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Abstract

We consider the adaptive refinement of bivariate quartic C^2-smooth box spline spaces on the three-directional (type-I) grid G. The polynomial segments of these box splines belong to a certain subspace of the space of quartic polynomials, which will be called the space of special quartics. Given a bounded domain $\Omega \subset \mathbb{R}^2$ and finite sequence $(G^\ell)_{\ell=0,\dots,N}$ of dyadically refined grids, we obtain a hierarchical grid by selecting mutually disjoint cells from all levels such that their union covers the entire domain. Using a suitable selection procedure allows to define a basis spanning the hierarchical box spline space. The paper derives a characterization of this space. Under certain mild assumptions on the hierarchical grid, the hierarchical spline space is shown to contain all C^2-smooth functions whose restrictions to the cells of the hierarchical grid are special quartic polynomials. Thus, in this case we can give an affirmative answer to the completeness questions for the hierarchical box spline basis.

Keywords: hierarchical splines, box splines, completeness, adaptive refinement, three-directional grid, type-I triangulation

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1. Introduction

Box splines and the functions contained in the spaces spanned by them form a very useful class of piecewise polynomial functions on regular grids. They possess a number of useful properties that make them well-suited for applications. It has been shown that box splines have small support (a few

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cells of the underlying grid), they are non-negative, they form a partition of unity, and that box splines are refinable, i.e., the box spline spaces on refined grids are nested [2, 7]. Monographs and survey articles about box splines include [6, 7, 3, 27]. Tensor-product B-splines are special instances of box splines also.

From the rich literature on box splines, we mention a few representative publications on three specific topics. Firstly, a substantial number of results on the *approximation power* of box splines is described in the literature, e.g. [23, 28]. Secondly, several publications discuss techniques for the *efficient manipulation* of box spline bases. A general stable evaluation algorithm is devised in [16]. In [15] the problem of efficient evaluation of box splines is addressed by making use of the local Bernstein representation of basis functions on each triangle. Also, *numerical integration* schemes, which are important for applications, based on quasi-interpolation have been considered in [20, 5].

Using *hierarchical splines* is a well-established approach to adaptive refinement in geometric modeling [10] and numerical analysis [26, 29, 34]. Kraft [18] introduced a basis for hierarchical tensor-product spline spaces using a selection mechanism for B-splines. More recently, a slight modification of this approach was shown to provide a basis with better properties, such as the partition of unity property, strong stability and full approximation power [12, 13, 32]. The hierarchical approach has been extended to Powell-Sabin splines, Zwart-Powell elements and B-spline-type basis functions for cubic splines on regular grids [14, 31, 37].

Computations of the dimensions of spline spaces on partitions of a domain and constructions of suitable bases for such spaces are classical problems considered in the theory of multivariate splines [35, 36]. The *completeness question* (i.e., how to construct a basis spanning the entire spline space on a partition of a domain) led to the introduction of polynomial splines over hierarchical T-meshes (PHT-splines) [8, 9, 21]. Bivariate splines over T-meshes are considered in [30]. A graphical summary of the related literature has been given in [24, Fig. 1]. Several publications have addressed the completeness questions for hierarchical spline spaces generated by tensor-product B-splines [1, 11, 25]. Given a hierarchical spline space, these publications derive sufficient conditions which guarantee that the associated basis (obtained by Kraft's selection mechanism) spans the entire spline space on the partition of the domain which is determined by the hierarchical space.

The paper explores the hierarchical spline spaces generated by C^2-smooth quartic box splines on nested type-I triangulations of a bounded domain $\Omega \subset \mathbb{R}^2$, cf. Fig. 1. These functions form the mathematical basis of Loop's subdivision scheme and are therefore used to construct the regular parts of the corresponding subdivision surfaces, cf. [22, 33]. Moreover, it is known that any C^2-smooth piecewise polynomial function of degree 4 on a type-I triangulation of $\mathbb{R}^2$ can be represented as a linear combination of box splines plus three (globally supported) truncated power (piecewise polynomial) functions of degree 4. However, the box splines suffice for representing all *locally supported* C^2-spline in the space, cf. [4]. On each cell of the triangulation, the space generated by the box splines spans a 12-dimensional subspace of quartic bivariate polynomials. This subspace, which will be called the space of *special quartics*, is known to contain the cubic polynomials. The current manuscript follows the approach presented in [25] in order to establish the completeness of hierarchical C^2-smooth quartic box splines with respect to special quartics.

Using a suitable selection procedure, which generalizes the hierarchical B-spline basis introduced by Kraft [17] to quartic C^2-smooth box splines, we define a basis spanning a hierarchical box spline space. We prove that the elements of the space can equivalently be characterized as the C^2-smooth functions whose restrictions to the cells of the hierarchical grid (which consists of mutually disjoint cells from different levels covering the entire domain) are special quartic polynomials.

The remainder of this paper consists of five sections and an appendix. The next section recalls existing results concerning bivariate spline spaces on regular grids and C^2-smooth quartic box splines. It also derives a characterization of C^2-contacts between segments of special quartic polynomials by their box spline coefficients. Section 3 is devoted to the spaces of C^2-smooth functions whose segments are special quartic polynomials on a multi-cell domain, where all cells belong to the same level. For certain multi-cell domains, which are said to be admissible, these spaces are spanned by the associated box splines. These domains are characterized by an offset condition in Section 4. Based on these results, the fifth section discusses the completeness of the hierarchical box spline basis on the associated hierarchical grids. Finally we conclude the paper. The appendix proves that the spaces spanned by the special quartic polynomials on each cell are simply restrictions of a globally defined subspace of the space of quartic polynomials.

2. Preliminaries

We recall existing results concerning C^2-smooth quartic splines on regular type-I triangulations and characterize C^2-smooth contacts between special quartic polynomials on adjacent triangular cells.

2.1. Bivariate splines on regular grids

We consider bivariate splines on a three-directional grid in the plane $\mathbb{R}^2$, see Fig. 1. Let us denote by $\mathcal{P}_d$ the linear space of polynomials in $\mathbb{R}[x, y]$ of degree less than $d + 1$.

Furthermore, we consider a partition G_Ω of a polygonal domain $\Omega \subset \mathbb{R}^2$ into mutually disjoint cells, where each cell is an open set and the closure of the union of all cells equals Ω. In addition we choose a finite-dimensional linear space T of functions on $\mathbb{R}^2$.

Definition 1. We define $S^r(G_\Omega, T)$ to be the space of C^r-smooth functions s on Ω with the property that their restrictions (denoted by the vertical bar $s|_\triangle$) to any cell $\triangle \in G_\Omega$ yields a function which is obtained by restricting a function in T to the cell, i.e.,

$$S^r(G_\Omega, T) = \{s \in C^r(\Omega) : s|_\triangle \in T|_\triangle \text{ for all cells } \triangle \in G_\Omega\}, \tag{1}$$

where the linear space $T|_\triangle$ contains the restrictions of the functions in T to the cell $\triangle$.

We will call the above space a *spline space*, and its elements spline functions or simply splines. This definition is quite general and applies to any partition of a planar domain in $\mathbb{R}^2$ and to any linear space T. A typical choice is $T = \mathcal{P}_d$, but other choices are also envisaged in this paper.

