Generalized Lyapunov Exponents of Homogeneous Systems

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***Abstract*— The paper deals the method of Lyapunov exponents for a class of a generalized homogeneous systems. Homogeneous systems may have some sup-exponential and super-exponential grows. In this case, the method of Lyapunov exponents becomes non-informative, e.g. all Lyapunov exponents may equal to zero but the system is globally uniformly asymptotically stable. In this paper we propose an approach which allows us to analyze a behavior of such homogeneous systems by means of the method of Lyapunov exponents.**

I. Introduction and Problem Statement

The concept of stability introduced by A.M. Lyapunov [1] considers some *nominal motion* $x^*(t,x_0)$ of a dynamic system

$$\dot{x} = f(x), \quad f:\mathbb{R}^n \to \mathbb{R}^n \tag{1}$$

and studies a behavior of the system for perturbed initial condition $x_0 + \Delta x_0$. If small perturbations always imply small deviations of the perturbed motion $x_{t_0,x_0+\Delta x_0}(t)$ from $x^*_{t_0,x_0}(t)$ then the *nominal motion* is called stable. The conventional approach to the corresponding analysis is based on linearizion of the system along the trajectory

$$\dot{e} = A(t)e + g(t,e), \tag{2}$$

where $e(t) = x(t) - x^*(t,x_0)$, $A(t) = f'(x^*(t,x_0))$ and $g = (g_1,...,g_n)^\top$ is given by

$$g_i(t,e) = e^\top\left(\int_0^1\int_0^1 s f_i''(x^*(t,x_0)+\tau s e)\,ds\,d\tau\right)e, \quad i=1,...,n.$$

The similar problem appears for observer design for nonlinear systems (see e.g. [2]).

The method of Lyapunov exponents [3], [4] is frequently utilized for the stability analysis of the latter system. This method allows stability analysis provided that the system satisfies some *regularity property*(see e.g. [3], [4]).

Homogeneity is a kind of symmetry when an object (a function, a vector field, a set etc) remains invariant in a certain sense with respect to a class of transformations called dilations [5], [6], [7], [8]. For example, if $\exists \nu \in \mathbb{R}$: $f(e^s x) = e^{\nu s} f(x)$ for all $s \in \mathbb{R}$ and for all x the the function f is called *standard homogeneous* (or homogeneous in Euler's sense). All linear and a lot of nonlinear models of mathematical physics are homogeneous in a generalized sense [9]. Homogeneous models of dynamical systems also appears as local approximation of nonlinear systems if linearization is not informative or simply impossible [10], [11], e.g. asymptotic stability of the zero solution of the system $\dot{x} = -x^3, x \in \mathbb{R}$ cannot be studied by means of linearization at zero, which has the form $\dot{x} = 0$. Below we also show that the conventional analysis based on Lyapunov exponents also does not allow us to study stability of this system.

The aim of this paper is to expand/adapt the method Lyapunov exponents to a class of homogeneous systems.

Notation. $\mathbb{R}$, $\mathbb{R}_+$,$\lambda_{\min}$, $\lambda_{\max}$, $\overline{co}$, sign; a class $\mathcal{K}$ function; $\lfloor A \rfloor := \inf_{x\neq \mathbf{0}} \frac{\|Ax\|}{\|x\|}$,$\|A\| := \sup_{x\neq\mathbf{0}} \frac{\|Ax\|}{\|x\|}$, where $A \in \mathbb{R}^{n\times n}$, $x \in \mathbb{R}^n$.

II. Generalized Homogeneity

A. Linear Dilations and Canonical Homogeneous Norm

Different generalization of standard (Euler's) homogeneity can be found in the literature [5], [12], [13], [14]. The generalized homogeneity studied in [9], [15], [16], [17] deals with the group of linear transformations (*linear dilations*).

Definition 1: [16] *A map* $\mathbf{d} : \mathbb{R} \to \mathbb{R}^{n\times n}$ *is called* ***dilation*** *in* $\mathbb{R}^n$ *if it satisfies*

- ***Group property:*** $\mathbf{d}(0) = I_n$ *and* $\mathbf{d}(t+s) = \mathbf{d}(t)\mathbf{d}(s) = \mathbf{d}(s)\mathbf{d}(t)$ *for all* $t,s \in \mathbb{R}$*;*
- ***Continuity property:*** $\mathbf{d}$ *is a continuous map, i.e.,*

$$\forall t\in\mathbb{R}, \forall\varepsilon>0, \exists\delta>0 : |s\text{-}t|<\delta \Rightarrow \|\mathbf{d}(s)\text{-}\mathbf{d}(t)\| \le \varepsilon;$$

- ***Limit property:*** $\lim\limits_{s\to-\infty}\|\mathbf{d}(s)x\| = 0$ *and* $\lim\limits_{s\to+\infty}\|\mathbf{d}(s)x\| = +\infty$ *uniformly on the unit sphere*

$$S := \{x \in \mathbb{R}^n : \|x\| = 1\}.$$

The dilation $\mathbf{d}$ is a continuous group of invertible linear maps $\mathbf{d}(s) \in \mathbb{R}^{n\times n}$, $\mathbf{d}(-s) = [\mathbf{d}(s)]^{-1}$. The matrix $G_{\mathbf{d}} \in \mathbb{R}^{n\times n}$

$$G_{\mathbf{d}} = \lim_{s\to 0} \tfrac{\mathbf{d}(s)-I}{s}$$

is known [18, Chapter 1] as the **generator** of the group $\mathbf{d}$. It satisfies the following properties

$$\tfrac{d\,\mathbf{d}(s)}{ds} = G_{\mathbf{d}}\mathbf{d}(s) \quad \text{and} \quad \mathbf{d}(s) = e^{G_{\mathbf{d}}s} = \sum_{i=0}^{+\infty} \tfrac{s^i G_{\mathbf{d}}^i}{i!}, \ s \in \mathbb{R}. \tag{3}$$

The latter implies $\mathbf{d}(s_1) - \mathbf{d}(s_2) = G_{\mathbf{d}} \int_{s_2}^{s_1} \mathbf{d}(s)ds$, $s_1, s_2 \in \mathbb{R}$. The most popular dilations in $\mathbb{R}^n$ are [19], [20]

