Characterization of Finite/Fixed-time Stability of Evolution Inclusions

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***Abstract*— Characterizations (necessary and sufficient conditions) of finite-time and fixed-time stability of evolution inclusions in Banach spaces are presented in terms of Lyapunov functionals. The present version of the paper refines some theorems from [1].**

I. Introduction

The method of Lyapunov function is the main tool for stability analysis and stabilization of both finite [2] and infinite dimensional non-linear systems [3], [4]. Characterization (necessary and sufficient conditions) of stability of evolution systems is an important problem in this context (see [5], [6]).

Finite-time and fixed-time control and estimation problems are in the scope control theory [7], [8]. Finite/Fixed-time estimation algorithms allow a separation principle to be realized in nonlinear systems [9]. Finite/fixed-time control algorithms allows us to fulfill some time constrains [10]. Finite-time stable of infinite dimensional systems are studied, for example, in [11], [12], [13], [14].

Some sufficient and some necessary conditions of finite-time and fixed-time stability of finite dimensional systems are given, for example, in [7], [15], [16]. This paper presents the conditions for a class of evolution systems (both finite and infinite dimensional).

Notation. $\mathbb{B}$ is a real Banach; $\mathbb{H}$ is a real Hilbert space; $\mathcal{L}(\mathbb{B},\mathbb{B})$ denotes the space of linear bounded operator on $\mathbb{B}$; L^2 denotes the Lebesgue space of quadratically integrable functions; $C_c^\infty$is a set of infinitely smooth functions with a compact support; H^p is a Sobolev space and H_0^p is a completion of C_c^∞ in the norm of H^p; $\mathbf{0}$ is the zero element of $\mathbb{B}$; $\mathcal{K}$ (resp. $\mathcal{K}_\infty$) is a set of strictly increasing positive functions $\sigma : [0,+\infty) \to [0,+\infty)$ (resp. $\sigma(s) \to +\infty$ as $s \to +\infty$); $B(r)$ is a ball of the radius $r > 0$ centered at $\mathbf{0}$.

II. System Description and Basic Assumptions

Let us consider the nonlinear system

$$\dot{x} - Ax \in F(t,x), \quad t > t_0, \tag{1}$$

where $t_0 > 0$ is an initial instant of time, $x(t) \in \mathbb{B}$ is the system state, $A : \mathcal{D}(A) \subset \mathbb{B} \to \mathbb{B}$ is a linear (possibly) unbounded closed densely defined operator which generates a strongly continuous semi-group Φ of linear bounded operators on $\mathbb{B}$ (see e.g. [17, Chapter 2] for more details) and $F : \mathbb{R}\times\mathbb{B} \rightrightarrows \mathbb{B}$ is a set-valued non-linear mapping such that $\mathbf{0} \in F(t,\mathbf{0})$. The latter means that the evolution inclusion always has the zero solution.

Definition 1: A continuous function $x : [t_0, t_0+T) \to \mathbb{B}$ is said to be a *mild solution* to (1) if there exists a selector (a single-valued mapping) $f \in L^1((t_0,t_0+T),\mathbb{B})$ such that

$$f(s) \in F(s,x(s)) \quad \text{for almost all} \quad s \in (0,T)$$

$$x(t) = \Phi(t)x_0 + \int_{t_0}^t \Phi(t-s)f(s)ds, \quad \forall t \in [t_0,t_0+T).$$

If this mild solution satisfies (1) for (almost) all $t \in (0,T)$ then x is called classical (strong) solution of (1).
The integral in the above definition is understood in the sense of Bochner (see e.g. [18], page 187).

The problem of existence of solutions for evolution models like (1) with singled-valued and set-valued F is studied in literature (see e.g. [17], [19], [20]). Notice that if $\mathbb{B} = \mathbb{R}^n$ then, to guarantee existence of solutions, the operator A must be bounded and, without lost of generality, we may assume $A = \mathbf{0}$, i.e. (1) becomes an ordinary differential inclusion.

Assumption 1: A mild solution x_{t_0,x_0} with the initial condition $x_0 \in \mathbb{B}$ exists and is defined, at least, on a time interval $[0,\bar{t})$, where $\bar{t} = +\infty$ or $\bar{t} < +\infty : \lim_{t\to\bar{t}} \|x_{t,x_0}(t)\| = +\infty$. The time instant $\bar{t}$ may depend on x_0 and/or on a concrete solution x_{t_0,x_0} if solutions are not unique.

The aim of this paper is to provide characterizations (necessary and sufficient condition) of finite-time and fixed-time stability of the system (1) in terms of Lyapunov functions.

Notice that the most of constructions presented below can be repeated for more general models of dynamical systems dynamical systems considered, for example, in [5] and [21].

III. Preliminaries

A. Stability Definitions

The concept of stability introduced by A.M. Lyapunov [22] considers some *nominal motion* $x^*_{t_0,x_0}(t)$ of a dynamic system and studies perturbations Δx_0 of the initial condition x_0, where the index t_0, x_0 indicates a dependence of a solution of the initial condition $x(t_0) = x_0$. If small perturbations imply small deviations of perturbed motions $x_{t_0,x_0+\Delta x_0}(t)$ from $x^*_{t_0,x_0}(t)$ then the *nominal motion* is called stable. Below we consider the zero solution (or, equivalently, the origin) of the system as the nominal motion.

Solutions to ODEs, differential inclusions and evolution systems may be non-unique in the general case. This implies two possible types of stability: *weak stability* (a property holds for *a solution*) and *strong stability* (a property holds for *all solutions*). Weak stability usually is not enough for

The research is partially supported by ANR Finite4SoS project (15-CE23-0007) and the Government of Russian Federation (Grant 08-08).

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control purposes. In this paper we deal only with strong stability.

Definition 2 (Lyapunov stability): The origin of the system (1) is said to be locally (globally) Lyapunov stable if there exists $\varepsilon_{t_0} \in \mathcal{K}$ (resp. $\mathcal{K}_\infty$) such that

$$\|x_{t_0,x_0}(t)\| \le \varepsilon_{t_0}(\|x_0\|), \qquad t > t_0$$

for any solution x_{t_0,x_0} of (1) and any $x_0 \in U$, where $U \subset \mathbb{B}$ is a neighborhood of the origin (resp. $U = \mathbb{B}$). If the function ε_{t_0} is independent of t_0 then the origin of the system (1) is locally (gloablly) uniformly Lyapunov stable.

Let us recall the well-known result about Lyapunov stable systems (see e.g. [7]).

Proposition 1: If the origin of the system (1) is Lyapunov stable then $x_{t_0,\mathbf{0}}(t) \equiv \mathbf{0}$ is the unique solution of the system (1) with the initial condition $x(t_0) = \mathbf{0}$.