In a slight abuse of notation we will also use this definition for the case $r = -1$, where we obtain the space $S^{-1}(G_\Omega, T)$ of disconnected splines on G_Ω. Strictly speaking, these spline functions are well-defined only on the interior of the cells, $\bigcup_{\triangle \in G_\Omega} \triangle$, since they may take multiple values on edges and vertices.

Example 2. If $T = \mathcal{P}_d$, we obtain the spline space formed by the C^r-smooth piecewise polynomials of (total) degree d. Typically, this space is used when considering triangulations of the domain. Alternatively one may consider bivariate tensor-product polynomials $T = \mathcal{P}_d \otimes \mathcal{P}_d$ of bidegree (d, d), which are used in connection with partitions of the domain into axis-aligned boxes. Clearly, it is also possible to consider trigonometric polynomials $T = \text{span}\,\{1, \cos kx, \sin kx \, : \, k = 1, \ldots, d\}$ or multivariate versions thereof. In this paper we are interested in a certain subspace $T = \hat{\mathcal{P}}$ of $\mathcal{P}_4$ which is generated by the polynomial segments of C^2-smooth quartic box splines on type-I triangulations. $\diamond$

Throughout this paper, we consider a particular triangulation that allows to construct splines with good properties. More precisely, we consider the bi-infinite grid in $\mathbb{R}^2$ with lines $\mathbb{R} \times \mathbb{Z}$ and $\mathbb{Z} \times \mathbb{R}$, and the triangulation obtained by adding the north-east diagonals in the squares of the grid, see Fig. 1. This produces a three-directional grid which we denote by G. The grid G is a set which contains the elementary triangles (which are called cells) as its elements, where each of the triangles is considered as an open subset of $\mathbb{R}^2$.

This type of grid is called a *type-I triangulation* (e.g. [19]), and spline spaces on triangulations of this type have been studied thoroughly in the rich literature on this subject. In particular, they include box spline spaces, which are interesting due to their elegant construction and simple refinement algorithm.

All results concerning splines on type-I triangulation remain valid under affine transformations of the underlying grid G. For instance, these transformations include scalings of the grid (and we will use this fact later when constructing hierarchical spline spaces), but also affine mappings that transform all triangles into equilateral ones, a particular case which reveals the built-in symmetries of these spline spaces.

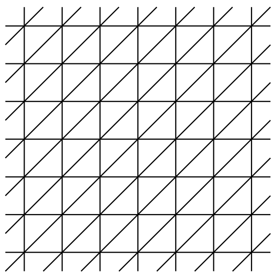

Figure 1: The three-directional grid G.

2.2. Quartic box splines

We restrict ourselves to polynomials $\mathcal{P}_4$ of degree up to four and we will denote this space simply by $\mathcal{P}$. For each triangle $\triangle \in G$, let us denote by $\mathcal{P}|_\triangle$ the linear space formed by the restrictions $f|_\triangle$ of all polynomials $f \in \mathcal{P}$ to $\triangle$, i.e.,

$$\mathcal{P}|_\triangle = \{f|_\triangle : f \in \mathcal{P}\}.$$

For a given triangle $\triangle$, any bivariate polynomial can be represented as a linear combination of the associated bivariate Bernstein polynomials on this triangle [19],

$$f|_\triangle = \sum_{i+j+k=4} c_{ijk} B^4_{ijk}\,, \tag{2}$$

with real coefficients c_{ijk}. Each Bernstein polynomial B^4_{ijk} has an associated anchor point, which possesses the barycentric coordinates $(i/4, j/4, k/4)$ with respect to the triangle. This representation of the polynomials is quite useful for the efficient evaluation of the functions and their derivatives at a given point [15, 19].

The coefficients in Fig. 2, which are placed according to the associated anchor points, define a piecewise polynomial function, whose support is the set of these triangles. The multiple 1/24 of this function is the symmetric quartic box spline (cf. [19] for more details). This box spline is a C^2-smooth function on the three-direction grid and will be our main object of interest. It will be denoted by $\mathcal{B}$. Note that this box spline forms the mathematical basis of Loop subdivision surfaces [22, 33].

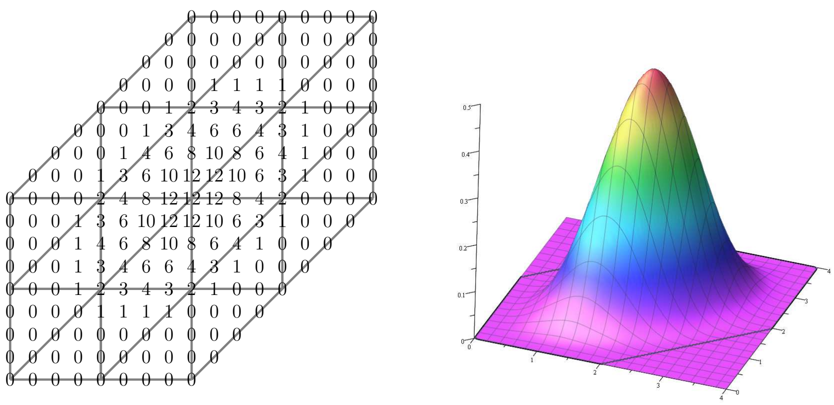

Figure 2: Left: Support and the Bernstein coefficients of the scaled box spline $24\mathcal{B}$. The central vertex is located at the origin. Right: Graph of $\mathcal{B}$ over $[0, 4]^2$.

The translates

$$\beta_{ij}(\cdot) = \mathcal{B}(\cdot - (i, j)), \quad (i, j) \in \mathbb{Z}^2$$

are known to form a locally linearly independent set

$$B = \{\beta_{ij} : (i, j) \in \mathbb{Z}^2\} \tag{3}$$

in the following sense: for any open subset $A \subset \mathbb{R}^2$, the translates

$$B_A = \{\beta_{ij} \in B : \operatorname{supp}(\beta_{ij}) \cap A \neq \emptyset\}$$

restricted to A are linearly independent [19]. Here $\operatorname{supp}(f)$ denotes the support of the function f.

2.3. Contact of polynomial pieces

By construction, each translated box spline β_{ij} is associated with the lattice point (i, j). For a cell $\triangle$ in G, let $\bar{\triangle}$ denote the closure of $\triangle$. We consider the translates whose support contains the given cell $\triangle$,

$$B_{\triangle} = \{\beta_{ij} : \operatorname{supp}(\beta_{ij}) \cap \triangle \neq \emptyset\}.$$

This set is formed by the 12 translates β_{ij}, which are associated with the vertices of the 1-ring neighborhood of $\triangle$ in the three directional grid, see Fig. 3.

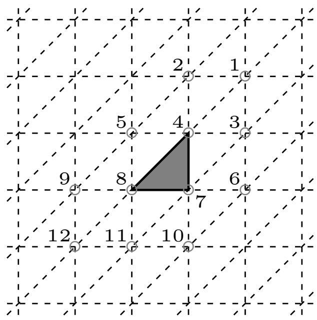

Figure 3: The 1-ring of a triangle.

We now consider the linear space spanned by the restrictions of these translates to the given triangle,

$$\mathbb{V}_{\triangle} = (\operatorname{span} B_{\triangle})|_{\triangle}. \tag{4}$$

Since this space is a genuine subset of quartic polynomials (its dimension is only 12), we will call it the *space of special quartics* on $\triangle$.