- *uniform (or standard) dilation* (L. Euler 17th century) :

$$\mathbf{d}_1(s) = e^s I_n, \quad s\in\mathbb{R},$$

- *weighted dilation* (Zubov 1958, [5]):

$$\mathbf{d}_2(s) = \begin{pmatrix} e^{r_1 s} & 0 & \dots & 0 \\ 0 & e^{r_2 s} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & e^{r_n s}\end{pmatrix}, \ s\in\mathbb{R}, \ r_i > 0, \ i=1,...,n$$

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They satisfy Definition 1 with $G_{\mathbf{d}_1}=I_n$ and $G_{\mathbf{d}_2}=\mathrm{diag}\{r_i\}$, respectively. In fact, any anti-Hurwitz[1] matrix $G_{\mathbf{d}} \in \mathbb{R}^{n\times n}$ defines a dilation $\mathbf{d}(s) = e^{G_{\mathbf{d}} s}$ in $\mathbb{R}^n$. The geometric dilation studied in [14], [13] is more general since it allows the dilation group to be nonlinear.

Definition 2: The dilation $\mathbf{d}$ *is said to be* ***strictly monotone*** *if there exists* $\beta > 0$ *such that* $\|\mathbf{d}(s)\| \leq e^{\beta s}$ *as* $s<0$.
Obviously, the monotonicity of a dilation may depend on the norm $\|\cdot\|$ in $\mathbb{R}^n$.

Theorem 1: [16] *Let* $\mathbf{d}$ *be a dilation in* $\mathbb{R}^n$*, then*

1) *all eigenvalues* λ_i *of the matrix* $G_{\mathbf{d}}$ *are placed in the right complex half-plane, i.e.,* $\Re(\lambda_i) > 0, i = 1, 2, ..., n$;
2) *there exists a matrix* $P \in \mathbb{R}^{n\times n}$ *such that*

$$PG_{\mathbf{d}} + G_{\mathbf{d}}^\top P \succ 0, \quad P = P^\top \succ 0; \tag{4}$$

3) *the dilation* $\mathbf{d}$ *is strictly monotone with respect to the weighted Euclidean norm* $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$ *induced by the inner product* $\langle x, z\rangle = x^\top P z$ *with* P *satisfying* (4).

Moreover,

$$\begin{array}{l} e^{\alpha s} \leq \lfloor \mathbf{d}(s) \rfloor \leq \|\mathbf{d}(s)\| \leq e^{\beta s} \quad \text{if} \quad s \leq 0, \\ e^{\beta s} \leq \lfloor \mathbf{d}(s) \rfloor \leq \|\mathbf{d}(s)\| \leq e^{\alpha s} \quad \text{if} \quad s \geq 0, \end{array} \tag{5}$$

where $\alpha = \frac{1}{2}\lambda_{\max}\left(P^{\frac{1}{2}} G_{\mathbf{d}} P^{-\frac{1}{2}} + P^{-\frac{1}{2}} G_{\mathbf{d}}^\top P^{\frac{1}{2}}\right)$ *and* $\beta = \frac{1}{2}\lambda_{\min}\left(P^{\frac{1}{2}} G_{\mathbf{d}} P^{-\frac{1}{2}} + P^{-\frac{1}{2}} G_{\mathbf{d}}^\top P^{\frac{1}{2}}\right)$.

The latter theorem proves that any dilation $\mathbf{d}$ is strictly monotone if $\mathbb{R}^n$ is equipped with the norm $\|x\|=\sqrt{x^\top P x}$, provided that the matrix $P\succ 0$ satisfies (4).

Definition 3: A continuous function $p : \mathbb{R}^n \to \mathbb{R}_+$ *is said to be* $\mathbf{d}$*-homogeneous norm if* $p(x) \to 0$ *as* $x \to \mathbf{0}$ *and* $p(\mathbf{d}(s)x) = e^s p(x) > 0$ *for* $x \in \mathbb{R}^n\backslash\{\mathbf{0}\}$ *and* $s \in \mathbb{R}$.
Obviously, the $\mathbf{d}$-homogeneous norm is neither a norm nor semi-norm in the general case, since the triangle inequality may not hold. However, many authors (see [11], [21], [17] and references therein) call a function satisfying the latter definition a homogeneous norm. We follow this tradition. *The* ***canonical homogeneous norm*** $\|\cdot\|_{\mathbf{d}} : \mathbb{R}^n \to \mathbb{R}_+$ *is defined as*

$$\|x\|_{\mathbf{d}} = e^{s_x} \text{ where } s_x \in \mathbb{R} : \|\mathbf{d}(-s_x)x\| = 1. \tag{6}$$

The map $\|\cdot\|_{\mathbf{d}} : \mathbb{R}^n \to [0, +\infty)$ is well defined and single-valued for monotone dilations [17]. In [22] such a homogeneous norm was called canonical because it is induced by the (canonical) norm in $\mathbb{R}^n$. Notice that

$$\lfloor \mathbf{d}(\ln \|x\|_{\mathbf{d}}) \rfloor \leq \|x\| \leq \|\mathbf{d}(\ln \|x\|_{\mathbf{d}})\| \quad \text{ for } \quad x \in \mathbb{R}^n,$$

and, due to (5), $\|\cdot\|_{\mathbf{d}}$ is continuous at zero.

Proposition 1 ([16]): If $\mathbf{d}$ *is strictly monotone then*

- *the canonical homogeneous norm* $\|\cdot\|_{\mathbf{d}}$ *is Lipschitz continuous on* $\mathbb{R}^n\backslash\{\mathbf{0}\}$*;*
- *if the norm* $\|\cdot\|$ *is smooth outside the origin then the homogeneous norm* $\|\cdot\|_{\mathbf{d}}$ *is also smooth outside the origin,* $\frac{d\|\mathbf{d}(-s)x\|}{ds} < 0$ *if* $s \in \mathbb{R}$, $x \in \mathbb{R}^n\backslash\{\mathbf{0}\}$ *and*

$$\frac{\partial \|x\|_{\mathbf{d}}}{\partial x} = \left.\frac{\|x\|_{\mathbf{d}} \left.\frac{\partial \|z\|}{\partial z}\right|_{z=\mathbf{d}(-s)x}}{\left.\frac{\partial \|z\|}{\partial z}\right|_{z=\mathbf{d}(-s)x} G_{\mathbf{d}} \mathbf{d}(-s)x}\right|_{s=\ln \|x\|_{\mathbf{d}}} \tag{7}$$

[1]The matrix $G_{\mathbf{d}} \in \mathbb{R}^n$ is anti-Hurwitz if $-G_{\mathbf{d}}$ is Hurwitz.