Proof. Suppose the contrary, i.e. there exists a solution $x_{t_0,\mathbf{0}}$ and a time instant $t' > t_0$ such that $x_{t_0,\mathbf{0}}(t') \neq \mathbf{0}$. Denote $\varepsilon := \|x_{t_0,\mathbf{0}}(t')\|/2 > 0$. From the definition of Lyapunov stability we have for any $\varepsilon > 0$ there exits $\delta > 0$ such that $\|x_0\| < \delta$ always implies $\|x_{t_0,x_0}(t)\| < \varepsilon$ for all $t > t_0$. Taking into account that, in our case, $\|x_0\| = \|\mathbf{0}\| = 0 < \delta$ we obtain the contradiction. ■

Definition 3 (Finite-time stability): The origin of the system (1) is said to be locally (globally) finite-time stable if it is locally (globally) Lyapunov stable and *finite-time attractive* in a neighborhood $U(t_0)$ (resp. $U(t_0) = \mathbb{B}$) of the origin, i.e. $\exists T : \mathbb{R} \times \mathbb{B} \to \mathbb{R}_+$ such that

$$\forall x_0 \in U(t_0) \quad \Rightarrow \quad x_{t_0,x_0}(t) = \mathbf{0}, \;\; \forall t \ge t_0 + T(t_0, x_0)$$

for any solution x_{t_0,x_0} of the system.

Obviously, if T satisfy the latter definition then the function $T + T^+$ also does for any nonnegative T^+. Therefore, it is reasonable to consider a minimal functional T.

Definition 4: A functional $T : \mathbb{R} \times \mathbb{B} \to \mathbb{R}_+$ is called the *settling-time function* of the finite-time stable system (1), if T satisfies Definition 3 and for any $\tilde{T} : \mathbb{R} \times \mathbb{B} \to [0, +\infty) :$ $\tilde{T} \neq \mathbf{0}$ the functional $T - \tilde{T}$ does not satisfy Definition 3 (with the same U).

Finite-time stability always implies asymptotic stability. The settling-time function T of time-invariant finite-time stable system is independent of t_0. However, in contrast to asymptotic and Lyapunov stability, finite-time stability of time-invariant system, in general, does not imply uniform finite-time stability [7].

Definition 5 (Uniform finite-time stability): The origin of the system (1) is said to be locally (globally) uniformly finite-time stable if it is finite-time stable in a time-invariant attraction domain U and the settling time function $T : \mathbb{R} \times U \to \mathbb{R}$ is *locally bounded uniformly on the first argument*, i.e. $\forall y \in U, \exists \varepsilon > 0$ such that $\sup_{t_0 \in \mathbb{R},\, \|x_0 - y\| < \varepsilon} T(t_0, x_0) < +\infty$.

Notice that uniformity of finite-time stability on initial condition is guaranteed by local boundedness of T on x, but uniformity on the initial instant of time comes from the uniform boundedness of T on t_0 (see [23] for more details).

Definition 6 (Fixed-time Stability): The origin of the system (1) is said to be locally (globally) fixed-time stable if it is locally (globally) uniformly finite-time stable and the settling-time functional T is globally bounded, i.e.

$$\exists T_{\max} > 0 : x_{t_0,x_0}(t) = 0, \quad \forall t \ge t_0 + T_{\max}, \forall x_0 \in U$$

where $U \subset \mathbb{B}$ is a neighborhood of the origin ($U = \mathbb{B}$).

Fixed-time stability can be discovered even for linear evolution systems (i.e. with $F \equiv \mathbf{0}$). For example, solutions of wave equations with the so-called transparent boundary conditions vanish in a fixed time (see e.g. [24] and [11]).

B. Generalized derivatives

A mild solution satisfies the evolution inclusion (1) in the integral sense (see Definition 1). Its time derivative may not exist and not satisfy (1). Generalized derivatives (see e.g. [25]) can be utilized in this case for Lyapunov analysis.

Let $\mathcal{I}$ be one of the following sets: $[a, b]$, (a, b), $[a, b)$ or $(a, b]$, where $a, b \in [-\infty, +\infty]$, $a < b$. Let $\mathbb{K}$ be the set of sequences of real non-zero numbers converging to zero:

$$\{h_n\} \in \mathbb{K} \quad \Leftrightarrow \quad h_n \to 0, \; h_n \neq 0.$$

Definition 7 (page 207, [25]): A number

$$D_{\{h_n\}}\varphi(t) = \lim_{n \to +\infty} \frac{\varphi(t+h_n) - \varphi(t)}{h_n}, \;\; \{h_n\} \in \mathbb{K} : t + h_n \in \mathcal{I}$$

is called derivative number of the function $\varphi : \mathcal{I} \to \mathbb{R}$ at the point $t \in \mathcal{I}$, if the above limit exists (both finite and infinite limits are admissible).

The set of all derivative numbers of the function φ at the point $t \in \mathcal{I}$ is called contingent derivative:

$$D_{\mathbb{K}}\varphi(t) = \bigcup_{\{h_n\} \in \mathbb{K} : \exists D_{\{h_n\}}\varphi(t)} \{D_{\{h_n\}}\varphi(t)\} \subset [-\infty, +\infty].$$

Obviously, if a function φ is differentiable at a point $t \in \mathcal{I}$ then $D_{\mathbb{K}}\varphi(t) = \dot{\varphi}(t)$.

Lemma 1 ([25], page 208): The set $D_{\mathbb{K}}\varphi(t) \subseteq [-\infty, +\infty]$ is nonempty for $\varphi : \mathcal{I} \subset \mathbb{R} \to \mathbb{R}$ and $t \in \mathcal{I}$.

If $\varphi : \mathcal{I} \to \mathbb{R}$ then the notation $D_{\mathbb{K}}\varphi(t) < 0$ means $y < 0$, $\forall y \in D_{\mathbb{K}}\varphi(t)$. In the similar component-wise sense we define$|D_{\mathbb{K}}\varphi(t)| < L$ with $L \in \mathbb{R}_+$, and $qD_{\mathbb{K}}\phi(t)$ with $q \in \mathbb{R}$. Definition 7 implies *if* $D_{\mathbb{K}}\phi_1(t) \le c_1$ *and* $D_{\mathbb{K}}\phi_2(t) \le c_2$ *for all* $t \in \mathcal{I}$*, where* $c_1, c_2 \in \mathbb{R}$*, then* $D_{\mathbb{K}}(\phi_1 + \phi_2)(t) \le c_1 + c_2$.