Remark 3. It can be shown that $\mathbb{V}_{\triangle} = \hat{\mathcal{P}}|_{\triangle}$ where

$$\hat{\mathcal{P}} = \operatorname{span}(\mathcal{P}_3 \cup \{x^4 - 2x^3y, y^4 - 2xy^3\}). \tag{5}$$

In particular, it should be noted that $\hat{\mathcal{P}}$ is independent of the chosen cell $\triangle$. The proof of this observation is postponed to the Appendix. $\diamond$

Any polynomial $f|_{\triangle} \in \mathbb{V}_{\triangle}$ has a unique representation

$$f|_{\triangle}(\mathbf{x}) = \sum_{\beta \in B_{\triangle}} \lambda_{\triangle}^{\beta}(f|_{\triangle})\, \beta(\mathbf{x}), \quad \mathbf{x} \in \triangle,$$

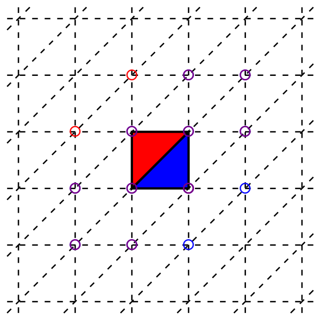

Figure 4: Active box splines on a square domain consisting of two triangles. Twelve functions are active (non-zero) on each triangle. Exactly ten of those are active on both triangles.

for certain coefficients $\lambda_{\triangle}^{\beta}(f|_{\triangle}) \in \mathbb{R}$.

Now we consider two cells $\triangle$ and $\triangle'$ which share an edge. The 1-rings around $\triangle$ and $\triangle'$ have 10 vertices in common, or equivalently, $B_{\triangle}$, and $B_{\triangle'}$ share 10 elements, see Fig. 4.

The following notion of contact will be important in the sequel for the definition of spline spaces.

Definition 4. We say that two polynomials $f|_{\triangle} \in \mathcal{P}_{\triangle}$ and $f'|_{\triangle'} \in \mathcal{P}_{\triangle'}$ have contact of order 2 (and write $f|_{\triangle} \sim f'|_{\triangle'}$), if

$$\forall \mathbf{x} \in \bar{\triangle} \cap \bar{\triangle}', \quad \frac{\partial^{i+j}}{\partial x^i \partial y^j} f|_{\triangle}(\mathbf{x}) = \frac{\partial^{i+j}}{\partial x^i \partial y^j} f'|_{\triangle'}(\mathbf{x}) \tag{6}$$

for all i and j with $0 \leq i+j \leq 2$. The derivatives and values at points on the boundary of a triangle are obtained by one-sided limits.

This property of contact, or continuity of higher order, is a desired property in applications. The following lemma will help us to characterize the situation of contact of quartic box splines.

Lemma 5. *Consider a domain* $\Omega = \overline{\triangle \cup \triangle'}$ *consisting of two triangles meeting on one edge. Then* $\dim S^2(G_\Omega, \hat{\mathcal{P}}) = 14$.

Proof. Let $\ell(x, y)$ be the (implicit) equation of the line that contains the common edge of the two triangles.

Let $(f|_{\triangle}, f'|_{\triangle'}) \in S^2(G_\Omega, \hat{\mathcal{P}})$, then

$$f|_{\triangle} - f'|_{\triangle'} = \ell^3(ax + by + c),$$

since $\ell(x,y)^3|(f-f')$ is equivalent to $f|_\triangle \sim f'|_{\triangle'}$. Therefore

$$f = f' + (Ax+By+C)^3(ax+by+c), \tag{7}$$

where $\ell(x,y) = Ax+By+C$, for $A,B,C \in \mathbb{R}$.

Since the grid G_Ω has only three directions, we only need to consider the cases: $(A,B,C) = (1,0,t)$, $(0,1,t)$ or $(1,-1,t)$, for $t \in \mathbb{R}$.

If $(A,B,C) = (1,0,t)$, then (7) becomes

$$f = f' + ax^4 + byx^3 + p(a,b,c)$$

for a polynomial $p \in \mathcal{P}_3$ that depends on a,b and c. Since $f,f' \in \hat{\mathcal{P}}$, from Remark 3 we deduce that $b=-2a$, leading to

$$f = f' + a(x^4 - 2x^3y) + p(a,c). \tag{8}$$

The case $(A,B,C) = (0,1,t)$ is analogous to latter due to the symmetry of the generators of $\hat{\mathcal{P}}$ with respect to x and y.

It remains to consider $(A,B,C) = (1,-1,t)$. In this case (7) becomes

$$f = f' + a(x^4-3x^3y+3x^2y^2-xy^3) + b(yx^3-3x^2y^2+3xy^3-y^4) + p(a,b,c) \tag{9}$$

for a polynomial $p \in \mathcal{P}_3$ that depends on a,b and c. Since $f,f' \in \hat{\mathcal{P}}$, from Remark 3 we know that $x^2y^2 \notin \hat{\mathcal{P}}$, then we deduce that $b=a$, leading to

$$f = f' + a(x^4-2x^3y) - a(y^4-2xy^3) + p(a,c).$$

Thus, it follows from (8) and (9) that $\dim S^2(G_\Omega, \hat{\mathcal{P}}) = 14$. $\square$

We now come to the characterization of contact of polynomial pieces.

Lemma 6 (Contact Characterization Lemma (CCL))**.** *Consider two special polynomials $f|_\triangle \in \mathbb{V}_\triangle$ and $f'|_{\triangle'} \in \mathbb{V}_{\triangle'}$ on two disjoint triangles $\triangle, \triangle'$, and assume that the two triangles share an edge $e = \bar{\triangle} \cap \bar{\triangle}'$. The two polynomials $f|_\triangle$ and $f'|_{\triangle'}$ have a contact of order 2 if and only if*

$$\forall\ \beta \in B\ :\ \beta|_e \neq 0|_e \implies \lambda^\beta_\triangle(f|_\triangle) = \lambda^\beta_{\triangle'}(f'|_{\triangle'}). \tag{10}$$

Proof. Firstly, we observe that $f|_\triangle \sim f'|_{\triangle'}$ is equivalent to

$$F = (f|_\triangle, f|_{\triangle'}) \in S^2(G_\Omega, \hat{\mathcal{P}}),$$

where $\Omega = \overline{\triangle \cup \triangle'}$. Secondly, noting that there are 14 box splines in B_Ω (see Figure 4) and in view of Lemma 5, we conclude that B_Ω is a basis for

$S^2(G_\Omega, \hat{\mathcal{P}})$. Consequently, $F \in S^2(G_\Omega, \hat{\mathcal{P}})$ if and only if there exist unique coefficients $c_\beta \in \mathbb{R}$ such that

$$F = \sum_{\beta \in B_{\triangle \cup \triangle'}} c_\beta \beta. \tag{11}$$

Thirdly, we consider the restrictions of F to the two triangles,

$$F|_\triangle = \sum_{\beta \in B_\triangle} \lambda_\triangle^\beta(f)\beta, \quad F|_{\triangle'} = \sum_{\beta \in B_{\triangle'}} \lambda_{\triangle'}^\beta(f')\beta.$$

Due to the local linear independence of the box splines, the existence of the representation (11) is equivalent to

$$\lambda_\triangle^\beta(f) = c_\beta = \lambda_\triangle^\beta(f') \quad \text{for} \quad \beta \in B_\triangle \cap B_{\triangle'}.$$

This completes the proof since $B_\triangle \cap B_{\triangle'}$ coincides with the set of box splines that do not vanish on the common edge e. □

Remark 7. CCL cannot be generalized to two triangles with vertex-vertex contact. In fact, the dimension of the corresponding spline space, which is 18, is then always larger than the number of box splines (either 16 or 17, depending on the type vertex-vertex contact).

