# Equivalent point-source modeling of small obstacles for electromagnetic waves

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**Abstract**

We develop reduced models to approximate the solution of scattering problem by electromagnetic obstacles that are small in comparison with the wavelength. Using the matched asymptotic expansions method, we investigate a meshless multi-scale approach where the scatterers are represented by equivalent point-sources. In the context of multiple scattering, we deduce from this a Foldy-Lax approximation whose accuracy and efficiency are illustrated with numerical simulations.

**Keywords:** Reduced models, Maxwell's equations, Multiple scattering

## Introduction

The propagation of time-harmonic electromagnetic waves of angular frequency $\omega > 0$ in a homogeneous and isotropic dielectric infinite medium of electric permittivity $\varepsilon > 0$ and magnetic permeability $\mu > 0$ is described by an incident wave

$$\mathrm{Re}\left(\mathbf{E}^{\mathrm{inc}}(x)\exp(-\mathrm{i}\omega t)\right), \qquad x \in \mathbb{R}^3, \quad t > 0.$$

In the presence of a small and smooth auto-similar obstacle $\boldsymbol{\omega}_\delta = \delta\boldsymbol{\omega} \subset \mathbb{R}^3$ centered at the origin, whose characteristic length $\delta$ is very small compared to the wavelength $\lambda = \frac{2\pi}{\omega\sqrt{\mu\varepsilon}}$, the wave is scattered and gives birth to electromagnetic fields $\mathbf{E}_\delta$ and $\mathbf{H}_\delta$ satisfying the time-harmonic Maxwell equations

$$\begin{cases} \nabla \times \mathbf{E}_\delta - \mathrm{i}\kappa \mathbf{H}_\delta = 0, \\ \nabla \times \mathbf{H}_\delta + \mathrm{i}\kappa \mathbf{E}_\delta = 0, \end{cases}$$

where $\kappa = \frac{2\pi}{\lambda}$ denotes the wave-number. For a perfectly conducting obstacle, the domain of propagation $\Omega_\delta$ is the exterior domain $\mathbb{R}^3 \setminus \overline{\boldsymbol{\omega}_\delta}$ and the boundary condition reads as

$$\mathbf{n} \times \mathbf{E}_\delta = -\mathbf{n} \times \mathbf{E}^{\mathrm{inc}} \qquad \text{on } \partial\boldsymbol{\omega}_\delta.$$

The existence of a unique solution is guaranteed by the hypothesis of outgoing wave at infinity given by the Silver-Müller radiation condition,

$$\lim_{|x|\to\infty} |x| \left(\mathbf{H}_\delta \times \widehat{x} - \mathbf{E}_\delta\right) = 0 \quad \text{unif. in } \widehat{x} = \frac{x}{|x|}.$$

Numerical techniques based on a discrete approximation of the geometry are limited and very expensive due to the smallness of the obstacle. To overcome these difficulties, we investigate two meshless approaches involving approximate solutions of the exterior Maxwell problem.

## Equivalent point-source modeling

The first one is a volumical approach based on the method of matched asymptotic expansions [1]. This method consists in constructing distinct expansions of the solution in different regions of the domain of propagation with appropriate scales, and matching them in an intermediate region called the matching area. Far from the obstacle, the obstacle is modeled like a dipolar source around the origin. As a result,

$$\mathbf{E}_\delta \underset{\delta\to 0}{\sim} \mathcal{E}_{\mathrm{elec}}[\mathbf{d}^{\mathrm{E}}_\delta] + \mathcal{E}_{\mathrm{mag}}[\mathbf{d}^{\mathrm{H}}_\delta], \tag{1}$$

where $\mathcal{E}_{\mathrm{elec}}[\mathbf{d}^{\mathrm{E}}_\delta]$ (resp. $\mathcal{E}_{\mathrm{mag}}[\mathbf{d}^{\mathrm{H}}_\delta]$) is the electric field generated by an electric (resp. magnetic) dipole of moment $\mathbf{d}^{\mathrm{E}}_\delta \in \mathbb{C}^3$ (resp. $\mathbf{d}^{\mathrm{H}}_\delta \in \mathbb{C}^3$), given by