Recall that a function $\varphi : \mathcal{I} \to \mathbb{R}$ is called *decreasing* on $\mathcal{I}$ if $\forall t_1, t_2 \in \mathcal{I} : t_1 \le t_2 \;\Rightarrow\; \varphi(t_1) \ge \varphi(t_2)$.

Lemma 2: The function $\varphi : \mathcal{I} \in \mathbb{R} \to \mathbb{R}$ is decreasing on $\mathcal{I}$ if and only if $D_{\mathbb{K}}\varphi(t) \le 0$ holds for all $t \in \mathcal{I}$.

Sufficiency is the straightforward consequence of Definition 7 (see also [25], page 266]). *Necessity* immediately follows from the definition of the decreasing function.

Lemma 3 ([25], page 212): If a function $\varphi : [a, b] \to \mathbb{R}$ is decreasing then it differentiable almost everywhere and

$$\phi(b) - \phi(a) \le \int_a^b \dot{\phi}(\tau) d\tau.$$

Lemma 4: Let a function $f : [0, T) \to \mathbb{R}$ be continuously differentiable with $f' \ge 0$, where $T \le +\infty$. If $\varphi : \mathcal{I} \to [0, T)$ is a decreasing function then $D_{\mathbb{K}} f(\varphi(t)) \le 0$ for all $t \in \mathcal{I}$ and $\frac{df(\varphi(t))}{dt} \le f'(\varphi(t))\dot{\varphi}(t)$ almost everywhere on $\mathcal{I}$.

Proof. Since φ is decreasing and f and differentiable with $f' \geq 0$ then for $h_n > 0$ we have $\frac{f(\varphi(t+h_n))-f(\varphi(t))}{h_n} = \frac{-\int_{\varphi(t+h_n)}^{\varphi(t)} f'(x)dx}{h_n} \leq \frac{\varphi(t+h_n)-\varphi(t)}{h_n} \min\limits_{x\in[\varphi(t+h_n),\varphi(t)]} f'(x) \leq 0$, but for $h_n < 0$ we derive $\frac{f(\varphi(t+h_n))-f(\varphi(t))}{h_n} = \frac{\int_{\varphi(t)}^{\varphi(t+h_n)} f'(x)dx}{h_n} \leq \frac{\varphi(t+h_n)-\varphi(t)}{h_n} \min\limits_{x\in[\varphi(t),\varphi(t+h_n)]} f'(x) \leq 0$. According to Lemma 3 the function φ is differentiable almost everywhere on $\mathcal{I}$. Hence, it is continuous almost everywhere on $\mathcal{I}$ and $\liminf\limits_{\varepsilon\to\mathbf{0}} f'(\varphi(t+\varepsilon))D_{\mathbb{K}}\phi(t) = f'(\varphi(t))\dot{\varphi}(t)$ almost everywhere on $\mathcal{I}$. ■

Lemma 5: Let $\varphi_1 : \mathcal{I} \to [0, C]$ with $C < +\infty$ and $\varphi_2 : \mathcal{I} \to \mathbb{R}$. If $D_{\mathbb{K}}\varphi_1(t) \leq -1$ and $D_{\mathbb{K}}\varphi_2(t) \leq 0$ then

$$D_{\mathbb{K}}(\varphi_1\varphi_2)(t) \leq -\varphi_2(t), \quad t \in \mathcal{I}.$$

Proof. Since $\frac{\varphi_1(t+h_n)\varphi_2(t+h_n)-\varphi_1(t)\varphi_2(t)}{h_n} = \frac{\varphi_1(t+h_n)\varphi_2(t+h_n)-\varphi_2(t)\varphi_1(t+h_n)+\varphi_2(t)\varphi_1(t+h_n)-\varphi_1(t)\varphi_2(t)}{h_n} = \frac{\varphi_1(t+h_n)(\varphi_2(t+h_n)-\varphi_2(t))}{h_n} + \varphi_2(t)\frac{\varphi_1(t+h_n)-\varphi_1(t)}{h_n}$ then under our assumptions we derive $D_{\{h_n\}}(\varphi_1\varphi_2)(t) \leq \varphi_2(t)D_{\{h_n\}}\varphi_1(t)$. ■

C. Positive Definiteness and Generalized Properness

Recall that a functional $W : \Omega \subset \mathbb{B} \to \mathbb{R}$ is said to be *positive definite* if $W(0) = 0$ and $W(x) > 0$ for $x \in \Omega\backslash\{\mathbf{0}\}$.

Lyapunov function candidates in $\mathbb{R}^n$ are usually positive definite and proper (see e.g. [26]). Recall that a mapping $f : \mathbb{R}^n \to \mathbb{R}$ is proper if the inverse image of any compact set is a compact set. In the general case, closedness and boundedness is not sufficient for compactness in Banach spaces, and the properness in the classical sense may be too strong condition for Lyapunov function candidate. For "generalized" proper functions given by Definition 8 an inverse image of any compact set belongs to a closed bounded set (which may be not compact in the general case). Below the word "generalized" is omitted for shortness.

Definition 8: A positive definite functional $V : \mathbb{R} \times \Omega \to [0, +\infty)$ is said to be proper at $\mathbf{0}$ (**locally proper**) if there exists $\underline{V}, \overline{V} \in \mathcal{K}$ such that $\underline{V}(\|x\|) \leq V(t, x) \leq \overline{V}(\|x\|)$ for all $x \in \Omega\backslash\{\mathbf{0}\}$, and all $t \in \mathbb{R}$, where $\Omega \subset \mathbb{B}$ is a neighborhood of $\mathbf{0}$.

A positive definite functional $V : \mathbb{B} \to [0, +\infty)$ is said to be **globally proper** if $\underline{V}, \overline{V} \in \mathcal{K}^{\infty}$, $\Omega = \mathbb{B}$.

The locally/globally proper positive definite functions are conventional Lyapunov function candidates for stability analysis of dynamical systems (see [2], [5]). To analyze a decay of such discontinuous function along trajectories of a system, the generalized derivatives considered above can be utilized.

Given number $r \in \mathbb{R}$ and given positive definite function $W : \Omega \subset \mathbb{B} \to \mathbb{R}$ let us denote *the level set* of the function W as follows $\Pi(W, r) := \{x \in \Omega : W(x) < r\}$.

IV. Characterization of Lyapunov stability

Lyapunov Function Method is founded on the so-called energetic approach to stability analysis. It considers a positive definite function as a possible energetic characteristic (energy) of a dynamical system and studies evolution of this function in time. If such a function is decreasing along any system trajectory then the system has some stability property. The following theorem can also be recovered from [27, Proposition 14.1].