3. Special quartic splines on multi-cell domains

We now turn our attention to a domain consisting of a collection of cells and we establish conditions for obtaining a basis of the special quartics on the domain.

3.1. Piecewise polynomial functions on multi-cell domains

In the three-directional grid G, we will consider a finite set of cells (triangles) $M \subset G$. Any such set M corresponds to a bounded domain $\mathcal{M}$, which is the closure of the union of its cells.

More precisely, we define the *union operator* U, which maps any element Q of the power set of $\mathbb{R}^2$ to a subset of $\mathbb{R}^2$,

$$U(Q) = \overline{\bigcup_{c \in Q} c} \tag{12}$$

We can now formally define $\mathcal{M}$ as

$$\mathcal{M} = U(M). \tag{13}$$

If the domain $\mathcal{M}$ is connected, then we will say that the set M of triangles is also connected. We need a stronger version of connectivity which excludes vertex-vertex contacts of triangles.

Definition 8. A set M of triangles is said to be $\star$-*connected* if it is connected and if additionally for any two triangles $\triangle$, $\triangle'$ in M, which have a common vertex v, $\bar{\triangle} \cap \bar{\triangle}' \supseteq \{v\}$, there is a chain of triangles $\triangle_0, \triangle_1, \ldots, \triangle_m$ all in M such that $\triangle_0 = \triangle$, $\triangle_m = \triangle'$ and $\bar{\triangle}_i \cap \bar{\triangle}_{i+1} = e_i$ for some edge $e_i \in M$ that contains the vertex v, i.e. $v \in e_i$, for $i = 0, \ldots, m-1$.

In particular, two triangles are $\star$-connected triangles if they possess a common edge.

We require the following condition.

Condition 9. *The set M is assumed to be a union of finitely many mutually disconnected finite sets of triangles, each of which is $\star$-connected.*

In particular, this condition on the set of triangles M implies that we do not allow "kissing vertices" in any connected component of $\mathcal{M}$; or in other words, M is a triangulation of a 2-manifold $\mathcal{M}$ with boundary.

Example 10. The domain $\mathcal{M} = U(M)$ in Fig. 5 has only one component and it is not $\star$-connected. In Fig. 6, the set of triangles M is modified in several ways by adding and deleting triangles and the different components (when more than one) are all $\star$-connected, and hence the domains satisfy Condition 9. $\diamond$

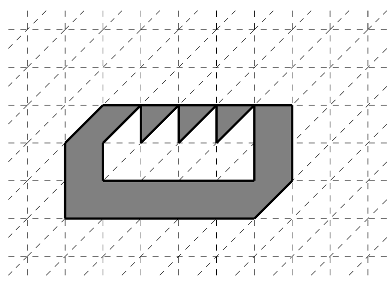

Figure 5: Example of a set M (in gray) which is connected but not $\star$-connected.

In the remainder of the paper, every set M is assumed to satisfy Condition 9. The set of the translates of box splines that act on the triangles M will be denoted as

$$B_M = \{\beta_{ij} \in B \,:\, \operatorname{supp} \beta_{ij} \cap \mathcal{M}^\circ \neq \emptyset\},$$

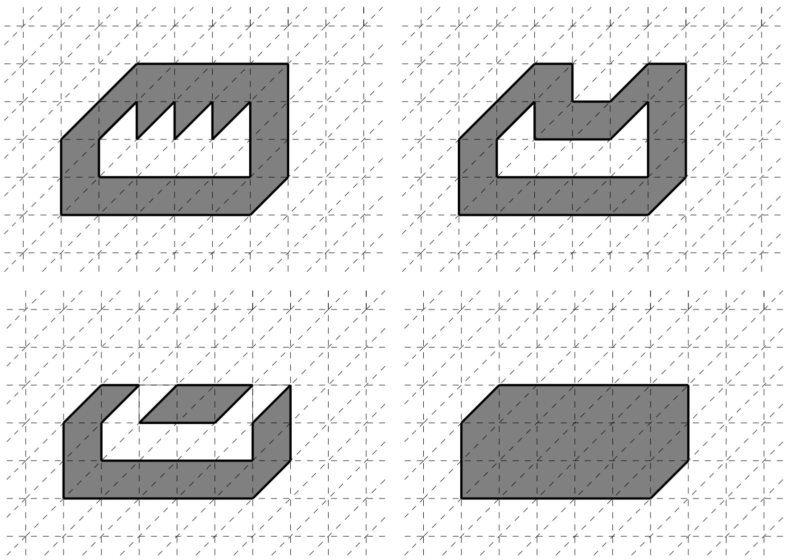

Figure 6: Example of sets of triangles obtained by adding and/or deleting certain triangles to the set M in Fig. 5. The additional triangles in each of the four cases appear in dark-gray. The four new sets satisfy Condition 9 i.e., each of its connected components is $\star$-connected.

where $\mathcal{M}^\circ$ denotes the interior of $\mathcal{M}$, see Fig. 7 for an example. These basis functions generate a space which we denote by

$$\mathbb{V}_M = \operatorname{span} B_M|_M. \tag{14}$$

In particular, when M contains just the single cell $\triangle$ then

$$\mathbb{V}_M = \mathbb{V}_{\{\triangle\}} = \mathbb{V}_\triangle = \hat{\mathcal{P}}|_\triangle \ ,$$

as in Remark 3.

For a finite set of triangles $M \subset G$, we consider the space of *disconnected quartics* $S^{-1}(M, \mathcal{P})$, and the space of *disconnected special quartics* $S^{-1}(M, \hat{\mathcal{P}})$. For $M = \{\triangle\}$, these two spaces coincide with $\mathcal{P}|_\triangle$ and with $\hat{\mathcal{P}}|_\triangle$, respectively. It is obvious that $S^{-1}(M, \hat{\mathcal{P}}) \subset S^{-1}(M, \mathcal{P})$ for any choice of M.

Given a disconnected special quartic $\mathbf{f} = (f|_\triangle)_{\triangle \in M} \in S^{-1}(M, \hat{\mathcal{P}})$, we have a local representation

$$f|_\triangle(\mathbf{x}) = \sum_{\beta \in B} \lambda_\triangle^\beta (f|_\triangle) \beta|_\triangle(\mathbf{x}), \ \mathbf{x} \in \triangle,$$

in terms of the restriction of the box splines, for any $\triangle \in M$. However, this representation is generally not available for general disconnected quartics.

