mathrm{star}(\sigma^{m-1})}$ is injective and open.*

For a simplex σ, let us denote by $\mathrm{int}(\sigma)$ its relative interior.

By definition of pseudo-manifolds, $\sigma^{m-1} \notin \partial\mathscr{C}$ implies that $\mathrm{star}(\sigma^{m-1})$ is the union of the relative interior $\mathrm{int}(\sigma_1^m)$ and $\mathrm{int}(\sigma_2^m)$ of exactly two proper cofaces (σ_1 and σ_2) and the relative interior $\mathrm{int}(\sigma^{m-1})$ of σ^{m-1} itself.

For $x \in \mathrm{int}(\sigma_i^m)$, $i = 1,2$, the restriction of F to $|\sigma_i^m|$ being a non-degenerate linear map, F is locally open at x. Consider now the case where $x \in \mathrm{int}(\sigma^{m-1})$. $F(x)$ belongs to the boundary of $F(|\sigma_1^m|)$. Moreover, $F(\mathrm{int}(\sigma^{m-1}))$ spans an hyperplane Π in $\mathbb{R}^m$ that separates the space in two closed half spaces H^-, and H^+, with $H^- \cap H^+ = \Pi$.

Then the fact that $\mathscr{C}$ is oriented and that F is simplexwise positive implies that $F(|\sigma^{m-1}|)$ appears with opposite orientations in the respective boundaries of $F(|\sigma_1^m|)$ and $F(|\sigma_2^m|)$.

We assume without loss of generality that $F(|\sigma_1^m|) \subset H^-$: it follows that $F(|\sigma_2^m|) \subset H^+$. Taking $\rho > 0$ smaller than the distance between $F(x)$ and the image of the boundary of the closed star $F(\partial\,\overline{\mathrm{star}}(\sigma^{m-1}))$ we have that:

$$B(x,\rho) \subset F\left(\mathrm{star}(\sigma^{m-1})\right)$$

and we have proven that $F|_{\mathrm{star}(\sigma^{m-1})}$ is open.

From the definition of simplexwise positiveness, the restriction of F to any simplex is injective. Because $\mathscr{C}$ is a pseudomanifold $\mathrm{star}(\sigma^{m-1}) \subset |\sigma_1^m| \cup |\sigma_2^m|$. Now suppose there was a pair $x,y \in \mathrm{star}(\sigma^{m-1})$ with $x \neq y$ and $F(x) = F(y)$. Let us assume without loss of generality that $x \in |\sigma_1^m|$. Then we must have $y \in |\sigma_2^m|$.

But since $F(|\sigma_1^m|) \subset H^-$, $F(|\sigma_2^m|) \subset H^+$ it follows that $F(x) = F(y) \in H^- \cap H^+ = \Pi$. Then $x,y \in |\sigma^{m-1}|$ but again, since F is one-to-one on each simplex, we get $x = y$. Claim 27 is proven.

Claim 28 *If $x,y \in |\mathscr{C} \setminus \partial\mathscr{C}|$ are such that $x \neq y$ and $F(x) = F(y)$ then if $x \in \mathrm{int}(\sigma_x)$ and $y \in \mathrm{int}(\sigma_y)$, we have* $\mathrm{star}(\sigma_x) \cap \mathrm{star}(\sigma_y) = \emptyset$.

Indeed otherwise there would be a simplex $\sigma \in \mathrm{star}(\sigma_x) \cap \mathrm{star}(\sigma_y)$ such that $x,y \in \sigma$ but we have $x \neq y$ and $F(x) = F(y)$ which contradicts the fact that the restriction of F to σ is injective.

Claim 29

$$x,y \in \mathbf{R} \setminus F(|\mathscr{C}^{m-1}|) \Rightarrow \#F^{-1}(x) = \#F^{-1}(y)$$

where $\#E$ denotes the cardinality of the set E.

Since $F(|\mathscr{C}^{m-2}|)$ is a finite union of simplices of codimension 2, it cannot disconnect the open set $\mathbf{R}$. In other words $\mathbf{R} \setminus F(|\mathscr{C}^{m-2}|)$ is path connected.

Consider x and y as in the claim. Since $\mathbf{R} \setminus F(|\mathscr{C}^{m-1}|) \subset \mathbf{R} \setminus F(|\mathscr{C}^{m-2}|)$ there exists a path $\gamma : [0,1] \to \mathbf{R} \setminus F(|\mathscr{C}^{m-2}|)$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Assume for a contradiction that for example $\#F^{-1}(x) > \#F^{-1}(y)$ and consider $t_0 = \sup\{t, \#F^{-1}(\gamma(t)) \geq \#F^{-1}(x)\}$. Since $\gamma(t_0) \in \mathbf{R} \setminus F(|\mathscr{C}^{m-2}|)$, any point $p \in F^{-1}(\gamma(t_0))$ belongs to the relative interior of a simplex σ of dimension m or $m-1$. In both cases, thanks to Claim 27, the restriction of F to the star of σ is open and injective. It follows that there are $\#F^{-1}(\gamma(t_0))$ distinct such stars of simplices whose image cover $\gamma(t_0)$. Two such stars $\mathrm{star}(\sigma_1)$ and $\mathrm{star}(\sigma_2)$ are disjoint by claim 28. But since the restriction of F to each of theses stars of simplices is open, there is an open neighborhood of $\gamma(t_0)$ which is covered at least $\#F^{-1}(\gamma(t_0))$ times, a contradiction with the definition of x, y, and t_0. Claim 29 is proven.

