A NURBS-based Discontinuous Galerkin method for CAD compliant flow simulations

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A NURBS-BASED DISCONTINUOUS GALERKIN METHOD FOR CAD COMPLIANT FLOW SIMULATIONS

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ABSTRACT

In this work, we explain how a classical nodal Discontinuous Galerkin (DG) method for conservation laws can be modified to be geometrically exact with respect to CAD (Computer-Aided Design) data. The proposed approach relies on the use of rational Bézier elements, that can exactly match geometries defined by NURBS (Non-Uniform Rational B-Splines) after some basic transformations. It has been found convenient to use the same basis to describe the solution, yielding a so-called *isogeometric* formulation. The resulting method exhibits optimal convergence rates and facilitates couplings with geometry, e.g. for local refinement, shape sensitivity analysis, or moving computational domains. Illustrations are provided for two-dimensional compressible Euler and Navier-Stokes equations.

NURBS-BASED DG METHOD

We consider a two-dimensional conservation law, that can be written in the conservative form as follows:

$$\frac{\partial \mathbf{W}}{\partial t} + \nabla \cdot \vec{F} = 0, \qquad (1)$$

where $\mathbf{W}$ are the conservative flow variables and $\vec{F} = (\mathbf{F}_x(\mathbf{W}), \mathbf{F}_y(\mathbf{W}))$ is the vector of fluxes, composed of inviscid fluxes for Euler equations and including both inviscid and viscous fluxes for Navier-Stokes equations. A DG method [1] is derived from the weak formulation, by multiplying Eq. (1) by an arbitrary test function ϕ and integrating by parts over an arbitrary element Ω_j:

$$\int_{\Omega_j} \frac{\partial \mathbf{W}}{\partial t} \phi \, d\Omega - \int_{\Omega_j} \vec{F}(\mathbf{W}) \cdot \vec{\nabla}\phi \, d\Omega + \int_{\partial\Omega_j} \vec{F}(\mathbf{W}) \cdot \vec{n} \, \phi \, d\Gamma = 0. \qquad (2)$$

Since the solution is *a priori* discontinuous at the interfaces, the normal flux is evaluated by a numerical flux function $F^\star(\mathbf{W}^+, \mathbf{W}^-, \vec{n})$, defined according to the values of the solution that prevail at each side of the interface and the local unit vector $\vec{n}$ directed outwards.

In order to match exactly a boundary defined by a NURBS curve, a straightforward appproach consists in choosing a rational Bézier patch as element Ω_j. Indeed, any NURBS curve can be splitted in a set of rational Bézier curves, by applying a classical CAD transformation known as *Bézier extraction* [2]. Such a patch is defined by a tensor product of rational Bernstein basis functions, associated to control points and weights:

$$\mathbf{x}(\xi, \eta) = \sum_{i_1=1}^{p+1} \sum_{i_2=1}^{p+1} R^p_{i_1 i_2}(\xi, \eta) \mathbf{X}_{i_1 i_2}, \qquad (3)$$

$$R^p_{i_1 i_2}(\xi, \eta) = \frac{w_{i_1 i_2} \, N^p_{i_1}(\xi) \, N^p_{i_2}(\eta)}{\sum_{j_1=1}^{n_1} \sum_{j_2=1}^{n_2} w_{j_1 j_2} N^p_{j_1}(\xi) N^p_{j_2}(\eta)}, \qquad (4)$$

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where $\mathbf{x} = (x,y)$ are the physical coordinates in the patch and $(\mathbf{X}_{i_1 i_2})_{i_1=1,\cdots,n_1\, i_2=1,\cdots,n_2}$ is the lattice of control points. $R^p_{i_1\,i_2}(\xi,\eta)$ represents the basis function of index $i_1 i_2$, defined thanks to the Bernstein polynomials $N^p_{i_1}$ and $N^p_{i_2}$ of degree p in parametric coordinates (ξ,η). The weights $(\omega_{i_1 i_2})_{i_1=1,\cdots,n_1\, i_2=1,\cdots,n_2}$, associated to each control point, are necessary to represent conic curves.

By using the basis functions $R^p_{i_1\,i_2}$ to represent the solution fields over each element (*isogeometric* framework [3]), and employing $R^p_{k_1\,k_2}$ as test function in Eq. (2), one obtains:

$$\sum_{i_1=1}^{p+1}\sum_{i_2=1}^{p+1} \frac{\partial \mathbf{W}_{i_1 i_2}}{\partial t} \int_{\widehat{\Omega}_j} R^p_{i_1 i_2} R^p_{k_1 k_2}\, |J_\Omega|\, d\widehat{\Omega} = \int_{\widehat{\Omega}_j} \vec{F}(\mathbf{W}) \cdot \nabla \vec{R}^p_{k_1 k_2}\, |J_\Omega|\, d\widehat{\Omega} - \int_{\partial\widehat{\Omega}_j} F^\star(\mathbf{W}^+,\mathbf{W}^-,\vec{n})\, R^p_{k_1 k_2}\, |J_\Gamma|\, d\widehat{\Gamma}, \quad (5)$$

where integrals are expressed in the parametric domain $\widehat{\Omega}_j$. This formulation is implemented by using Gauss quadratures and explicit Runge-Kutta methods for time integration. Numerical tests using analytical solutions exhibit optimal convergence rates [4].

ILLUSTRATING RESULTS

As example, we consider the inviscid transonic flow in a nozzle, whose geometry is defined as a cubic NURBS curve. A very coarse discretization of the computational domain is first constructed using a set of 8×4 rational Bézier patches, which allow to represent exactly the nozzle geometry. An adaptive refinement strategy is then implemented, controled by a measure of the interface jumps as error indicator. Thanks to the representation used, the refinement process is achieved without alteration of the geometry, because rational Bézier patches can be identically subdivided [2]. Here, HLL method is employed to evaluate the inviscid fluxes while shock capturing is achieved using subcell artificial viscosity [5] and LDG method [1] for the resulting viscous fluxes. As illustrated in Fig. (1), coarse elements are maintained in regular regions, matching however the nozzle geometry exactly, whereas refinement allows a sharp capture of the shock.

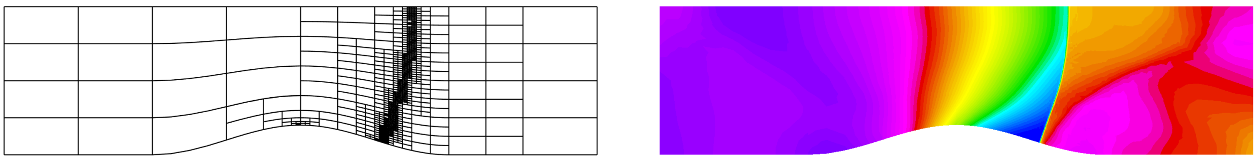

FIGURE 1: transonic flow in a nozzle using cubic NURBS elements.

REFERENCES

[1] J. S. Hesthaven and T. Warburton. *Nodal Discontinuous Galerkin Methods.* Springer, 2008.

[2] L. Piegl and W. Tiller. *The NURBS book.* Springer-Verlag, 1995.

[3] J. Cottrell, T. Hughes, and Y. Bazilevs. *Isogeometric analysis : towards integration of CAD and FEA.* John Wiley & sons, 2009.

[4] R. Duvigneau. *Isogeometric analysis for compressible flows using a Discontinuous Galerkin method.* Computer Methods in Applied Mechanics and Engineering, 333(443-461), 2018.

[5] P.-O. Persson and J. Peraire. *Sub-cell shock capturing for Discontinuous Galerkin methods.* AIAA Paper 2006-112, 2006.