Below we use the notation $\|\cdot\|_{\mathbf{d}}$ only for the canonical homogeneous norm induced by the weighted Euclidean norm $\|x\| = \sqrt{x^\top P x}$ with a matrix $P \succ 0$ satisfying (4). The unit sphere S is defined using the same norm. In this case the formula (7) becomes

$$\frac{\partial \|x\|_{\mathbf{d}}}{\partial x} = \|x\|_{\mathbf{d}} \frac{x^\top \mathbf{d}(-\ln \|x\|_{\mathbf{d}}) P \mathbf{d}(-\ln \|x\|_{\mathbf{d}})}{x^\top \mathbf{d}(-\ln \|x\|_{\mathbf{d}}) P G_{\mathbf{d}} \mathbf{d}(-\ln \|x\|_{\mathbf{d}}) x}.$$

B. Homogeneous vector-fields and flows

Vector fields, which are homogeneous with respect to dilation $\mathbf{d}$, have many properties useful for control design and state estimation of linear and nonlinear plants as well as for analysis of convergence rates [14], [23], [8], [24].

Definition 4: [16] *A vector field* $f : \mathbb{R}^n \to \mathbb{R}^n$ *(resp. a function* $h : \mathbb{R}^n \to \mathbb{R}$*) is said to be* $\mathbf{d}$*-homogeneous if there exists* $\nu \in \mathbb{R}$

$$f(\mathbf{d}(s)x) = e^{\nu s}\mathbf{d}(s)f(x), \ \forall x \in \mathbb{R}^n\backslash\{\mathbf{0}\}, \ \forall s \in \mathbb{R}. \tag{8}$$

$$\text{(resp. } h(\mathbf{d}(s)x) = e^{\nu s} h(x), \ \forall x \in \mathbb{R}^n\backslash\{\mathbf{0}\}, \ \forall s \in \mathbb{R}.) \tag{9}$$

The number $\nu \in \mathbb{R}$ is called the homogeneity degree of f (resp. h). *Let* $\mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$ *(resp.* $\mathbb{H}_{\mathbf{d}}(\mathbb{R}^n)$*) be the set of vector fields* $\mathbb{R}^n \to \mathbb{R}^n$ *(resp. functions* $\mathbb{R}^n \to \mathbb{R}$*) satisfying the identity* (8) *(resp.* (9)*), which are* ***continuous on*** $\mathbb{R}^n\backslash\{\mathbf{0}\}$*. Let* $\deg_{\mathbb{F}_{\mathbf{d}}}(f)$ *(resp.* $\deg_{\mathbb{H}_{\mathbf{d}}}(h)$*) denote the homogeneity degree of* $f \in \mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$ *(resp.* $h \in \mathbb{H}_{\mathbf{d}}(\mathbb{R}^n)$*).*

The homogeneity allows local properties (like smoothness) of vector fields (functions) to be extended globally [5], [6].

Lemma 1 ([16]): The vector field $f \in \mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$ is Lipschitz continuous (smooth) on $\mathbb{R}^n\backslash\{\mathbf{0}\}$ if and only if it satisfies Lipschitz condition (smooth) on the unit sphere $S = \{x \in \mathbb{R}^n : x^\top P x = 1\}$, where P satisfies (4).
The next results is a corollary of Euler's Homogeneous Function Theorem.

Lemma 2: If $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable on S and $\tilde{f}(e^s x) = \tilde{f}(x)$ for all $x \in \mathbb{R}^n$ and all $s \in \mathbb{R}$ then

$$\tilde{f}'(x)x = 0, \qquad \forall x \in \mathbb{R}^n\backslash\{\mathbf{0}\},$$

$$e^{-s}\tilde{f}'(x) = \tilde{f}'\left(e^s x\right), \qquad \forall x \in \mathbb{R}^n\backslash\{\mathbf{0}\}, \forall s \in \mathbb{R}.$$

The result follows, for example, from [16, Corollary 2].

Homogeneity may simplify the analysis of differential equations. The most important property of $\mathbf{d}$-homogeneous systems is the symmetry of solutions [5], [13], [7],[21], [8].

Theorem 2 ([16]): If $\varphi(\cdot, x_0) : [0, T) \to \mathbb{R}^n$ is a solution to

$$\dot{x} = f(x), \quad f \in \mathbb{F}_{\mathbf{d}}(\mathbb{R}^n) \tag{10}$$

with the initial condition $x(0) = x_0 \in \mathbb{R}^n$, then $\varphi(\cdot, \mathbf{d}(s)x_0) : [0, e^{-\nu s}T) \to \mathbb{R}^n$ defined as

$$\varphi(t, \mathbf{d}(s)x_0) = \mathbf{d}(s)\varphi(te^{\nu s}, x), \quad s \in \mathbb{R}$$

is a solution to (10) with the initial condition $x(0) = \mathbf{d}(s)x_0$, where $\nu = \deg_{\mathbb{F}_{\mathbf{d}}}(f)$.
The next result shows that any generalized homogeneous system is equivalent to a standard homogeneous one.

Theorem 3: The systems (10) and

$$\dot{z} = \|z\|^{1+\deg_{\mathbf{d}}(f)}\tilde{f}(z), \tag{11}$$

$$\tilde{f}(z)=\left(\frac{(I_n-G_{\mathbf{d}})z^\top zP}{z^\top PG_{\mathbf{d}}z}+I_n\right)f\left(\frac{z}{\|z\|}\right)$$

with $\|z\|=\sqrt{z^\top Pz}$ and P satisfying (4), are *topologically equivalent* on $\mathbb{R}^n\backslash\{\mathbf{0}\}$. Moreover, these systems are topologically equivalent on $\mathbb{R}^n$ if the origin of one of these systems is Lyapunov stable.