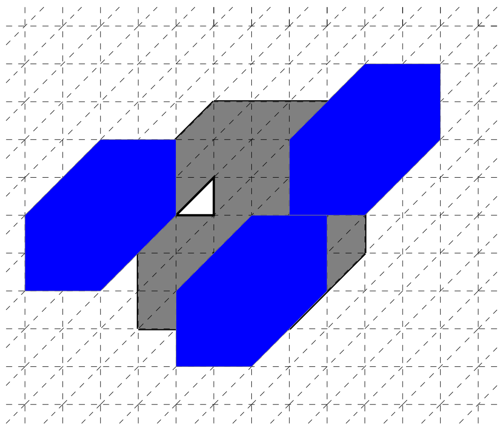

Figure 7: Supports (shown in blue) of three translates of the box splines $\mathcal{B}$ that act on a domain $\mathcal{M}$ (grey).

Lemma 11. *For a finite set of triangles M, and the corresponding disconnected space $S^{-1}(M, \hat{\mathcal{P}})$, the space $S^2(M, \hat{\mathcal{P}})$ is precisely*

$$S^2(M, \hat{\mathcal{P}}) = \{\mathbf{f} \in S^{-1}(M, \hat{\mathcal{P}}) \, : \, \forall \triangle, \triangle' \in M, \, f|_{\triangle} \sim f|_{\triangle'}\}$$

where the relation $\sim$ is defined in Definition 4.

Proof. By the definition of the space of C^2 splines given in (1), it is easy to deduce that $S^2(M, \hat{\mathcal{P}}) \subset S^{-1}(M, \hat{\mathcal{P}})$. Let $\mathbf{f} \in S^{-1}(M, \hat{\mathcal{P}})$, then $\mathbf{f} \in S^2(M, \hat{\mathcal{P}})$ if and only if $\mathbf{f}$ is a C^2 function on M, or equivalently, if and only if for all pair of cells $\triangle$ and $\triangle'$ in M satisfy (6) in Definition 4. □

The space $S^2(M, \hat{\mathcal{P}})$ will be referred to as the *special spline space on M*.

As we shall see later, the special spline space $S^2(M, \hat{\mathcal{P}})$ can be generated by box splines with support on $\mathcal{M}$, but one may need to use several copies of some of these box splines, as shown in the following Example.

Example 12. The domain in Fig. 8 consists of two $\star$-disconnected triangles. The space of disconnected special quartics consists of pairs of special polynomials, where the first and the second entry of each pair is associated with the first and the second triangle. Since the two triangles are disconnected, the special spline space is equal to the space of disconnected special quartics. Consequently, it has dimension 24 and is therefore not spanned by the 18 box splines whose support intersects this domain. ◇

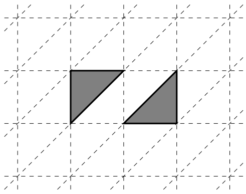

Figure 8: A domain with $\mathbb{V}_M \subsetneq S^2(M, \hat{\mathcal{P}})$.

Definition 13. For a spline $\beta \in B_M$, the *coefficient graph* Γ_β associated to β is defined as follows:

- The vertices of the graph Γ_β are the cells $\triangle \in M$ such that $\triangle \subseteq \operatorname{supp} \beta$.
- Two vertices of Γ_β are connected by an edge if the corresponding cells $\triangle, \triangle'$ have a common edge and $\beta|_{\bar{\triangle} \cap \bar{\triangle}'} \neq 0|_{\bar{\triangle} \cap \bar{\triangle}'}$.

We will write $\triangle \in \Gamma_\beta$ to indicate that $\triangle$ corresponds to a vertex of Γ_β.

Example 14. Let us consider the domain in Fig. 9, and the box splines β_i ($i = 1, 2, 3$). The coefficient graphs Γ_{β_i} corresponding to these box splines are given in Fig. 10. $\diamond$

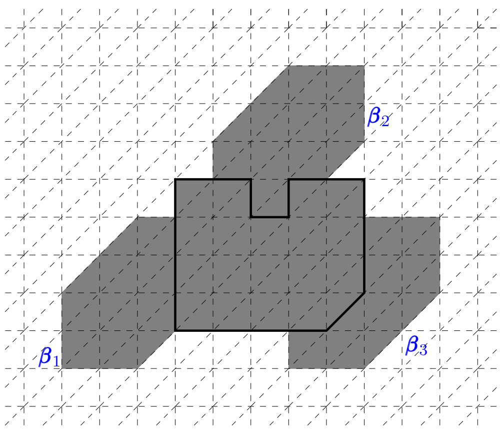

Figure 9: A multi-cell, and three examples of box splines β_i.

Proposition 15. *An element* $\mathbf{f} \in S^{-1}(M, \hat{\mathcal{P}})$ *is in* $S^2(M, \hat{\mathcal{P}})$ *if and only if the coefficients satisfy* $\lambda^\beta_\triangle(f|_\triangle) = \lambda^\beta_{\triangle'}(f|_{\triangle'})$, *for all* $\beta \in B_M$, *and all pair of cells* $\triangle, \triangle'$ *belonging to the same component of* Γ_β.

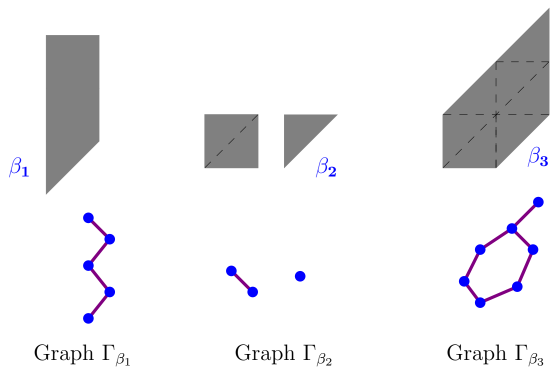

Figure 10: Coefficient graphs associated to β_i, $i = 1, 2, 3$ from Fig. 9.

Proof. Suppose $\mathbf{f} \in S^2(M, \hat{\mathcal{P}})$, and β in $\mathcal{B}$. If $\triangle = \triangle_0$ and $\triangle' = \triangle_{d+1}$ are two cells in M corresponding to vertices in the same component of Γ_β, then there is a chain of vertices $v_1, \ldots, v_d$ in Γ_β corresponding to cells $\triangle_1, \ldots, \triangle_d$ in M, such that $\triangle_i$ and $\triangle_{i+1}$ intersect in an edge, for $i = 0, \ldots, d$. By Lemma 6, $\lambda^\beta_{\triangle_i}(f|_{\triangle_i}) = \lambda^\beta_{\triangle_{i+1}}(f|_{\triangle_{i+1}})$, and since this is valid for every $0 \leq i \leq d$, then the same follows for $\triangle$ and $\triangle'$.

Conversely, from a similar argument as before, if for any pair of triangles $\triangle$ and $\triangle'$ in M with an edge e in common we have that $\lambda^\beta_\triangle(f|_\triangle) = \lambda^\beta_{\triangle'}(f|_{\triangle'})$ for every basis function $\beta \in B$ such that $\beta|_e \neq 0$, then by Lemma 6 every linear combination of the basis functions β is in $S^2(M, \hat{\mathcal{P}})$. $\square$

3.2. Box spline bases on multi-cell domains

Definition 16. For every $\beta \in B$ and every connected component Φ of Γ_β we define the function

$$\beta_\Phi(\mathbf{x}) = \begin{cases} \beta(\mathbf{x}) & \text{if} \quad \mathbf{x} \in U(\Phi), \\ 0 & \text{otherwise.} \end{cases}$$

The set of these functions, for the different connected components of the graph Γ_β, is denoted by Λ, more precisely,

$$\Lambda = \bigcup_{\beta \in B} \{\beta_\Phi | \ \Phi \ \text{ is a connected component of } \ \Gamma_\beta\}.$$

Theorem 17. *The set Λ, when restricted to $\mathcal{M}$, forms a locally linearly independent basis for $S^2(M, \hat{\mathcal{P}})$.*

The proof is analogous to that of Theorem 2.12 in [25].

