Correct and Efficient Antichain Algorithms for Refinement Checking

Maurice Laveaux, Jan Friso Groote, Tim Willemse

Correct and Efficient Antichain Algorithms for Refinement Checking

Maurice Laveaux[0000−0001−8732−7580] (✉), Jan Friso Groote[0000−0003−2196−6587], and Tim A.C. Willemse[0000−0003−3049−7962]

Eindhoven University of Technology, Eindhoven, The Netherlands
{m.laveaux, j.f.groote, t.a.c.willemse}@tue.nl

Abstract. Refinement checking plays an important role in system verification. This means that the correctness of the system is established by showing a refinement relation between two models; one for the implementation and one for the specification. In [21], Wang *et al.* describe an algorithm based on antichains for efficiently deciding stable failures refinement and failures-divergences refinement. We identify several issues pertaining to the correctness and performance in these algorithms and propose new, correct, antichain-based algorithms. Using a number of experiments we show that our algorithms outperform the original ones in terms of running time and memory usage.

1 Introduction

Refinement is often an integral part of a mature engineering methodology for designing a (software) system in a stepwise manner. It allows one to start from a high-level specification that describes the permitted and desired behaviours of a system and arrive at a detailed implementation that behaves as originally specified. While in many settings, refinement is often used rather informally, it forms the mathematical cornerstone in the theoretical development of the process algebra CSP (Communicating Sequential Processes) by Hoare [12, 17, 18].

This formal view on refinement—as a mathematical relation between a specification and its implementation—has been used successfully in industrial settings [10], and it has been incorporated in commercial Formal Model-Driven Engineering tools such as *Dezyne* [3]. In such settings there are a variety of refinement relations, each with their own properties. In particular, each notion of refinement offers specific guarantees on the (types of) behavioural properties of the specification that carry over to correct implementations. For the theory of CSP, the—arguably—most prominent refinement relations are *stable failures refinement* [2, 18] and *failures-divergences refinement* [18]. Both are implemented in the tool FDR [6] for specifying and analysing CSP processes.

Both stable failures refinement and failures-divergences refinement are computationally hard problems; deciding whether there is a refinement relation between an implementation and a specification, both represented by CSP processes or labelled transition systems, is PSPACE-hard [13]. In practice, however, tools such as FDR are able to work with quite large state spaces. The basic algorithm for deciding a stable failures refinement or a failures-divergences refinement between an implementation

and a specification relies on a *normalisation* of the specification. This normalisation is achieved by a subset construction that is used to obtain a deterministic transition system which represents the specification.

As observed in [21] and inspired by successes reported, *e.g.*, in [1], *antichain* techniques can be exploited to improve on the performance of refinement checking algorithms. Unfortunately, a closer inspection of the results and algorithms in [21], reveals several issues. First, the definitions of stable failures refinement and failures-divergences refinement used in [21] do not match the definitions of [2, 18], nor do they seem to match known relations from the literature [8].

Second, as we demonstrate in Example 2 in this paper, the results [21, Theorems 2 and 3] claiming correctness of their algorithms for deciding both refinement relations are incorrect. We do note that their algorithm for checking stable failures refinement correctly decides the refinement relation defined by [2, 18].

Third, unlike claimed by the authors in [21], their algorithms violate the antichain property as we demonstrate in Example 4. Fourth, their algorithms suffer from severely degraded performance due to suboptimal decisions made when designing the algorithms, leading to an overhead of a factor $|\Sigma|$, where Σ is the set of events. When using a FIFO queue to realise a breadth-first search strategy instead of the stack used by default for a depth-first search this factor is even greater, *viz.* $|\Sigma|^{|S|}$, where S is the set of states of the implementation, see our Example 3. Note that there are compelling reasons for using a breadth-first strategy [17]; *e.g.*, the conciseness of counterexamples to refinement.

The contributions of the current paper are as follows. Apart from pointing out the issues in [21], we propose new antichain-based algorithms for deciding stable failures refinement and failures-divergences refinement and we prove their correctness. We compare the performance of the stable failures refinement algorithm of [21] to ours. Due to the flaw in their algorithm for checking failures-divergences refinement, a comparison for this refinement relation makes little sense. Our results indicate a small improvement in run time performance for practical models when using depth-first search, whereas our experiments using breadth-first search illustrate that decision problems intractable using the algorithm of [21] generally become quite easy using our algorithm.

The remainder of this paper is organised as follows. We recall the necessary mathematics in Section 2 and we describe the essence of refinement checking algorithms in Section 3. In Section 4, we analyse the algorithms of [21] and in Section 5, we propose new antichain-based refinement algorithms, claim their correctness and provide proof sketches. In Section 6, we compare the performance of our algorithm to that of [21]. Full proofs of our claims can be found in a technical report [14], which also contains additional experiments, showing that further speed improvements can be obtained by applying divergence-preserving branching bisimulation [7] minimisation before checking refinement.

2 Preliminaries

In this section the preliminaries of labelled transition systems, stable failures refinement and failures-divergences refinement checking are introduced.

2.1 Labelled Transition Systems

Let Σ be a finite set of event labels that does not contain the constant τ, modelling *internal* events.

Definition 1. *A labelled transition system is a tuple $\mathcal{L} = (S, \iota, Act, \rightarrow)$ where S is a set of states; $\iota \in S$ is an initial state; $Act = \Sigma$ or $Act = \Sigma \cup \{\tau\}$ is a set of actions and $\rightarrow \subseteq S \times Act \times S$ is a labelled transition relation.*

The following definitions are in the context of a given labelled transition system $\mathcal{L} = (S, \iota, Act, \rightarrow)$. We typically use letters s, t, u to denote states, U, V to denote *sets* of states, a to denote an arbitrary action, e to denote an arbitrary event and $\sigma, \rho \in Act^*$ to denote a sequence of actions.