Proof. Topological equivalence means that there exists a homeomorphism[2] between solution sets of two systems. In [16] its is shown that the change of variables

$$z=\Phi(x)$$

where $\Phi:\mathbb{R}^n\to\mathbb{R}^n$ is given by

$$\Phi(x)=\|x\|_{\mathbf{d}}\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x,\quad x\in\mathbb{R}^n, \tag{12}$$

transforms the system (10) to the system (11). Its inverse Φ^{-1} is as follows

$$\Phi^{-1}(z)=\mathbf{d}(\ln\|z\|)\frac{z}{\|z\|},\quad z\in\mathbb{R}^n.$$

Proposition 1 implies that Φ defined by (12) is a diffeomorphism (i.e. continuously differentiable homeomorphism) on $\mathbb{R}^n\backslash\{\mathbf{0}\}$. This means that a set of solutions of the system (10) defined on $\mathbb{R}^n\backslash\{\mathbf{0}\}$ homeomorphic to the set of solutions of the system (11) defined on $\mathbb{R}^n\backslash\{\mathbf{0}\}$. The latter means that if one of the systems is Lyapunov stable then the another one is also Lyapunov stable. Since Lyapunov stability implies uniqueness of the zero solution, then the stable systems (10) and (11) are topologically equivalent on $\mathbb{R}^n$. ■

The next proposition shows that if the vector-field f satisfies a Lipschitz condition on the unit sphere, then certain equivalence can be established between homogeneous and globally Lipschitz system.

Lemma 3: Let us consider the system

$$\dot{y}=\|y\|\tilde{f}(y), \tag{13}$$

where $\tilde{f}$ is defined by the formula (11), and let $\tilde{f}$ satisfy Lipschitz condition of the unit sphere S. Then the system (13) has a unique solution $y(\cdot,y_0):[0,+\infty)\to\mathbb{R}^n$ for each $y_0\in\mathbb{R}^n$ and for any $y_0\neq\mathbf{0}$ one has

$$z\left(\int_0^t\frac{ds}{\|y(s,y_0)\|^{\deg_{\mathbb{F}_{\mathbf{d}}}f}},y_0\right)=y(t,y_0),\ t>0$$

$$z\left(\tau,y_0\right)=y\left(\int_0^\tau\|z(s,y_0)\|^{\deg_{\mathbb{F}_{\mathbf{d}}}f}\,ds,y_0\right),\ \tau\in[0,\tau^{\max}(y_0)),$$

where $z(\cdot,y_0):[0,T(y_0))\to\mathbb{R}^n$ is a unique solution of (11) with the initial condition $z(0)=y_0$ and

$$T(y_0)=\int_0^{+\infty}\|y(s,y_0)\|^{-\deg_{\mathbb{F}_{\mathbf{d}}}f}ds.$$

Proof. Notice that by construction

$$\sup_{y\in\mathbb{R}^n}\|\tilde{f}(y)\|=\overline{K}<+\infty.$$

Since $\tilde{f}$ satisfies Lipschitz condition on the unit sphere S then due to Lemma 1, the function $y\to\|y\|^{1+\deg_{\mathbb{F}_{\mathbf{d}}}f}\tilde{f}(y)$ is Lipschitz continuous on $\mathbb{R}^n\backslash\{\mathbf{0}\}$ and the function $y\to\|y\|\tilde{f}(y)$ is Lipschitz continuous on $\mathbb{R}^n$. This means that the system (11) has unique solutions on $\mathbb{R}^n\backslash\{\mathbf{0}\}$ and the system (13) has unique solutions $y(t,y_0)$ for all $y_0\in\mathbb{R}^n$. Moreover, $0<\|y(t,y_0)\|<+\infty$ for all $t>0$ provided that $y_0\neq 0$. Indeed, finite-time blow up is impossible because $\|\tilde{f}\|\leq\overline{K}$ but Lipschitz continuity at the origin does not allow finite-time extinction.

Let us denote $\tau=\int_0^t\|y(s,y_0)\|^{-\deg_{\mathbb{F}_{\mathbf{d}}}(f)}ds$ and show that $z(\tau,y_0)$ defined above with $\tau\in[0,\tau^{\max}(y_0))$ is a unique solution of the system (10) with the initial condition $z(0)=y_0$. Indeed,

$$\|z(\tau,y_0)\|\tilde{f}(z(\tau,y_0))=\|y(t,y_0)\|\tilde{f}(y(t,y_0))=\frac{dy(t,y_0)}{dt}=$$

$$\frac{d}{dt}z\left(\int_0^t\|y(s,y_0)\|^{-\deg_{\mathbb{F}_{\mathbf{d}}}f}ds,y_0\right)=\frac{dz(\tau)}{d\tau}\frac{d\tau}{dt}=$$

$$\frac{dz(\tau)}{d\tau}\|y(t,y_0)\|^{-\deg_{\mathbb{F}_{\mathbf{d}}}f}=\frac{dz(\tau)}{d\tau}\|z(\tau,y_0)\|^{-\deg_{\mathbb{F}_{\mathbf{d}}}f}.$$

The second claimed identity can be proven similarly. ■

The latter lemma guarantees that if some stability result is obtained for the system (13) then the similar result holds for the system (10) with the only difference is a convergence rate, e.g. the case $\deg_{\mathbb{F}_{\mathbf{d}}}f<0$ would correspond to a finite-time convergence. Indeed, the negative homogeneity is necessary and sufficient condition of finite-time stability of an asymptotically stable homogeneous system (see e.g. [16]). Below we use Lemma 3 in order to define an analog of Lyapunov exponents for homogeneous systems.

Finally, let us recall the following result, which is utilized in the later constructions.

Theorem 4 ([16]): The two claims are equivalent

1) *The origin of the system* (10) *is asymptotically stable.*

2) *For any matrix $P\in\mathbb{R}^{n\times n}$ satisfying* (4) *there exists a map $\Xi\in C^\infty(\mathbb{R}^n\backslash\{\mathbf{0}\},\mathbb{R}^{n\times n})$ such that $\det(\Xi(z))\neq 0$, $\frac{\partial\Xi(z)}{\partial z_i}z=0$, $\Xi(e^sz)=\Xi(z)$ for $z=(z_1,...,z_n)^\top\in\mathbb{R}^n\backslash\{\mathbf{0}\}$, $s\in\mathbb{R}$, $i=1,...,n$*

$$z^\top\Xi^\top(z)P\Xi(z)\tilde{f}(z)<0, \tag{14}$$

where $\tilde{f}$ is given by (11).