Theorem 1: The origin of the system (1) is locally (globally) uniformly Lyapunov stable if and only if there exists a locally (globally) proper positive definite functional $V : \mathbb{R} \times \Omega \subset \mathbb{B} \to \mathbb{R}$ such that the inequality

$$D_{\mathbb{K}}V(t, x_{t_0,x_0}(t)) \leq 0, \qquad \forall t > t_0 \tag{2}$$

holds for any $t_0 \in \mathbb{R}$ and for any mild solution x_{t_0,x_0} of (1) as long as $x_{t_0,x_0}(t) \in \Omega\backslash\{\mathbf{0}\}$, where Ω is a neighborhood of the origin (resp. $\Omega = \mathbb{B}$).

Proof. *Sufficiency*. Let $\varepsilon_0 > 0$ be such that $B(\varepsilon_0) \subset \Omega$, where $B(r)$ as before denotes an open ball in $\mathbb{B}$ of the radius $r > 0$. Notice that such ε_0 always exists since $\mathbf{0} \in \operatorname{int}\Omega$.

Since V is locally proper, then there exist $\underline{V}, \overline{V} \in \mathcal{K}$ such that $\underline{V}(\|x\|) \leq V(t, x) \leq \overline{V}(\|x\|)$ for all $x \in \Omega\backslash\{\mathbf{0}\}$ and all $t \in \mathbb{R}$. This implies that for any $\varepsilon > 0$ here exists a sufficiently small $\lambda_\varepsilon > 0$ such that $\Pi(V, \lambda_\varepsilon) \subset B(\varepsilon)$ and there exists $q_\varepsilon \in (0, \min\{\varepsilon_0, \varepsilon\})$ such that $\underline{V}(q_\varepsilon) = \lambda_\varepsilon$. On the other hand, since the functional V is locally proper then $\exists\delta \in (0, q_\varepsilon)$ such that $V(t, x) < \underline{V}(q_\varepsilon)$ for all $x \in B(\delta)$, all $t \in \mathbb{R}$ and $B(\delta) \subset \Pi(V, \lambda_\varepsilon) \subset B(\varepsilon)$.

Given $x_0 \in B(\delta)$ let us consider an arbitrary mild solution $x_{t_0,x_0}(t)$ of (1). The inequality (2) and Lemma 2 imply that $t \to V(t, x_{t_0,x_0}(t))$ is decreasing on $t > t_0$ as long as $x_{t_0,x_0}(t) \in \Omega$.

Let us show that $\|x_{t_0,x_0}(t^*)\| < q_\varepsilon < \varepsilon$ for all $t > t_0$ provided that $\|x_0\| < \delta < q_\varepsilon$. Suppose the contrary, i.e. $\exists t^* > t_0 : \|x_{t_0,x_0}(t^*)\| = q_\varepsilon$ and $\|x_{t_0,x_0}(t^*)\| < q_\varepsilon$ for $t \in (t_0, t^*)$. From the definition of the function $\underline{V}$ we derive $V(t, x(t^*, t_0, x_0)) \geq \underline{V}(\|x(t^*, t_0, x_0)\|) = \underline{V}(q_\varepsilon) = \lambda_\varepsilon$. On the other hand, $x_{t_0,x_0}(t) \in B(q_\varepsilon) \subset B(\varepsilon_0) \subset \Omega$ for all $t \in (t_0, t^*)$, i.e. $V(t, x_{t_0,x_0}(t))$ is decreasing on this interval and $V(t, x_{t_0,x_0}(t)) \leq V(t_0, x_0) < \lambda_\varepsilon$. We obtain the contradiction.

Necessity. Let S_{t_0,x_0} denote the set of all solutions of the system (1) with the initial condition $x(t_0) = x_0$. Let the origin of the system (1) be uniformly Lyapunov stable.In this case, the functional $V : \mathbb{R} \times \Omega \subset \mathbb{B} \to \mathbb{B}$ given by

$$V(t_0, x_0) := \sup_{x_{t_0,x_0}\in S_{t_0,x_0}} \sup_{t\geq t_0} \|x_{t_0,x_0}(t)\|.$$

is locally (globally) proper. Indeed, by construction V is non-negative and $\underline{V}(\|x_0\|) := \|x_0\| \leq V(t, x_0)$, but the definition of local (global) uniform Lyapunov stability implies that V is well defined and locally bounded in some neighborhood Ω of $\mathbf{0}$ (resp. $\Omega = \mathbb{B}$) and there exists $\overline{V} \in \mathcal{K}$ (resp. $\mathcal{K}_\infty$) such that $V(t, x) \leq \overline{V}(\|x\|)$, e.g. $\bar{V}(s) = \frac{s+1}{s}\sup_{t\in\mathbb{R},\|x\|=s} V(t, x)$. Finally, due to uniform Lyapunov stability of the origin, $V(t, x_{t_0,x_0}(t))$ is decreasing and, by Lemma 2, $D_{\mathbb{K}}V(t, x_{t_0,x_0}(t)) \leq 0$ as long as $x_{t_0,x_0}(t) \in \Omega\backslash\{\mathbf{0}\}$. ■

Notice that, in many cases (see e.g. [23]), we do not need to know solutions x_{t_0,x_0} to check the inequality (2).

V. Characterization of Finite-Time stability

The finite-time stability combines two properties: Lyapunov stability and finite-time convergence. The corresponding Lyapunov theorem must characterize both of them.

Theorem 2: The origin of the system (1) is locally (globally) uniformly finite-time stable if and only if there exist a locally (globally) proper positive definite functional $V : \mathbb{R}\times\Omega\subset\mathbb{B}\to\mathbb{R}$ and a locally bounded uniformly on the first argument functional $T : \mathbb{R}\times\Omega\subset\mathbb{B}\to[0,+\infty)$ such that the inequalities

$$D_{\mathbb{K}}V(t,x_{t_0,x_0}(t))\leq 0,\quad D_{\mathbb{K}}T(t,x_{t_0,x_0}(t))\leq -1,\quad t>t_0 \tag{3}$$

hold for any $t_0\in\mathbb{R}$ and for any mild solution x_{t_0,x_0} of (1) as long as $x_{t_0,x_0}(t)\in\Omega\backslash\{\mathbf{0}\}$, where Ω is a neighborhood of the origin (resp. $\Omega=\mathbb{B}$). Moreover, the functional T estimates the settling time, i.e. $x_{t_0,x_0}(t)=\mathbf{0}$ for $t\geq t_0+T(t_0,x_0)$.