Corollary 18. *If the intersection of the support of each β with the multi-cell domain $\mathcal{M}$ is $\star$-connected, then the functions in B_M, when restricted to $\mathcal{M}$, form a basis of $S^2(M, \hat{\mathcal{P}})$.*

Proof. If the condition is satisfied, then for each $\beta \in B_M$ the coefficient graph Γ_β has either one component or it is empty. The result thus follows from Theorem 17. □

Example 19. The graph Γ_β, associated to every β with non-empty intersection with the interior of $\mathcal{M}$ in Fig. 7, has only one component. From the previous Corollary, it follows that the functions in B_M, restricted to $\mathcal{M}$, form a basis for the special spline space $S^2(M, \hat{\mathcal{P}})$ on the domain $\mathcal{M}$. ◇

4. Admissible multi-cell domains

In view of the discussion in the previous section, we give the following definition.

Definition 20. A domain $\mathcal{M} = U(M)$ is said to be *admissible*, if the intersection of the support of any box spline with $\mathcal{M}$ is $\star$-connected.

The following result is then obvious from Corollary 18:

Corollary 21. *For any admissible domain $\mathcal{M}$, the functions in B_M when restricted to $\mathcal{M}$ form a basis of $S^2(M, \hat{\mathcal{P}})$.*

A subset of admissible domains can be characterized by the offsets of their boundaries.

Definition 22. We define the offset curve of a multi-cell domain M as follows: Consider any cell (triangle) in $G \setminus M$. If the boundary of this triangle shares a vertex with $\mathcal{M}$, but both incident edges are not part of the boundary of $\mathcal{M}$, then the opposite edge is added to the offset curve. We say that a domain $\mathcal{M}$ satisfies the offset condition if its offset is a simple closed curve or a collection of simple closed curves.

Proposition 23. *If a domain satisfies the offset condition, then it is also admissible.*

Proof. The proof follows from a careful case-by-case analysis. □

Remark 24. For the domain on the left in Fig. 11 the box splines in B_M form a basis for $S^2(M, \hat{\mathcal{P}})$. In this situation, when the holes in the domain are *"sufficiently small"*, they do not split the support of any basis function $\beta \in B_M$ and the result follows by Corollary 18. Consequently, the offset condition is not necessary for admissibility. ◇

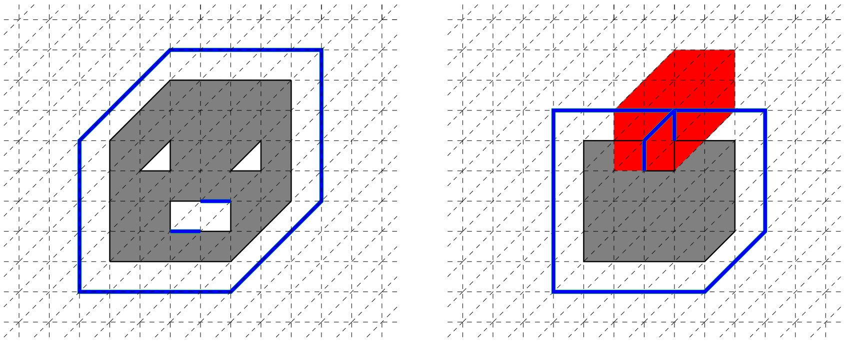

Figure 11: The offsets of the two domain boundaries in the picture are non-simple curves. The domain on the left is admissible, however one of the holes has a discontinuous offset. The domain on the right does not possess a simple offset, and furthermore it is not admissible since there are box splines (such as the one whose support is indicated) with a disconnected coefficient graph, see Fig. 10.

5. Hierarchical Box splines

We define the special spline spaces on a hierarchical grid and a hierarchical basis. We then prove the completeness of this basis under certain assumptions on the domain hierarchy.

5.1. Special spline spaces on hierarchical grids

Given an integer $N \geq 0$, we consider the hierarchical grids

$$G^\ell, \quad \ell = 1, \ldots, N$$

such that $G^{\ell+1}$ is obtained from G^ℓ by one global, uniform dyadic refinement step, where the coarsest grid $G^1 = G$ is the one with vertices $\mathbb{Z}^2$ which has been described in Section 2. More precisely, the grid $G^{\ell+1} = \frac{1}{2}G^\ell$ is obtained by dividing all edges of triangles of G^ℓ into two edges and adding three new interior edges, thus every triangle is split into four smaller ones, see Fig. 12. The index ℓ will be called the *level* of the grid, and the number N specifies the number thereof. Each grid G^ℓ is a set of triangles of the same shape, and these triangles are open sets.

In addition to the union operator U, see (12), which transforms a set Q of subsets of $\mathbb{R}^2$ into the closed subdomain $U(Q) \subset \mathbb{R}^2$ covered by it, we define the *triangulation operators* T^ℓ which restrict the grid of level ℓ to a given subset $\mathcal{Q}$ of $\mathbb{R}^2$,

$$T^\ell(\mathcal{Q}) = \{\triangle \in G^\ell \, : \, \triangle \subseteq \mathcal{Q}\}.$$

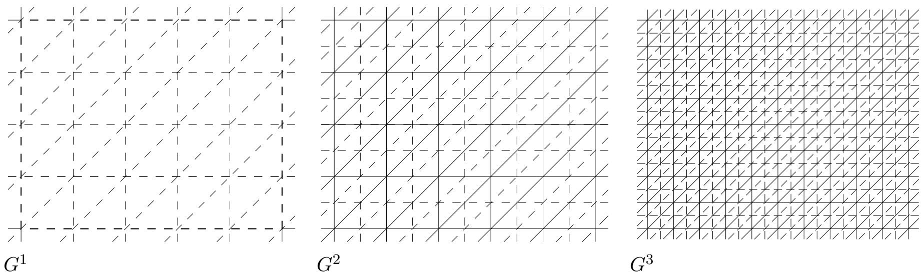

Figure 12: Three levels of hierarchical grids.

When applied to subsets of the corresponding grids, these operators are the inverses of the union operator, i.e. for any $M^\ell \subset G^\ell$ we have $T^\ell(U(M^\ell)) = M^\ell$.

Let Ω be a domain of $\mathbb{R}^2$ aligned with level N. More precisely, its boundary $\partial\Omega$ is a union of edges from the grid G^N, see the top right of Fig. 13. The hierarchical grid is defined by an inversely nested sequence of subdomains $\mathcal{M}^\ell = U(M^\ell)$ thereof,

$$\emptyset = \mathcal{M}^0 \subseteq \mathcal{M}^1 \subseteq \cdots \subseteq \mathcal{M}^N = \Omega, \tag{15}$$

which correspond to subsets of the corresponding grids, $M^\ell \subset G^\ell$. Thus the boundary $\partial\mathcal{M}^\ell$ is a union of edges of the grid G^ℓ of the same level. These subdomains were called rings in [11]. They are multi-cell domains (with respect to the grid) of level ℓ.