Notice that the properties of the matrix-valued function Ξ imply that $V:\mathbb{R}^n\to[0,+\infty)$ given by

$$V(z)=z^\top\Xi^\top(z)P\Xi(z)z$$

is a Lyapunov function for the system (11).

III. Method of Lyapunov Exponents for Homogeneous Systems

A. Lyapunov exponents

This approach was introduced by A.M. Lyapunov [1] and developed/generalized in later in many works, e.g. [3], [4].

Definition 5 ([3], page 17): Let $\mathbb{G}$ be a set of functions $[0,+\infty)\to\mathbb{R}^n$. An upper (lower) Lyapunov exponent of a function $x\in\mathbb{G}$ is a number $\chi(x)$ defined as follows

$$\chi^+(x)=\limsup_{t\to+\infty}\frac{1}{t}\ln\|x(t)\|, \tag{15}$$

[2] Homeomorphism is a continuous invertible mapping with a continuous inverse.

$$(\text{resp. } \chi^-(x) = \liminf_{t\to+\infty} \frac{1}{t}\ln\|x(t)\|). \tag{16}$$

For functionals χ^+ and χ^- one holds (see [3])

- if $\|x(t)\| \le \|y(t)\|$ for all $t>0$ then $\chi^{\pm}(x) \le \chi^{\pm}(y)$
- $\chi^{\pm}(\alpha x) = \chi^{\pm}(x)$ for each $x \in G$ and $\alpha \in \mathbb{R}\backslash\{0\}$;
- $\chi^+(x+y) \le \max\{\chi^+(x), \chi^+(y)\}$ for any $x, y \in G$;
- $\chi^+(xy) \le \chi^+(x) + \chi^+(y)$ for any $x, y \in G$;
- $\chi^{\pm}(0) = -\infty$ (*normalization property*).
- $\varepsilon > 0$, $\exists C(\varepsilon) > 0 : \|g(t)\| \le C(\varepsilon)e^{(\chi^+(g)+\varepsilon)t}$ for $t \ge 0$;
- $\forall \varepsilon > 0$ one has $\|g(t)\|e^{-(\chi^+(g)+\varepsilon)t} \to 0$ as $t \to +\infty$.

For example, if there exist $\overline{K} > 0$ such that the vector field $f : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ satisfies the inequality

$$\|f(t,x)\| \le \overline{K}\|x\| \quad \text{for all } x \in \mathbb{R}^n \tag{17}$$

for some $\overline{K} > 0$ and the system

$$\dot{x} = f(t,x)$$

has a solution $x(\cdot, x_0) : [0, +\infty) \to \mathbb{R}^n$ with any $x_0 \in \mathbb{R}^n$: $x(0) = x_0$ defined on whole $(0, +\infty)$, then $|\chi^+(x(\cdot, x_0))| \le \bar{K}$. If $\sup_{x(\cdot,x_0)} \chi^+(x(\cdot, x_0)) < 0$ then the origin of the system is globally exponentially attractive.

B. Lyapunov exponents of linear systems

Notice that for any linear system

$$\dot{x} = A(t)x$$

with $A(t) \in \mathbb{R}^{n\times n}$, attractivity implies stability, i.e negativeness of Lyapunov exponents of the linear time-varying system implies its exponential stability. *For linear systems*

$$\dot{x} = A(t)x, \quad t > 0 \tag{18}$$

a number of disjoint Lyapunov exponents is finite [3], [4]:

$$\{-\infty\} \le \chi_1^+ < \chi_2^+ < \ldots < \chi_s^+, \quad s \le n.$$

and the numbers χ_i^+ can be calculated or, at least, estimated directly from $A(t)$. For each $1 \le i \le s$, define

$$E_i = \{x_0 \in \mathbb{R}^n : \chi(x(\cdot, x_0)) \le \chi_i^+\}$$

Put $E_0 = \{\mathbf{0}\}$. It can be easily shown (see e.g. [4], page 9) that E_i is a linear subspace of $\mathbb{R}^n$, and

$$\{\mathbf{0}\} = E_0 \subset E_1 \subset \ldots \subset E_s = \mathbb{R}^n, \quad E_i \ne E_j \text{ if } i \ne j.$$

A collection $\mathbb{E} = \{E_i, i = 0, \ldots, s\}$ satisfying the latter property is called a *filtration* of $\mathbb{R}^n$ [4].

Definition 6: The values χ_i^+ with the multiplicities $\dim(E_i) - \dim(E_{i-1})$ define a *spectrum* $\sigma(A)$ of the system (18) (see [4] for more details).

Stability of Lyapunov exponents a linear system with respect to perturbation of its parameters is introduced by the following definition.

Definition 7: [25, page 136] *The characteristic exponents $\xi_i^+(A)$ of A are said to be structurally stable if for any $\varepsilon > 0$ the exists $\delta > 0$ such that* $\sup_{t>0}\|Q(t)\| < \delta$ *implies*

$$|\xi_i^+(A) - \xi_i^+(A+Q)| \le \varepsilon, \qquad i = 1, 2, \ldots, s \le n.$$

C. Lyapunov exponents of homogeneous systems

In many cases homogeneous systems behave similarly to linear (or, at least, Lipschitz) ones. We may expect that Lyapunov exponents of homogeneous system have some similar properties. The examples given below show that this is not true in the general case and some specific constructions are required in order to use the conventional methodology of Lyapunov exponents in the context of homogeneous systems.