Proof. *Sufficiency.* Theorem 1 implies that the origin of the system (1) is locally (globally) Lyapunov stable. This means that any solution $x_{t_0,x_0}(t)$ remains bounded for all $t>t_0$ and $x_{t_0,x_0}(t)\in\Omega$ provided that x_0 belongs to some neighborhood U of $\mathbf{0}$ (resp. $U=\mathbb{B}$ in the case of global Lyapunov stability). Hence, $t\to T(t,x_{t_0,x_0}(t))$ is uniformly bounded $\forall t>t_0$.

Let us consider the interval $[t_0,t_1+\varepsilon],\varepsilon>0,t_1=t_0+T(t_0,x_0)$ and show that there exists an instant of time $t^*\in[t_0,t_1+\varepsilon]$ such that $x_{t_0,x_0}(t^*)=\mathbf{0}$. Suppose the contrary, i.e. $x_{t_0,x_0}(t)\neq 0$ for $\forall t\in[t_0,t_1+\varepsilon]$. The inequality (3) and Lemma 2 imply that the function $T(\cdot,x_{t_0,x_0}(\cdot))$ is decreasing on $[t_0,t_1+\varepsilon]$ and, consequently, differentiable almost everywhere on $[t_0,t_1]$. Then using Lemma 3 we derive

$$T(t_1+\varepsilon,x_{t_0,x_0}(t_1+\varepsilon))-T(t_0,x_{t_0,x_0}(t_0))\leq$$
$$\int_{t_0}^{t_1+\varepsilon}\dot T(\tau,x_{t_0,x_0}(\tau)))d\tau\leq-(t_1+\varepsilon-t_0)=-T(t_0,x_0)-\varepsilon,$$

i.e., $T(t_1+\varepsilon,x_{t_0,x_0}(t_1+\varepsilon))\leq-\varepsilon$. This contradicts to $T\geq 0$ for all $x\in U\subset\Omega$ and all $t\in\mathbb{R}$. Therefore, there exists $t^*\in[t_0,t_1+\varepsilon]$ such that $x_{t_0,x_0}(t^*)=\mathbf{0}$. Since $\varepsilon>0$ can be selected arbitrary small then $t^*\in[t_0,t_1]$, i.e. $t^*\leq t_0+T(t_0,x_0)$. Proposition 1 implies that the origin of the system (1) is uniformly finite-time stable.

Necessity. Let the origin of the system (1) be locally (globally) uniformly finite-time stable. Notice that Proposition 1 implies that $x_{t_0,\mathbf{0}}\equiv\mathbf{0}$ is the unique solution of our system with the zero initial condition. The existence of a functional V follows from Theorem 1.

Let S_{t_0,x_0} denote the set of all solutions of the system (1) with the initial condition $x(t_0)=x_0\in U$, where $U\subset\mathbb{B}$ is a domain of finite-time attraction ($U=\mathbb{B}$ in the case of global finite-time stability). If $z\in S_{t_0,x_0}$ then z is defined on $[t_0,+\infty)$.

Let us introduce the functional $T:\mathbb{R}\times U\subset\mathbb{B}\to\mathbb{R}$ as follows

$$T(t_0,x_0)=\sup_{x\in S_{t_0,x_0}}\inf_{\tau>0\,:\,x(t_0+\tau)=\mathbf{0}}\tau,\quad x_0\in U.$$

Since the origin is uniformly finite-time stable then $T(t,x)\geq 0$ and T is locally bounded on $x\in U$ uniformly on $t\in\mathbb{R}$.

Let x be an arbitrary solution such that $x(t)\in U$ for all $t>t^*$ and let $h>0$ be such that $x(p)\neq\mathbf{0}$ with $p\in[t_0,t_0+h]$. In this case, there exists $\tilde x\in S_{t_0,x(t_0)}$ such that $\inf_{\tau>0\,:\,\tilde x(t_0+\tau)=\mathbf{0}}\tau\geq h$ and

$$T(t_0,x(t_0))-h=\sup_{x\in S_{t_0,x(t_0)}}\inf_{\tau>0\,:\,x(t_0+\tau)=\mathbf{0}}\tau-h=$$
$$\sup_{x\in S_{t_0,x(t_0)}}\inf_{\tilde\tau>-h\,:\,x(t_0+h+\tilde\tau)=\mathbf{0}}\tilde\tau=$$
$$\sup_{x\in S_{t_0,x(t_0)}}\inf_{\tilde\tau>0\,:\,x(t_0+h+\tilde\tau)=\mathbf{0}}\tilde\tau.$$

Let us denote

$$S^h_{t_0,x_0}=\left\{z_{[t_0+h,+\infty)}\in C([t_0+h,+\infty),\mathbb{B}):z\in S_{t_0,x_0}\right\},$$

where $h>0$ and $z_{[t_0+h,+\infty)}$ denotes restriction of a function $z:[t_0,+\infty)\to\mathbb{B}$ to the time interval $[t_0+h,+\infty)$. Since $S_{t_0+h,x(t_0+h)}\subset S^h_{t_0,x(t_0)}$ then

$$T(t_0+h,x(t_0+h))=\sup_{x\in S_{t_0+h,x(t_0+h)}}\inf_{\tilde\tau>0\,:\,x(t_0+h+\tilde\tau)=\mathbf{0}}\tilde\tau\leq$$
$$\sup_{x\in S^h_{t_0,x(t_0)}}\inf_{\tilde\tau>0:x(t_0+h+\tilde\tau)=\mathbf{0}}\tilde\tau=\sup_{x\in S_{t_0,x(t_0)}}\inf_{\tilde\tau>0:x(t_0+h+\tilde\tau)=\mathbf{0}}\tilde\tau.$$

Hence, we obtain $h^{-1}(T(t_0+h,x(t_0+h))-T(t_0,x(t_0)))\leq-1$ for any sufficiently small $h>0$. Similarly, one can be shown $h^{-1}(T(t_0,x(t_0))-T(t_0-h,x(t_0-h)))\leq-1$ for $h>0$. Therefore, $D_{\mathbb{K}}(T(t,x(t)))\leq-1$ as long as $\mathbf{0}\neq x(t)\in U$. ■

Notice that *if the system* (1) *is autonomous (i.e. F is independent of t) then V and T in Theorems 1 and 2 are independent of t.*

If the settling-time function T is continuous at the origin then the function T can be excluded from Theorem 2 and the settling-time can be estimated using V.

Corollary 1: The origin of the autonomous system (1) is locally (globally) uniformly finite-time stable with a continuous at the origin settling-time function if and only if there exists a locally (globally) proper positive definite functional $V:\Omega\subset\mathbb{B}\to[0,+\infty)$ such that the inequalities

$$D_{\mathbb{K}}V(x_{t_0,x_0}(t))\leq-1,\qquad t>t_0, \tag{4}$$

hold for any $t_0\in\mathbb{R}$ and for any mild solution x_{t_0,x_0} of (1) as long as $x_{t_0,x_0}(t)\in\Omega\backslash\{\mathbf{0}\}$, where Ω is a neighborhood of the origin (resp. $\Omega=\mathbb{B}$).