$$\mathcal{E}_{\mathrm{elec}}[\mathbf{d}](x) = \frac{\exp(\mathrm{i}\kappa r)}{r}\left[\left(\frac{2}{r^2} - \frac{2\mathrm{i}\kappa}{r}\right)(\mathbf{d}\cdot\widehat{x})\widehat{x} + \left(-\frac{1}{r^2} + \frac{\mathrm{i}\kappa}{r} + \kappa^2\right)(\widehat{x}\times\mathbf{d})\times\widehat{x}\right],$$

$$\mathcal{E}_{\mathrm{mag}}[\mathbf{d}](x) = \frac{\exp(\mathrm{i}\kappa r)}{r}\left(\frac{1}{r} - \mathrm{i}\kappa\right)(\mathrm{i}\kappa\mathbf{d})\times\widehat{x},$$

where $r = |x|$. For the single-scattering case, the dipole moments will depend on the incident field, size and location of the scatterer. In particular, for a spherical obstacle, we have $\mathbf{d}^{\mathrm{E}}_\delta = \delta^3\mathbf{E}^{\mathrm{inc}}(0)$ and $\mathbf{d}^{\mathrm{H}}_\delta = -\frac{\delta^3}{2}\mathbf{H}^{\mathrm{inc}}(0)$. For the multiple-scattering case, each obstacle $\boldsymbol{\omega}^k_\delta = c_k + \delta\boldsymbol{\omega}$ ($k = 1, \dots, \mathsf{N}_{\mathrm{obs}}$) is modeled as a dipolar source around its center $c_k$. Following (1), the electric field is approximated by

$$\sum_{k=1}^{\mathsf{N}_{\mathrm{obs}}} \mathcal{E}_{\mathrm{elec}}[\mathbf{d}^{\mathrm{E}}_{\delta,k}](x - c_k) + \mathcal{E}_{\mathrm{mag}}[\mathbf{d}^{\mathrm{H}}_{\delta,k}](x - c_k).$$

According to Foldy-Lax theory, the dipole moments depend not only on the incident fields, but also on all the other scattered fields,

$$\mathbf{d}^{\mathrm{E}}_{\delta,k} = \delta^3\alpha(\delta)\Big\{\mathbf{E}^{\mathrm{inc}}(c_k) + \sum_{\ell=1,\,\ell\neq k}^{\mathsf{N}_{\mathrm{obs}}} \mathcal{E}_{\mathrm{elec}}[\mathbf{d}^{\mathrm{E}}_{\delta,\ell}](c_k - c_\ell) + \mathcal{E}_{\mathrm{mag}}[\mathbf{d}^{\mathrm{H}}_{\delta,\ell}](c_k - c_\ell)\Big\},$$

with a similar expression for the magnetic moments $\mathbf{d}^{\mathrm{H}}_{\delta,k}$. These expressions lead to a vectorial formulation involving $6\mathsf{N}_{\mathrm{obs}}$ unknowns. The coefficient $\alpha(\delta)$ differs with different levels of approximation. We define

$$\alpha(\delta) = \begin{cases} 1 & : \text{ first Foldy model,} \\ 1 + \dfrac{3(\kappa\delta)^2}{10} & : \text{ collected Foldy model,} \\ \dfrac{3\mathrm{i}}{2(\kappa\delta)^3}\,\dfrac{j_1(\kappa\delta)}{h_1^{(1)}(\kappa\delta)} & : \text{ modified Foldy model,} \end{cases}$$

where $j_1$ and $h_1^{(1)}$ denote the bessel function and the hankel function of the first kind, of order 1. The Born approximations are defined by neglecting the interactions between the obstacles.

### Spectral method: the reference solution

The electric field has the integral representation

$$\mathbf{E}_\delta(x) = \sum_{k=1}^{\mathsf{N}_{\mathrm{obs}}} \nabla \times \int_{\partial\boldsymbol{\omega}^k_\delta} \Phi(x,y)\mathbf{p}_k(y)\mathrm{d}s_y,\ x \in \Omega_\delta,$$

where $\Phi(x,y) = \frac{\exp(\mathrm{i}\kappa|x-y|)}{4\pi|x-y|}$ denotes the Green function associated to the Helmholtz equation. The tangential densities $\mathbf{p}_k$ solve the following boundary integral equations