The difference of two successive subdomains

$$\mathcal{D}^\ell = \overline{\mathcal{M}^\ell \setminus \mathcal{M}^{\ell-1}}$$

and the associated subset $D^\ell = T^\ell(\mathcal{D}^\ell)$ of the grid of level ℓ is called the *refinement domain of level* ℓ, see Fig 14. In particular we have $D^1 = M^1$ and $\mathcal{D}^1 = \mathcal{M}^1$.

Finally we collect the triangulations of the refinement area and arrive at the *hierarchical grid*

$$H = \bigcup_{\ell=1}^{N} D^\ell, \tag{16}$$

see Fig. 15. This hierarchical grid, which collects triangles from all levels, provides a representation of the domain $\Omega = U(H)$ as a hierarchical multi-cell domain. We use it to define the *hierarchical special spline space*

$$\mathbb{H} = S^2(H, \hat{\mathcal{P}}).$$

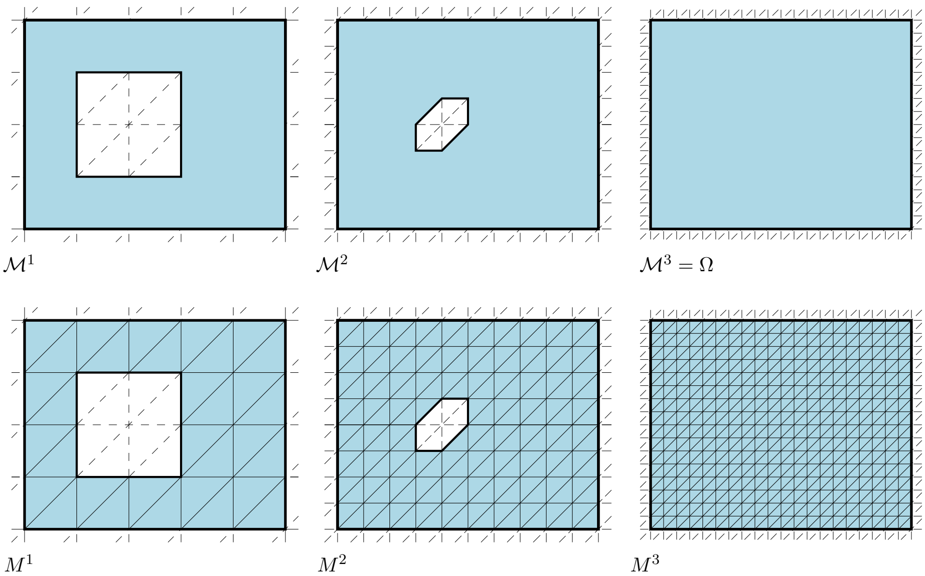

Figure 13: Nested sequence of domains $\mathcal{M}^\ell$ and the corresponding multi-cell domains M^ℓ, $\ell = 1, 2, 3$ from Fig. 12.

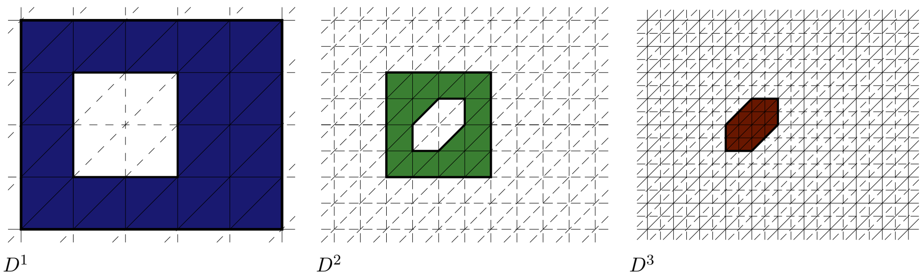

Figure 14: Refinement domains corresponding to the multi-cell domains M^ℓ in Fig. 13, $\ell = 1, 2, 3$.

Equivalently, the spline functions $s \in \mathbb{H}$ can be characterized by the fact that their restrictions to the subdomains belong to the corresponding spline spaces:

Lemma 25. *A function $s : \Omega \to \mathbb{R}$ is an element of $\mathbb{H}$ if and only if*

$$s|_{\mathcal{M}^\ell} \in S^2(M^\ell, \hat{\mathcal{P}}).$$

holds for all $\ell = 1, \ldots, n$.

The proof of this simple observation has been omitted.

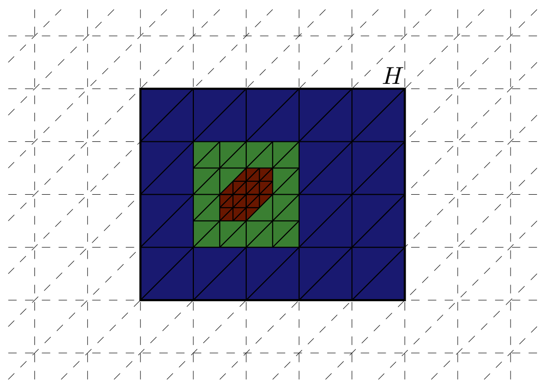

Figure 15: Hierarchical multi-cell domain $\Omega = U(H)$, obtained as the union of refinement areas D^i $(i = 1, 2, 3)$ from Fig. 14.

5.2. Completeness of the hierarchical spline basis

We consider the box splines B^ℓ, which are associated with the grids G^ℓ, and the spaces spanned by them. Similar to the grids, these box splines can be obtained by a dilation,

$$B^{\ell+1} = \{\beta(2\,\cdot) \;:\; \beta \in B^\ell\}$$

from the coarsest basis $B^1 = B$ which has been defined in Section 2. For each subdomain M^ℓ we consider the box splines whose supports possess a non-empty intersection with $\mathcal{M}^\ell$,

$$B^\ell_{M^\ell} = \{\beta \in B^\ell \;:\; \operatorname{supp} \beta \cap \mathcal{M}^\ell \neq \emptyset\}.$$

The hierarchical spline basis is defined by a selection procedure which was proposed by Kraft [18] for tensor-product B-splines. We select box splines from all levels,

$$K^\ell = \left\{\beta^\ell \in B^\ell_{M^\ell} \,:\, \operatorname{supp} \beta^\ell \cap \mathcal{M}^{\ell-1} = \emptyset\right\},$$

and collect them into the *hierarchical spline basis*

$$K = \bigcup_{\ell=1}^{N} K^\ell.$$

The linear independence of this set of functions is implied by the local linear independence of the box splines at each level, see [11, 18].

Finally we provide a sufficient condition for the completeness of the hierarchical spline basis.

Theorem 26. *The hierarchical spline basis K spans the hierarchical spline space $\mathbb{H}$ if each subdomain M^ℓ is admissible with respect to the grid of level ℓ.*

Proof. The proof follows standard arguments already presented in [11, 25], for the case of hierarchical tensor B-spline bases. Any spline function $s \in \mathbb{H}$ admits a representation

$$s = (h^1 + \cdots + h^N)|_\Omega,$$

where $h^\ell \in \operatorname{span} B^\ell_{M^\ell}$ with the property that

$$h^\ell|_{\mathcal{M}^\ell} = s|_{\mathcal{M}^\ell} - (h^1 + \ldots + h^{\ell-1})|_{\mathcal{M}^\ell}. \tag{17}$$

for $\ell = 1, \ldots, N$. This is proved by induction with respect to ℓ.