Proof. *Sufficiency* can be proved similarly to Theorem 2 using V instead of T in all considerations. *Necessity*. According to Theorem 2 we have two functions V and T such that V is locally proper, T is locally bounded and $D_{\mathbb{K}}V(x_{t_0,x_0}(t))\leq 0$, $D_{\mathbb{K}}T(x_{t_0,x_0}(t))\leq-1$. Since T is continuous at $\mathbf{0}$ and $T(\mathbf{0})=0$ for any finite-time stable system, then the function $V^{new}:=V+T$ is locally (globally) proper and $D_{\mathbb{K}}V^{new}(x_{t_0,x_0}(t))\leq-1$. ■

The finite-time stability can be characterized by means of the coercive Lyapunov function (see definitions in [5]).

Corollary 2: The origin of the autonomous system (1) is locally (globally) uniformly finite-time stable with a continuous at the origin settling-time function if and only if there exist a locally (globally) proper positive definite functionals $V,W:\Omega\subset\mathbb{B}\to[0,+\infty)$ such that the inequalities

$$D_{\mathbb{K}}V(x_{t_0,x_0}(t))\leq-1-W(x_{t_0,x_0}(t)),\qquad t>t_0, \tag{5}$$

hold for any $t_0 \in \mathbb{R}$ and for any mild solution x_{t_0,x_0} of (1) as long as $x_{t_0,x_0}(t) \in \Omega\backslash\{\mathbf{0}\}$, where Ω is a neighborhood of the origin (resp. $\Omega = \mathbb{B}$).

Proof. *Sufficiency* immediately follows from Corollary (2). *Necessity*. According to Theorem 2 we have two functions V and T such that V is locally proper, T is locally bounded and $D_{\mathbb{K}}V(x_{t_0,x_0}(t)) \leq 0$, $D_{\mathbb{K}}T(x_{t_0,x_0}(t)) \leq -1$. Since T is continuous at $\mathbf{0}$ and $T(\mathbf{0}) = 0$ for any finite-time stable system, then the function $V^{new} := T + (T+1)V$ is locally (globally) proper. Using Lemma 5 we derive $D_{\mathbb{K}}V^{new}(x_{t_0,x_0}(t)) \leq -1 - V$. ■

A. Characterization of Fixed-time Stability

If the system is fixed-time stable then it finite-time stable. Stability theorems for such systems are expected to be similar.

Theorem 3: The origin of the system (1) is locally (globally) fixed-time stable if and only if there exist a locally (globally) proper positive definite functional $V : \mathbb{R}\times\Omega \subset \mathbb{B} \to \mathbb{R}$ and a uniformly bounded functional $T : \mathbb{R}\times\Omega \subset \mathbb{B} \to [0,+\infty)$ such that $T(t,\mathbf{0}) = 0$, $T(t,x) \leq T_{\max} < +\infty$ and the inequalities $D_{\mathbb{K}}V(t, x_{t_0,x_0}(t)) \leq 0$, $D_{\mathbb{K}}T(t, x_{t_0,x_0}(t)) \leq -1$ with $t > t_0$ hold for any $t_0 \in \mathbb{R}$ and for any mild solution x_{t_0,x_0} of (1) as long as $x_{t_0,x_0}(t) \in \Omega\backslash\{\mathbf{0}\}$, where Ω is a neighborhood of the origin (resp. $\Omega = \mathbb{B}$).

Proof. The proof literally repeats the proof of Theorem 2 using the property of boundedness of T (i.e. $\exists T_{\max} > 0$ such that $T(t,x) \leq T_{\max}$ for all $t \in \mathbb{R}$ and all $x \in \Omega$). ■

If the settling-time function T is continuous at the origin then fixed-time stability can be analyzed by means of a Lyapunov function V only.

Corollary 3: The origin of the system (1) is locally (globally) fixed-time stable with a continuous at the origin settling-time function if there exist a locally (globally) proper functional $V : \Omega \subset \mathbb{B} \to \mathbb{R}$ and a number $q > 0$ such that the inequalities $D_{\mathbb{K}}V(x_{t_0,x_0}(t)) \leq 0$ with $t > t_0$, and

$$\dot{V}(x_{t_0,x_0}(t)) \overset{a.e.}{\leq} -q(1 + V^2(x_{t_0,x_0}(t))), \quad t > t_0 \tag{6}$$

hold for any $t_0 \in \mathbb{R}$ and for any mild solution x_{t_0,x_0} of (1) as long as $x_{t_0,x_0}(t) \in \Omega\backslash\{\mathbf{0}\}$, where $\Omega \subset \mathbb{B}$ is a neighborhood of the origin. The settling-time function admits the estimate $T(x_0) \leq \frac{\pi}{2q}$ for all $x_0 \in \mathbb{B}$ (resp. $\Omega = \mathbb{B}$).

Proof. Theorem 1 implies that the origin of the system (1) is locally (globally) Lyapunov stable. In particular this means that any solution initiated in U exists for all $t > 0$, where U is a neighborhood of the origin (resp. $U = \mathbb{B}$). Notice that $V(x_{t,x_0})$ is a monotone function and it is differentiable almost everywhere.

Let us prove fixed-time convergence and show a settling time bounded by $T_{\max} = 0.5\pi/q$. Suppose the contrary, i.e. there exists a solution x_{t_0,x_0} which exists on $[0,+\infty)$ and does not vanish after the time $T_{\max}$. Let us consider the functional $T : \mathbb{B} \to [0,+\infty)$ defined as follows $T(x) = \frac{\arctan(V(x))}{q} \leq T_{\max}$. Obviously, T is locally proper. Since $\arctan$ is a differentiable function then, using Lemma 4 we derive $D_{\mathbb{K}}T(x_{t_0,x_0}(t)) \leq 0$ for all $t \geq$ $[t_0, t_0 + T_{\max}]$ and $\dot{T}(x_{t_0,x_0}(t)) \leq \frac{\dot{V}(x_{t_0,x_0}(t))}{q(1+V^2(x_{t_0,x_0}(t+\varepsilon))} \leq -1$ for almost all $t \in [t_0, t_0 + T_{\max}]$. Hence, using Lemma 3 we derive $T(x_{t_0,x_0}(t_0+T_{\max})) - T(x_{t_0,x_0}(t_0)) \leq \int_{t_0}^{t_0+T_{\max}} \dot{T}(x_{t_0,x_0}(\tau))d\tau \leq -T_{\max}$, i.e. $T(x_{t_0,x_0}(t_0 + T_{\max})) = 0$. Proposition 1 implies that $x_{t_0,x_0}(t) = 0$ for all $t \geq t_0 + T_{\max}$. ■

The conditions mentioned in the latter corollary are necessary for the local fixed-time stability. Existence of a globally proper Lyapunov function satisfying (6) for a globally fixed-time stable system is still an open problem in the general case. Particular cases are studied in [16], [15] for finite-dimensional systems, where it is assumed that the settling-time function is, at least, locally proper. This is not the case for an unbounded operator A in (1) (see Section VI-B for more details).