Let $x(\cdot, x_0)$ be a solution of the system (10). Formally we can define the Lyapunov exponent for $x(\cdot, x_0)$ by a formula (15). However, we cannot guarantee that the corresponding number is finite, well-defined and provides some information about convergence/stability of the system. For example, the standard homogeneous Cauchy problem

$$\dot{x} = -x^3, \quad t > 0, \quad x(0) = x_0 \ne 0, \quad x \in \mathbb{R} \tag{19}$$

has the unique solution solution

$$x(t, x_0) = \tfrac{\operatorname{sign}(x_0)}{\sqrt{2(t + x_0^{-2}/2)}}.$$

Obviously, the origin of the system is globally asymptotically stable but $\chi^+(x(\cdot, x_0)) = \chi^-(x(\cdot, x_0)) = 0$. This happens because Lyapunov exponents "sense" only exponential convergence, while the convergence rate of this system is sub-exponential close to the origin and hyper exponential close to ∞. Similarly the unstable system

$$\dot{x} = x^{\frac{1}{3}}, \quad t > 0, \quad x(0) = x_0 \ne 0, \quad x \in \mathbb{R},$$

has the solution

$$x(t, x_0) = \left(\tfrac{2}{3}t + |x_0|\right)^{\frac{3}{2}} \operatorname{sign}(x_0)$$

and, again, $\chi^+(x(\cdot, x_0)) = \chi^-(x(\cdot, x_0)) = 0$. Therefore, the Lyapunov exponent given by (15) is well defined and informative only if $\deg_{\mathbb{F}_{\mathbf{d}}} f = 0$ (the case of exponential convergence/divergence). Some constructions are required in order to expand the method of Lyapunov exponents to homogeneous systems with $\deg_{\mathbb{F}_{\mathbf{d}}} f \ne 0$.

According to Theorem 3, the system (10) is equivalent to the system (11). Topological equivalence of systems implies a certain equivalence of Lyapunov exponents [3, Chapter 10]. Moreover, Lemma 3 establishes some relation between solutions of the systems (13) and (11). Since $\tilde{f}(\lambda z) = \tilde{f}(z)$ for any $\lambda > 0$ and any $z \in \mathbb{R}^n\backslash\{\mathbf{0}\}$ then

$$\|\tilde{f}(z)\| \le \overline{K} = \sup_{\|z\|=1} \tilde{f}(z)$$

and the inequality (17) holds for the system (13). Therefore, Lyapunov exponents for solutions of the system (13) are finite and belongs to the interval $[-\overline{K}, \overline{K}]$.

The key idea is to define Lyapunov exponents for the equivalent system (13).

For example, the system (19) is $\mathbf{d}$-homogeneous of degree 1 with $\mathbf{d}(s) = e^s, s \in \mathbb{R}$. The equivalent system (13) has the form

$$\dot{y} = -y, \quad t > 0, \quad y \in \mathbb{R}$$

and its Lyapunov exponents are $\chi^+(y(\cdot, y_0)) = \chi^-(y(\cdot, y_0)) = -1$ also characterize stability of the original (equivalent) system (19). This truck can also be utilized for multidimensional systems.

Theorem 5: Let $f \in \mathbb{F}_{\mathbf{d}}$ be a continuously differentiable on the unit sphere $S = \{x \in \mathbb{R}^n : \|x\| = 1\}$, $\|x\| = \sqrt{x^\top P x}$, where P satisfies (4). *The origin of the system* (10) *is globally uniformly asymptotically stable if and only if*

$$\sup_{y_0 \in S} \chi^+(y(\cdot, y_0)) < 0,$$

where $y(\cdot, y_0)$ denote a solution of the system (13) *with $y(0) = y_0$.*

Proof. *Sufficiency.* The system (13) is $\mathbf{d}$-homogeneous of degree zero with the dilation $\mathbf{d}(s) = e^s I_n$, $s \in \mathbb{R}$. Theorem 2 implies that

$$\chi^+(y(\cdot, e^s y_0)) = \chi^+(y(\cdot, y_0)), \quad \forall s \in \mathbb{R}.$$

Therefore, the condition $\chi^+(y(\cdot, y_0)) < 0$ for any $y_0 \in S$ implies $\chi^+(y(\cdot, y_0)) < 0$ for any $y_0 \in \mathbb{R}^n$, i.e. each trajectory of the system (13) converge to the origin exponentially. Theorem 3, Lemma 3 implies that the trajectories of the equivalent system (10) converge to the origin asymptotically. Finally, taking into account that asymptotic attractivity implies stability for homogeneous systems [8] we finish the proof.

Necessity. Equivalence between systems (13) and (10) is established above. We just need to demonstrate that asymptotic stability of (13) implies $\chi^+(y(\cdot, y_0)) < 0, \forall y_0 \in S$. For this purpose let us consider the Lyapunov function given by

$$V(y) = y^\top \Xi^\top(y) P \Xi(y) y,$$

where Ξ and P are defined in Theorem 4. Obviously, that

$$c_1 \|y\|^2 \leq V(y) \leq c_2 \|y\|^2, \quad y \in \mathbb{R}^n$$

where

$$c_1 = \inf_{y \in S} \lambda_{\min}\left(\Xi^\top(y) P \Xi(y)\right) > 0,$$
$$c_2 = \sup_{y \in S} \lambda_{\max}\left(\Xi^\top(y) P \Xi(y)\right) < +\infty,$$

where $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote minimal and maximal eigenvalues of a symmetric matrix. According to Theorem 4 for the system (13) we derive

$$\dot{V}(y) = \|y\| y^\top \Xi^\top(y) P \Xi(y) \tilde{f}(y) < 0.$$

Since Ξ and $\tilde{f}$ are continuous on S and $\Xi(e^s y) = \Xi(y)$, $\tilde{f}(e^s y) = \tilde{f}(y), s \in \mathbb{R}, y \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ then

$$c_3 = \inf_{y \in \mathbb{R}^n} -\tfrac{y^\top}{\|y\|} \Xi^\top(y) P \Xi(y) \tilde{f}(y) > 0$$

and

$$\dot{V}(y) \leq -c_3 \|y\|^2 \leq -\tfrac{c_3}{c_2} V(y).$$

Hence,

$$\|y(t, y_0)\|^2 \leq \tfrac{V(y(t, y_0))}{c_1} \leq \tfrac{e^{-\frac{c_3}{c_2} t} V(y_0)}{c_1} \leq e^{-\frac{c_3}{c_2} t} \tfrac{c_2 \|y_0\|^2}{c_1}.$$

The latter means that $\sup_{y_0 \in S} \chi^+(y(\cdot, y_0)) \leq -\frac{c_3}{2c_2}$. ■

The proven result implies the following conclusion. *Together with homogeneity degree the Lyapunov exponents of the system* (13) *completely characterize the asymptotic stability/instability and convergence rages of the original homogeneous system* (10).

The key question now : is it possible to calculate these Lyapunov exponents using solutions of the original system? The next proposition positively answers this question.