Claim 30 *The restriction of F to $\mathscr{C} \setminus \partial\mathscr{C}$ is open.*

Consider a k-simplex σ^k , $0 \leq k \leq m$ of $\mathscr{C}$ which is not in $\partial\mathscr{C}$: $\sigma^k \in \mathscr{C} \setminus \partial\mathscr{C}$.

Take $p \in \mathrm{int}(\sigma^k)$, where $\mathrm{int}(\sigma^k)$ is the relative interior of σ^k. We denote by $L = \overline{\mathrm{star}(\sigma^k)}$ the simplicial complex closure of the star of σ_k. Since L is a subcomplex of $\mathscr{C}$, it inherits from $\mathscr{C}$ its property to be an oriented pseudo-manifold on which F is simplexwise positive. Also, since $\sigma^k \notin \partial\mathscr{C}$ we have $\sigma^k \notin \partial L$. The restriction of F to any m-simplex is injective and since any $x \in \partial L$ belongs to some m-simplex containing also p we have that $F(p) \notin F(\partial L)$. Since $F(\partial L)$ is compact, there exists an open neigborhood $U \ni q = F(p)$ such that $U \cap F(\partial L) = \emptyset$. We can then apply Claim 28 to the complex L with U playing the role of $\mathbf{R}$: the number of inverse image in $U \setminus F(L^{m-1})$ is constant. But since the image of any m-simplex in L, being a non degenerate full-dimensional simplex containing q, intersects U, this number is at least one.

It follows that $F(L) \supset U \setminus F(L^{m-1})$. But L being compact $F(L)$ is compact and therefore $F(L) \supset \overline{U \setminus F(L^{m-1})} \supset U$. Since $U \cap F(\partial L) = \emptyset$ we have proven that $U \subset F(L \setminus \partial L)$.

Given a set A, a point p and $\varepsilon > 0$ denotes by $\mathscr{H}_{p,\varepsilon}(A) = \{p + \varepsilon(a-p), a \in A\}$ the image of A by the homothety with center p and ratio ε. Since F is piecewise linear, for any $\varepsilon > 0$, one has:

$$F\left(\mathscr{H}_{p,\varepsilon}\left(|L \setminus \partial L|\right)\right) = \mathscr{H}_{F(p),\varepsilon}\left(F\left(|L \setminus \partial L|\right)\right) \supset \mathscr{H}_{F(p),\varepsilon}\left(U\right).$$

It follows that the image of an arbitrary small neighborhood of p covers an open neighborhood of $q = F(p)$. We have shown that F is open at p. But since any point $p \in \mathscr{C} \setminus \partial\mathscr{C}$ belongs to the relative interior of some simplex in $\mathscr{C} \setminus \partial\mathscr{C}$ this result apply to any $p \in \mathscr{C} \setminus \partial\mathscr{C}$. We have thus shown that the restriction of F to $\mathscr{C} \setminus \partial\mathscr{C}$ is open and the claim is proven. Notice that, since $\mathrm{star}(\sigma^k)$ is open, we get as a consequence of the claim that the image by F of the star of any k-simplex $\sigma^k \in \mathscr{C} \setminus \partial\mathscr{C}$, $0 \leq k \leq m$, is open. Hence

$$\sigma^k \in \mathscr{C} \setminus \partial\mathscr{C} \Rightarrow F\left(\mathrm{star}(\sigma^k)\right) \text{ is open.} \tag{17}$$

Our last claim will end the proof of the lemma:

Claim 31 *If there is $q \in \mathbf{R} \setminus F(|\mathscr{C}^{m-1}|)$ such that $F^{-1}(q)$ is a single point then the restriction of F to $F^{-1}(\mathbf{R})$ is injective.*

For a contradiction, assume there are $x, y \in |\mathscr{C} \setminus \partial\mathscr{C}|$ such that $x \neq y$ and $F(x) = F(y)$. There are two simplices $\sigma_x, \sigma_y \in \mathscr{C} \setminus \partial\mathscr{C}$ such that $x \in \mathrm{int}(\sigma_x)$ and $y \in \mathrm{int}(\sigma_y)$.

It follows from (17) that there are two open sets U_x and U_y respective open neighborhoods of $F(x) = F(y)$ covered respectively by $F(\mathrm{star}(\sigma_x))$ and $F(\mathrm{star}(\sigma_y))$. Since, from claim 28, $\mathrm{star}(\sigma_x) \cap \mathrm{star}(\sigma_y) = \emptyset$ it follows that the points in $U = U_x \cap U_y$ are covered twice. But since there is $q \in \mathbf{R} \setminus F(|\mathscr{C}^{m-1}|)$ such that $F^{-1}(q)$ is a single point, we get from claim 28 that any point in $\mathbf{R} \setminus F(|\mathscr{C}^{m-1}|)$ is covered once, a contradiction since $U \cap \mathbf{R} \setminus F(|\mathscr{C}^{m-1}|) \neq \emptyset$.

So to conclude, we have proven that under the conditions of the lemma, the restriction of F to $F^{-1}(\mathbf{R})$ is injective (claim 31) and open (claim 30). Being a one-to-one continuous and open map the restriction of F to $F^{-1}(\mathbf{R})$ is an homeomorphism on its image $\mathbf{R}$. □