Corollary 4: If the origin of the autonomous system (1) is locally (globally) fixed-time stable with a continuous at the origin settling-time function then there exist a locally proper functional $V : \Omega \subset \mathbb{B} \to \mathbb{R}$ and a number $q > 0$ such that the inequalities $D_{\mathbb{K}}V(x_{t_0,x_0}(t)) \leq 0$ with $t > t_0$, and

$$\dot{V}(x_{t_0,x_0}(t)) \overset{a.e.}{\leq} -q(1 + V^2(x_{t_0,x_0}(t))), \quad t > t_0$$

hold for any $t_0 \in \mathbb{R}$ and for any mild solution x_{t_0,x_0} of (1) as long as $x_{t_0,x_0}(t) \in \Omega\backslash\{\mathbf{0}\}$, where $\Omega \subset \mathbb{B}$ is a neighborhood of the origin ($\Omega = \mathbb{B}$).

Proof. Since the system is locally (globally) fixed-time stable then it locally(globally) uniformly finite-time and Lyapunov stable. According to Theorem 3 there exist the Lyapunov functional V and the settling-time T such that $D_{\mathbb{K}}V(x_{t_0,x_0}(t)) \leq 0$, $D_{\mathbb{K}}T(x_{t_0,x_0}(t)) \leq -1$ as long as $x_{t_0,x_0}(t) \in \Omega$ and $T(x_0) \leq T_{\max} \in (0,+\infty)$ for all $x_0 \in \Omega$.

Let us consider the functional $T_0 : \Omega \to [0,+\infty)$ defined as follows $T_0(x) = T(x) + \left(1 - \frac{1}{V(x)+1}\right), x \in \Omega$. Using Lemma 4 we derive $D_{\mathbb{K}}T_0(x_{t_0,x_0}(t)) \leq -1$ as long as $x_{t_0,x_0}(t) \in \Omega$ and T_0 is locally proper, i.e. $\exists \underline{T}_0, \overline{T}_0 \in \mathcal{K}$ such that $\underline{T}_0(\|x\|) \leq T_0(x) \leq \overline{T}_0(\|x\|)$. Moreover, we have

$$\underline{T}_0(s) \to 1 \quad \text{and} \quad \overline{T}_0(s) \to 1 + T_{\max} \quad \text{as} \quad s \to +\infty.$$

The functional $V_0 : \mathbb{B} \to \mathbb{R}$ given by $V_0(x) = \tan\left(\frac{\pi}{2(1+T_{\max})}T_0(x)\right), x \in \Omega$. is locally proper. Using Lemma 4 we derive $V_0(x_{t_0,x_0}(t)) \leq 0$ for all $t \in [t_0, t_0 + T_{\max}]$ and $D_{\mathbb{K}}V_0(x_{t_0,x_0}(t)) \leq -\frac{\pi(1+V_0^2(x_{t_0,x_0}(t)))}{2(1+T_{\max})}$ for almost all $t \in [t_0, t_0 + T_{\max}]$. ■

The claims given above are proven without any assumption about continuous dependence of solutions on initial conditions. Making such assumptions we can derive continuity (or even Lipschitz continuity) of V and T.

VI. Examples

A. Example of finite/fixed-time stable evolution system in a Hilbert space

Let us consider the evolution inclusion

$$\dot{x} - Ax \in F(x), \quad x \in \mathbb{H}, \quad t > t_0 \tag{7}$$

$$F(x) = -\begin{cases} (\|x\|^{-1} + \alpha\|x\|^2)x, & \text{if } x \neq \mathbf{0}, \\ B(1), & \text{if } x = \mathbf{0}, \end{cases} \quad \alpha \geq 0$$

and $A:\mathcal{D}(A)\subset\mathbb{H}\to\mathbb{H}$ is a closed densely defined dissipative operator $\langle Ax,x\rangle\leq 0,\ x\in\mathcal{D}(A)$ on a real Hilbert space $\mathbb{H}$ with an inner product $\langle\cdot,\cdot\rangle$ and the norm $\|x\|=\sqrt{\langle x,x\rangle}, x\in\mathbb{H}$.

Existence of solution. Let us show that the considered evolution inclusion has a mild solution for any initial condition $x(0)=x_0\in\mathbb{H}$ and a classical solution for any initial condition $x(0)=x_0\in\mathcal{D}(A)\backslash\{\mathbf{0}\}$. Let us consider the evolution equation

$$\dot{x}=Ax-f(x)-\alpha\|x\|^2x,\qquad x\in\mathbb{H}\tag{8}$$

with $f(x)=\rho\left(\frac{\|x\|^4}{\delta^4}\right)\frac{x}{\|x\|}$, where $\delta>0$ and $\rho:[0,+\infty)\to[0,1]$ is the "cut-off" function given by

$$\rho(r)=\begin{cases}0 & \text{if}\quad r=0,\\ e^{-\frac{1}{r}(e-e^{-\frac{1}{1-r}})} & \text{if}\quad 0<r<1,\\ 1 & \text{if}\quad r>1.\end{cases}$$

The function ρ is smooth and $0=\rho'(0)=\rho'(r)$ for $r\geq 1$. Tthe norm is continuously Frechét differentiable on $\mathbb{H}\backslash\{\mathbf{0}\}$, $(D\|x\|)(h)=\frac{\langle h,x\rangle}{\|x\|},\quad x\in\mathbb{H}\backslash\{\mathbf{0}\},\quad h\in\mathbb{H},$. Hence, we derive the Frechét derivative $Df(x)\in\mathcal{L}(\mathbb{H},\mathbb{H})$ of f

$$(Df(x))(h)=\frac{4\langle h,x\rangle\rho'\left(\frac{\|x\|^4}{\delta^4}\right)\|x\|^2}{\delta^4}x-\frac{\langle h,x\rangle\rho\left(\frac{\|x\|^4}{\delta^4}\right)}{\|x\|^3}x+\frac{\rho\left(\frac{\|x\|^4}{\delta^4}\right)}{\|x\|}h.$$