Proposition 2: Let $f \in \mathbb{F}_{\mathbf{d}}$ be continuously differentiable on the unit sphere $S = \{x \in \mathbb{R}^n : \|x\| = 1\}$, $\|x\| = \sqrt{x^\top P x}$, where P satisfies (4). *Let $x(\cdot, x_0)$ and $y(\cdot, y_0)$ be solutions of the systems* (10) *and* (13) *with initial conditions $x(0) = x_0 \neq \mathbf{0}$, $y(0) = y_0 := \Phi(x_0)$, respectively. Then*

$$\xi^+(y(\cdot, y_0)) = \limsup_{\tau \to T(x_0)} \frac{\ln \|x(\tau, x_0)\|_{\mathbf{d}}}{\int_0^\tau \|x(\delta, x_0)\|_{\mathbf{d}}^{\deg_{\mathbb{F}_{\mathbf{d}}}} d\delta},$$

where $T(x_0) = +\infty$ if $0 < \|x(\tau, x_0)\|_{\mathbf{d}} < +\infty$ for $\tau > 0$ or $T(x_0) < +\infty$ is the first instant of time such that

$$\lim_{\tau \to T(x_0)} \|x(\tau, x_0)\|_{\mathbf{d}} = 0 \quad or \quad \lim_{\tau \to T(x_0)} \|x(\tau, x_0)\|_{\mathbf{d}} = +\infty\}.$$

The proof immediately follows from Lemma 3, the identity $\|x(\tau, x_0)\|_{\mathbf{d}} = \|z(\tau, \Phi(x_0))\|$ and the definition of the Lyapunov exponent, where Φ is, as before, given by (12).

D. "Linearization" of homogeneous systems around a trajectory

In some cases it is important to investigate a stability of a concrete (e.g. periodic) solution $x^*(t)$ of a nonlinear system (10). The usual approach in this case is a linearizion of the system around the corresponding trajectory (see introduction) combined with an analysis of negativeness of Lyapunov exponents for A in (2). In the previous section we have discovered that this conventional approach could be too conservative.

Theorem 6: Let $f \in \mathbb{F}_{\mathbf{d}}$ be twice continuously differentiable on the unit sphere $S = \{x \in \mathbb{R}^n : \|x\| = 1\}$, $\|x\| = \sqrt{x^\top P x}$, where P satisfies (4). Let a matrix-valued function $A : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ be defined as follows

$$\tilde{A}(y) = \tfrac{(\deg_{\mathbb{F}_{\mathbf{d}}} f + 1) \tilde{f}(y) y^\top P}{\|y\|} + \tilde{f}'\left(\tfrac{y}{\|y\|}\right), \quad y \in \mathbb{R}^n, \tag{20}$$

where $\tilde{f}$ is given by (11).

Let $x(\cdot, x^*)$ and $x(\cdot, x_0)$ be solutions of (10) with $x^* \neq \mathbf{0}$, $x_0 \neq 0$ and $y(\cdot, y^*)$ and $y(\cdot, y_0)$ be solutions of the system (13) with $y^* = \Phi(x^*)$ and $y_0 = \Phi(x_0)$. Then the difference

$$e(t) = y(t, y_0) - y(t, x^*) =$$
$$\Phi\left(x\left(\int_0^t \tfrac{ds}{\|y(s, y_0)\|^{\deg_{\mathbb{F}_{\mathbf{d}}} f}}, x_0\right)\right) - \Phi\left(x\left(\int_0^t \tfrac{ds}{\|y(s, y^*)\|^{\deg_{\mathbb{F}_{\mathbf{d}}} f}}, x^*\right)\right),$$

$x_0 \in \mathbb{R}^n$ satisfies the equation

$$\dot{e} = \tilde{A}(y(t, y^*)) e + g(y(t, y^*), e), \tag{21}$$

where $g = (g_1, .., g_n)^\top$ is given by

$$g_i(y(t, y^*), e) = e^\top \left(\int_0^1 \int_0^1 s F_i''(y(t, y^*) + \tau s e) \, ds \, d\tau\right) e,$$

and $F_i(y) = \|y\| \tilde{f}_i(y)$ with $i = 1, ..., n$, provided that

$$\mathbf{0} \notin \overline{co}\{y(t, y^*)), y(t, y_0)\}, \quad \forall t > 0. \tag{22}$$

Proof. Since f is twice continuously differentiable on S then F is twice continuously differentiable on $\mathbb{R}^n\backslash\{\mathbf{0}\}$ and continuous on $\mathbb{R}^n$. According to Lemma 3 the functions $y(\cdot, y^*)$ and $y(\cdot, y_0)$ satisfy

$$\dot{y} = F(y).$$

Moreover, it is easy to see that $\tilde{A}(y) = F'(y)$. Hence, the error e satisfy (21) provided that $y(t, y^*)+\tau se(t) \neq \mathbf{0}$ for all $t > 0$ and all $s, \tau \in [0, 1]$. The origin is excluded because it is the only possible discontinuity point of F''. ■

If the maximum characteristic exponent from the spectrum (see Definition 6) of $\tilde{A}(y(t, y^*))$ is negative then under some additional restrictions [4] a local exponential stability of the solution $y(\cdot, x^*)$ can be proven. Indeed, the matrix-valued function $\tilde{A}$ is globally bounded, so the Lyapunov exponents of the system (21) with $g = 0$ are finite and $g = O(\|e\|^2)$ for all e from a sufficiently small neighborhood of $\mathbf{0}$.

Corollary 1: Let all conditions of Theorem 6 hold and

$$\chi_{\max} := \max_{\chi\in\sigma\left(\tilde{A}(y(s,y^*))\right),} \chi < 0,$$

where σ denotes the spectrum of a time-varying matrix (see Definition 6). If Lyapunov exponents of $\tilde{A}\left(y(s, y^*)\right))$ are structurally stable and

$$\sup_{t\geq 0} \int_0^1\int_0^1 \frac{s\|e(t)\|\, ds\, \tau}{\|y(t, y^*)+\tau se(t)\|} = C < +\infty \tag{23}$$

then $e(t)$ converge to zero exponentially provided that the number

$$F''_{\max} = \max_{i\in\{1,2,\ldots,n\}} \sup_{y\in S} F''_i(y)$$

is sufficiently small.