For any given level ℓ, all functions $h^k|_{\mathcal{M}^\ell}$ of lower levels $k < \ell$ are contained in $S^2(M^\ell, \hat{\mathcal{P}})$, since

$$\operatorname{span} B^k_{M^k}|_{M^\ell} \subset S^2(M^\ell, \hat{\mathcal{P}}).$$

Lemma 25 implies $s|_{\mathcal{M}^\ell} \in S^2(M^\ell, \hat{\mathcal{P}})$. Consequently, the right-hand side of Eq. (17) is contained in $S^2(M^\ell, \hat{\mathcal{P}})$. Since the subdomain $\mathcal{M}^\ell$ is admissible, we conclude that $h^\ell \in \operatorname{span} B^\ell_{M^\ell}$ according to Corollary 21. In particular, choosing $\ell = N$ in Eq. (17) implies the first equation.

Moreover, the construction of the functions h^ℓ ensures that

$$h^\ell|_{\mathcal{M}^{\ell-1}} = 0|_{\mathcal{M}^{\ell-1}}.$$

Since the box splines possess the property of local linear independence we can conclude that $h^\ell \in \operatorname{span} K^\ell$. This completes the proof. $\square$

6. Conclusion

We extended the discussion of the completeness of hierarchical spline spaces from [25] to the case of hierarchies of bivariate quartic C^2-smooth box splines on type-I triangulations. There are two main differences to the original approach, which was formulated for tensor-product splines.

First, since box splines do not span the whole space of quartic polynomials, a special polynomial subspace – the special quartics – had to be introduced. In some sense this situation generalizes the tensor-product case, where the B-splines span a polynomial space of a given (coordinate-wise) bi-degree, instead of the the space of bivariate polynomials of a given total degree.

Second, the constraints on the domains are entirely different, due to the differences in the characterization of contacts between polynomial pieces. For bivariate tensor-product splines, both edge-edge and vertex-vertex contacts could be characterized easily by the equality of spline coefficients. In the present case, however, this was possibly solely for edge-edge contacts. Consequently, the completeness of hierarchical splines requires more severe restrictions to the hierarchical grid.

The hierarchical box spline basis does not form a partition of unity. Similar to the approach presented in [12], this property can be recovered with the help of a suitable truncation procedure. Also, in [25] it is described a decoupling procedure that allows to relax the assumptions regarding the hierarchical grid. This approach can be extended to the box spline case as well. Finally it is also possible to combine truncation and decoupling as in [24].

Appendix

We show that the space $\mathbb{V}_\triangle$ defined in Eq. (4) is the restriction of a global space to the triangle, as pointed out in Remark 3. In order to prove this result we use the notation introduced in Section 5. The following proof is not restricted to quartic box splines.

Proposition 27. *Consider a global polynomial $f \in \mathcal{P}$ defined in $\mathbb{R}^2$. If $f|_\triangle \in \mathbb{V}_\triangle$ for some $\triangle \in G^\ell$, then $f|_{\triangle'} \in \mathbb{V}_{\triangle'}$ for any other cell $\triangle'$ in the grid G^ℓ.*

Proof. We observe that if $\triangle, \triangle' \in G^\ell$ then both cells are contained in a *bigger* triangle $\tilde{\triangle}$ of a grid which we denote as G^0, see Fig. 16. We denote by $\mathbb{V}^0_{\tilde{\triangle}}$ the restriction of the span of the corresponding box splines (associated with the grid G^0) to the cell $\tilde{\triangle}$. Similarly we denote with $\mathbb{V}^\ell_\triangle$ and $\mathbb{V}^\ell_{\triangle'}$ the span of the correspondiung box splines (associated with the grid G^ℓ) restricted to these triangles. Clearly, we have that

$$\dim \mathbb{V}^0_{\tilde{\triangle}} = \dim \mathbb{V}^\ell_\triangle = \dim \mathbb{V}^\ell_{\triangle'}$$

because of the symmetry and the scaling invariance of the box spline construction. The box spline spaces on the different levels are nested, hence

$$\mathbb{V}^0_{\tilde{\triangle}}|_\triangle \subseteq \mathbb{V}^\ell_\triangle \quad \text{and} \quad \mathbb{V}^0_{\tilde{\triangle}}|_{\triangle'} \subseteq \mathbb{V}^\ell_{\triangle'}.$$

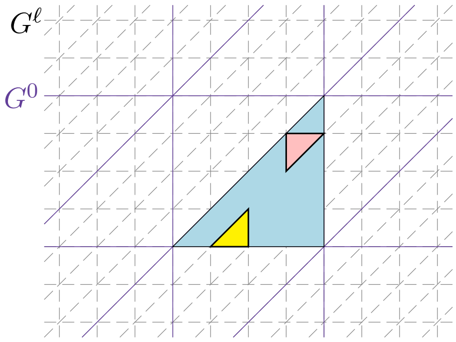

Figure 16: The two cells in the fine grid G^ℓ are both contained in a cell of the grid G^0.

Since $\mathbb{V}^0_{\hat{\triangle}}$ is a subspace of a polynomial space we also know that

$$\dim \mathbb{V}^0_{\hat{\triangle}} = \dim \mathbb{V}^0_{\hat{\triangle}}|_{\triangle} = \dim \mathbb{V}^0_{\hat{\triangle}}|_{\triangle'}.$$

Combining these observations confirms that $\mathbb{V}^\ell_{\triangle} = \mathbb{V}^0_{\hat{\triangle}}|_{\triangle}$ and $\mathbb{V}^\ell_{\triangle'} = \mathbb{V}^0_{\hat{\triangle}}|_{\triangle'}$, as we wanted to show. □

We may therefore define $\hat{\mathcal{P}}$ to be this global polynomial space. In the special case of C^2 quartic box splines, the above result may be seen directly by using the representation of the polynomial pieces in the monomial basis. For the sake of completeness we report the coefficients of $12\beta_i$, $\beta_i \in \beta_{\triangle}$ with respect to the basis (5) of $\hat{\mathcal{P}}$ in the matrix

	1	y	y^2	y^3	x	xy	xy^2	x^2	x^2y	x^3	x^4-2x^3y	y^4-2xy^3
β_1	0	0	0	0	0	0	0	0	0	0	0	−1
β_2	0	0	0	2	0	0	0	0	0	0	0	1
β_3	0	0	0	0	0	0	0	0	0	2	−1	2
β_4	1	2	0	−6	2	6	6	0	0	−4	2	−1
β_5	1	4	6	0	−2	−6	−6	0	0	2	−1	−1
β_6	0	0	0	0	0	0	0	0	0	0	1	−1
β_7	1	−2	0	6	4	−6	−12	6	6	−4	−1	−1
β_8	6	0	−12	0	0	12	12	−12	−12	8	−1	2
β_9	1	2	0	0	−4	−6	0	6	6	−4	1	0
β_{10}	0	0	0	−2	0	0	6	0	−6	2	−1	1
β_{11}	1	−4	6	0	2	−6	−6	0	12	−4	2	−1
β_{12}	1	−2	0	0	−2	6	0	0	−6	2	−1	0

Each row corresponds to a box spline, following the numbering in Figure 3. The grid size is 1 and the origin is located at vertex no. 8.

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