Since $g:\mathbb{H}\to[0,+\infty)$ defined as $g(x):=\|x\|^{-3}\rho\left(\frac{\|x\|^4}{\delta^4}\right)$ is continuous on $\mathbb{H}$, $g(\mathbf{0})=\mathbf{0}$ then and Df is continuous with respect to x in the operator norm. According to [17, Theorem 1.5., page 187] we conclude that the considered evolution equation (8) has classical solutions for any initial condition $x_0\in\mathcal{D}(A)$. Notice that in this case any mild solution with $x_0\in\mathbb{H}$ is uniform (on a compact interval of time) limit of classical solutions (see e.g. [17, Theorem 2.7, page 108]). Moreover, these classical solutions satisfy (7) for $\|x\|\geq\delta$. Tending $\delta\to 0$ we prove existence of classical solutions of (7) on $\mathcal{D}(A)\backslash\{\mathbf{0}\}$ and mild solutions on $\mathbb{H}\backslash\{\mathbf{0}\}$. Since $\mathbf{0}\in F(\mathbf{0})$ then in the view of Definition 1 the evolution inclusion (7) has the trivial solution $x(t)=\mathbf{0}$ for the initial condition $x(0)=\mathbf{0}$, i.e. (7) has solutions for all initial conditions in $\mathbb{H}$.

Stability Analysis. The functional $V:\mathbb{H}\to[0,+\infty)$ given by $V(x)=\|x\|$ is Frechét differentiable on $\mathbb{H}\backslash\{\mathbf{0}\}$ and globally proper. If x is a classical solution of (7) with $x_0\in D(A)\backslash\{\mathbf{0}\}$ then

$$\begin{aligned}\dot{V}(x(t))&=DV(x(t))\big(Ax(t)-\big(\|x(t)\|^{-1}+\alpha\|x(t)\|^2\big)x(t)\big)\\&=\frac{1}{\|x(t)\|}\left\langle Ax(t)-\frac{x(t)}{\|x(t)\|}-\alpha\|x(t)\|^2x(t),x(t)\right\rangle\\&\leq-1-\alpha\|x(t)\|^2=-1-\alpha V(x(t))^2,\end{aligned}$$

as long as $x(t)\neq\mathbf{0}$. Again any mild solution $\tilde{x}$ is a uniform (on compact intervals of time) limit of classical solutions $x_i\to\tilde{x}$ as $i\to+\infty$ (see e.g. [17, Theorem 2.7, page 108]). Since V is continuously Frechét differentiable then any $D_{\{h_n\}}V(\tilde{x}(t))$ is a limit of $\dot{V}(x_i(t))$ with some properly selected sequence $\{x_i\}$. Taking into account, $\dot{V}(x_i(t))\leq-1-\alpha V(x_i(t))^2$ we conclude $D_{\mathbb{K}}(\tilde{x}(t))\leq-1-\alpha V(\tilde{x}(t))^2$.

Therefore, according to Corollaries 2 and 3 the evolution inclusion (7) is globally uniformly finite-time stable for $\alpha\geq 0$ and globally fixed-time stable for $\alpha>0$.

B. Settling-time function with zero lower estimate

Let us consider the evolution inclusion (7) with $\alpha=0$, $\mathbb{H}=H^0((-1,1),\mathbb{R})$, $A=\frac{\partial^2}{\partial z^2}, D(A)=H^1_0((-1,1),\mathbb{R})\cap H^2((-1,1),\mathbb{R})$. In this case the model (7) describes some heat system. The term F can be tread as distributed feedback control [28], [13]. Recall that the inner product in $H^0((-1,1),\mathbb{R})=L^2((-1,1),\mathbb{R})$ is given by $\langle x,y\rangle=\int_{-1}^1 x(s)y(s)\,ds$, the eigenvalues of the operator A are $\lambda_i=i^2\pi^2, i=1,2,...$ and the eigenfunctions are given by $\phi_i(z)=\sin(i\phi), i=1,2,....$ define a basis in H^0. In this case, any classical solution x of (7) on $\mathbb{H}\backslash\{\mathbf{0}\}$ can be represented as

$$(x(t))(z)=\sum_{i=1}^{+\infty}x_i(t)\phi(z),$$

where $x_i:\mathbb{R}\to\mathbb{R}$ are some continuous functions. Taking into account $\langle\phi_i,\phi_j\rangle=0$ for $i\neq j$ and $\langle\phi_i,\phi_i\rangle=1$ we derive $\langle\phi_i,Ax(t)\rangle=-\lambda_i x_i$ and $\langle\phi_i,F(x(t))\rangle=-\frac{x_i(t)}{\sqrt{\langle x(t),x(t)\rangle}}=-\frac{x_i(t)}{\sqrt{\sum_{i=1}^{+\infty}x_i^2(t)}}$ al long as $x(t)\neq 0$. Hence, we have

$$\dot{x}_i(t)=-\lambda_i x_i(t)-\frac{x_i}{\sqrt{\sum_{i=1}^{+\infty}x_i^2}}.$$

Let us consider now the sequence of initial conditions $x(0)=r\phi_i$, where $r>0$ is an arbitrary constant. On the one hand, such a selection of the initial conditions guarantees that $x_j(t)=0$ for all $t\geq 0$, $j\neq i$ and $\dot{x}_i(t)=-\lambda_i x_i(t)-\frac{x_i(t)}{|x_i(t)|}$, $x_i(0)=1$. Hence, we derive $x_i(t)\to 0\quad$ as $\quad t\to\frac{\ln(r\lambda_i+1)}{\lambda_i}$, i.e. the settling time T tends to zero as $i\to+\infty$. On the other hand, $\|x(0)\|$ remains constant for any $i>1$, since $\|\phi_i\|=1$. Therefore, for any $r>0$ we derive $\inf_{\|x\|=r}T(x)=0$, i.e. the settling-time function cannot be estimated from below by a class $\mathcal{K}$ function. This is a specific feature of infinite dimensional finite-time stable systems with unbounded operators. Such a situation is impossible in the finite dimensional case (see [7]).

VII. Comments and Acknowledgments

In [1], Theorem 1 was claimed to be true for the time-invariant V. However, its proof contains an error in the necessity part. It works only for time-invariant case of (1) even if $\mathbb{B}=\mathbb{R}^n$. The author would like to thank an anonymous reviewer of the paper [29] who pointed out this error. Notice that the error has impacted some other Lyapunov theorems presented in [1] and utilized in [30]. The necessity parts in the mentioned theorems are true only for the time-invariant case $F(t,x)=F(x)$. This fact does not impact the results of [30] since the converse Lyapunov theorems are utilized there only for autonomous systems.

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