$$\sum_{\ell=1}^{\mathsf{N}_{\mathrm{obs}}} \mathcal{M}^{k\ell}_{\Gamma}\mathbf{p}_\ell = -\mathbf{n} \times \mathbf{E}^{\mathrm{inc}} \qquad \text{on } \partial\boldsymbol{\omega}^k_\delta, \qquad (2)$$

where the magnetic potentials $\mathcal{M}^{k\ell}_{\Gamma}\boldsymbol{\lambda}$ are defined as an extension of

$$\mathbf{n}(x_\Gamma) \times \lim_{x\to x_\Gamma}\left(\nabla \times \int_{\partial\boldsymbol{\omega}^\ell_\delta} \Phi(x,y)\boldsymbol{\lambda}(y)\,\mathrm{d}s_y\right),$$

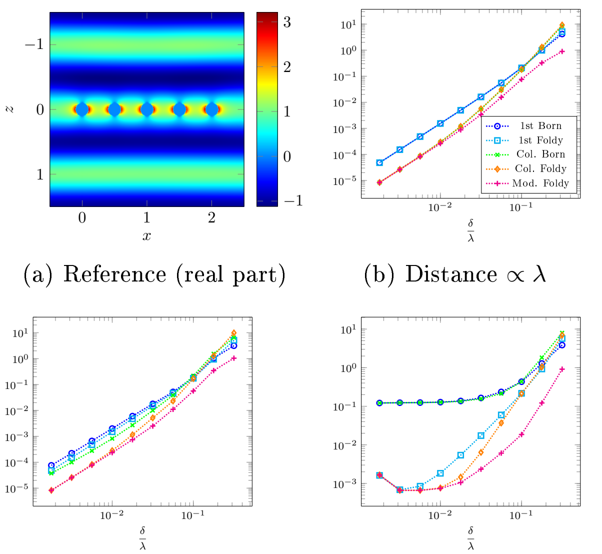
with $x_\Gamma \in \partial\boldsymbol{\omega}^k_\delta$. The spectral method [2] consists in discretizing (2) into a local spectral basis associated with the vectorial Laplace-Beltrami operator with $\mathsf{N}_{\mathrm{mod}}$ modes. For spherical obstacles, the basis is composed of the vector spherical harmonics $\nabla_{\mathbb{S}}Y_{n,m}$, $\mathrm{curl}_{\mathbb{S}}Y_{n,m}$,

$$\mathbf{p}_\ell = \sum_{n=1}^{\mathsf{N}_{\mathrm{mod}}}\sum_{m=-n}^{n} p^{\ell,\perp}_{n,m}\nabla_{\mathbb{S}}Y^\ell_{n,m} + p^{\ell,\times}_{n,m}\mathrm{curl}_{\mathbb{S}}Y^\ell_{n,m},$$

where $Y^\ell_{n,m}(\widehat{x}) = Y_{n,m}(\widehat{x - c_\ell})$. This formulation leads to the linear system developed in [3] with $2\mathsf{N}_{\mathrm{mod}}(\mathsf{N}_{\mathrm{mod}}+2)\mathsf{N}_{\mathrm{obs}}$ degrees of freedom. The matrix becomes more ill-conditionned as the number of obstacles grows or the size of obstacles decreases. We make use of linear algebra tools, preconditionners and iterative solvers to perform simulations with thousands of spheres.

### Numerical tests

The asymptotic models are validated with the spectral method, itself validated with finite element solutions provided by Montjoie code, in spherical geometries. Figure 1 shows the performance of the reduced models. The incident wave is an electromagnetic plane wave of wavelength $\lambda = 1.0$ and the medium contains five aligned spheres of radius $\delta$ varying between $10^{-0.5}$ and $10^{-2.75}$ . The reference solution is the spectral solution truncated at the order $\mathsf{N}_{\mathrm{mod}} = 10$.




(a) Reference (real part) (b) Distance $\propto \lambda$

(c) Distance $\propto \sqrt{\delta}$ (d) Distance $\propto \delta$

Figure 1: Relative $\mathbf{L}^2$-error depending on the obstacle size computed in the domain delimited by spheres of respective radius $15\lambda$ and $16\lambda$.

### References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