Proof. Since $F'(e^s y) = F'(y)$ for all $s \in \mathbb{R}$ and all $y \in \mathbb{R}^n\backslash\{\mathbf{0}\}$ then from Lemma 2 we derive $F''_i(y) = F''_i\left(\frac{\|y\|}{\|y\|}y\right) = \frac{1}{\|y\|}F''_i\left(\frac{y}{\|y\|}\right)$ and

$$e^{\top} F''_i(y(t, y^*)+\tau se)\, e = \frac{e^{\top} F''_i\left(\frac{y(t,y^*)+\tau se}{\|y(t,y^*)+\tau se\|}\right)e}{\|y(t,y^*)+\tau se\|}.$$

Since F''_i is continuous on S then there exists is a class $\mathcal{K}$ function σ such

$$|g_i(y(t, y^*), e(t))| \leq \sigma(F''_{\max})C\|e(t)\|, \quad \forall t > 0,$$

provided that (23) is fulfilled. This means that $\|g\|_\infty$ admits the linear estimate $\|g(y(t, y^*), e(t))\|_\infty \leq \sigma(F''_{\max})C\|e(t)\|$, and from $\chi_{\max} < 0$ and structural stability of Lyapunov exponents we conclude that $e(t) \to 0$ as $t \to \infty$ for sufficiently small $F''_{\max}$. ■

IV. Conclusions

In this paper we propose a possible approach for application of the well-known method of Lyapunov exponents to a generalized homogeneous systems. The key difficulty is that transitions of homogeneous systems may be sub- or super exponential while the analysis by means of Lyapunov exponents "sense" only an exponential growth.

The method of Lyapunov exponents is utilized, in particular, for the design of observers for nonlinear systems [2]. The interesting direction of future research is application of the developed technique for observers/controllers design for some classes of nonlinear homogeneous systems.

References

[1] A. M. Lyapunov, *The general problem of the stability of motion*. Taylor & Francis, 1992.

[2] J. Frank and S. Zhuk, "A detectability criterion and data assimilation for non-linear differential equations," *ArXiV:1711.05039v1*.

[3] D. Bylov, R. Vinograd, D. Grobman, and V. Nemyckii, *Theory of Lyapunov exponents and its application to problems of stability*. Nauka, Moscow, 1966, in Russian.

[4] L. Barreira and Y. Pesin, *Lyapunov Exponents and Smooth Ergodic Theory*. AMS: Providence, Rhode Island, 2002.

[5] V. Zubov, "On systems of ordinary differential equations with generalized homogenous right-hand sides," *Izvestia vuzov. Mathematica (in Russian).*, vol. 1, pp. 80–88, 1958.

[6] H. Hermes, "Nilpotent approximations of control systems and distributions," *SIAM Journal of Control and Optimization*, vol. 24, no. 4, p. 731, 1986.

[7] L. Rosier, "Homogeneous Lyapunov function for homogeneous continuous vector field," *Systems & Control Letters*, vol. 19, pp. 467–473, 1992.

[8] S. P. Bhat and D. S. Bernstein, "Geometric homogeneity with applications to finite-time stability," *Mathematics of Control, Signals and Systems*, vol. 17, pp. 101–127, 2005.

[9] A. Polyakov, D. Efimov, E. Fridman, and W. Perruquetti, "On homogeneous distributed parameters equations," *IEEE Transactions on Automatic Control*, vol. 61, no. 11, pp. 3657–3662, 2016.

[10] A. F. Filippov, *Differential Equations with Discontinuous Right-hand Sides*. Kluwer Academic Publishers, 1988.

[11] V. Andrieu, L. Praly, and A. Astolfi, "Homogeneous Approximation, Recursive Observer Design, and Output Feedback," *SIAM Journal of Control and Optimization*, vol. 47, no. 4, pp. 1814–1850, 2008.

[12] V. V. Khomenuk, "On systems of ordinary differential equations with generalized homogenous right-hand sides," *Izvestia vuzov. Mathematica (in Russian).*, vol. 3(22), pp. 157–164, 1961.

[13] M. Kawski, "Geometric homogeneity and stabilization," in *Proc. IFAC Nonlinear Control Symposium*, A. Krener and D. Mayne, Eds., Lake Tahoe, CA, 1995, pp. 164–169.

[14] L. Rosier, "Etude de quelques problèmes de stabilization," *PhD Thesis, Ecole Normale Superieure de Cachan (France)*, 1993.

[15] A. Polyakov, "Quadratic-like stabilizability of homogeneous systems," in *Proc. Conference on Decision and Control*, Melbourne, Australia, 2017, pp. 577–582.

[16] ——, "Sliding mode control design using canonical homogeneous norm," *International Journal of Robust and Nonlinear Control*, vol. 29, no. 3, pp. 682–701, 2018.

[17] A. Polyakov, J.-M. Coron, and L. Rosier, "On homogeneous finite-time control for linear evolution equation in hilbert space," *IEEE Transactions on Automatic Control*, vol. 63, no. 9, pp. 3143–3150, 2018.

[18] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer, 1983.

[19] A. Levant, "Homogeneity approach to high-order sliding mode design," *Automatica*, vol. 41, no. 5, pp. 823–830, 2005.

[20] Y. Orlov, "Finite time stability and robust control synthesis of uncertain switched systems," *SIAM Journal of Control and Optimization*, vol. 43, no. 4, pp. 1253–1271, 2005.

[21] L. Grune, "Homogeneous state feedback stabilization of homogeneous systems," *SIAM Journal of Control and Optimization*, vol. 38, no. 4, pp. 1288–1308, 2000.

[22] A. Polyakov, J.-M. Coron, and L. Rosier, "On finite-time stabilization of evolution equations: A homogeneous approach," in *Conference on Decision and Control*, 2016, pp. 3143–3148.

[23] L. Praly, "Generalized weighted homogeneity and state dependent time scale for linear controllable systems," in *Proc. IEEE Conference on Decision and Control*, San Diego, USA, 1997, pp. 4342–4347.

[24] W. Perruquetti, T. Floquet, and E. Moulay, "Finite-time observers: application to secure communication," *IEEE Transactions on Automatic Control*, vol. 53, no. 1, pp. 356–360, 2008.

[25] L. Y. Adrianova, *Introduction to Linear Systems of Differential Equations*. American Mathematical Society, Providence, Rhode Island, 1995.