We adopt the following conventions and notation. Whenever $(s, a, t) \in \rightarrow$, we write $s \xrightarrow{a} t$; we write $s \xrightarrow{a}$ just whenever there is some t such that $s \xrightarrow{a} t$, and $s \not\xrightarrow{a}$ holds iff not $s \xrightarrow{a}$. The set of actions that can be executed in s is given by the set $\mathsf{enabled}(s)$, defined as $\mathsf{enabled}(s) = \{a \in Act \mid s \xrightarrow{a}\}$. We generalise the relation $\rightarrow$ in the usual manner to sequences of actions as follows: $s \xrightarrow{\epsilon}\!\!\!\!\rightarrow t$ holds iff $s = t$, and $s \xrightarrow{a\sigma}\!\!\!\!\rightarrow t$ holds iff there is some u such that $s \xrightarrow{a} u$ and $u \xrightarrow{\sigma}\!\!\!\!\rightarrow t$. Finally, the *weak* transition relation $\rightsquigarrow \subseteq S \times \Sigma^* \times S$ is the least relation satisfying:

- $s \overset{\epsilon}{\rightsquigarrow} t$ if there is some $\sigma \in \tau^*$ such that $s \xrightarrow{\sigma}\!\!\!\!\rightarrow t$,
- if $s \xrightarrow{a} t$ then $s \overset{a}{\rightsquigarrow} t$,
- if $s \overset{\rho}{\rightsquigarrow} t$ and $t \overset{\sigma}{\rightsquigarrow} u$ then $s \overset{\rho\sigma}{\rightsquigarrow} u$.

Definition 2. *Traces, weak traces and reachable states are defined as follows:*

- *$\sigma \in Act^*$ is a* trace *starting in s iff $s \xrightarrow{\sigma}\!\!\!\!\rightarrow t$ for some t. We denote the set of all traces starting in s by $\mathsf{traces}(s)$, and we define $\mathsf{traces}(\mathcal{L}) = \mathsf{traces}(\iota)$,*
- *$\sigma \in \Sigma^*$ is a* weak trace *starting in s iff $s \overset{\sigma}{\rightsquigarrow} t$ for some t. The set of all weak traces starting in s is denoted by $\mathsf{traces}_w(s)$, and we define $\mathsf{traces}_w(\mathcal{L}) = \mathsf{traces}_w(\iota)$,*
- *the set of states, reachable from s is defined as $\mathsf{reachable}(s) = \{t \in S \mid \exists \sigma \in \Sigma^* : s \overset{\sigma}{\rightsquigarrow} t\}$; we define $\mathsf{reachable}(\mathcal{L}) = \mathsf{reachable}(\iota)$.*

The semantics of the CSP process algebra builds on observations of *failures* and *divergences*. A failure is a set of event labels that the system observably refuses following an experiment on that system, *i.e.*, after executing a weak trace on that system.

By assumption, refusals can only be observed when the system has *stabilised*. Formally, a state s is *stable*, denoted $\mathsf{stable}(s)$, if and only if $s \not\xrightarrow{\tau}$. A divergence can be understood as the potential inability of a system to stabilise. In effect, this means that a divergence is an infinite sequence of τ-actions; formally, a state s is a diverging state, denoted $\mathsf{div}(s)$, if and only if there is an infinite sequence of states s_i such that $s \xrightarrow{\tau} s_1 \xrightarrow{\tau} s_2 \xrightarrow{\tau} \cdots$. For a set of states U, we write $\mathsf{div}(U)$ iff $\mathsf{div}(s)$ for some $s \in U$.

Definition 3. *Let $s \in S$ be a stable state. The* refusals *of s are defined as*[1] *the set* $\mathsf{refusals}(s) = \mathcal{P}(\Sigma \setminus \mathsf{enabled}(s))$. *For a set of (not necessarily stable) states U, we define* $\mathsf{refusals}(U) = \{X \subseteq \Sigma \mid s \in U \wedge \mathsf{stable}(s) \wedge X \in \mathsf{refusals}(s)\}$.

We are now in a position to formally define the set of divergences and the set of failures of an LTS; we here follow the standard conventions and definitions of [9, 4, 18]. Note that in [14] we instead have adopted the notational conventions of [21] to allow for an easier comparison of our results to theirs.

Definition 4. *The set of all* divergences *of a state s, denoted by* $\mathsf{divergences}(s)$*, is defined as* $\{\sigma\rho \in \Sigma^* \mid \exists t \in S : s \overset{\sigma}{\rightsquigarrow} t \wedge \mathsf{div}(t)\}$. *We define* $\mathsf{divergences}(\mathcal{L}) = \mathsf{divergences}(\iota)$.

Observe that a divergence is not only a weak trace that can reach a diverging state, but also any suffix of a weak trace that can reach a diverging state. This is based on the assumption that divergences lead to *chaos*. In theories in which divergences are considered chaotic, such chaos obscures all information about the behaviours involving a diverging state; we refer to this as obscuring *post-divergences details*.

Definition 5. *The set of all* stable failures *of a state s, denoted* $\mathsf{failures}(s)$*, is defined as* $\{(\sigma, X) \in \Sigma^* \times \mathcal{P}(\Sigma) \mid \exists t \in S : s \overset{\sigma}{\rightsquigarrow} t \wedge \mathsf{stable}(t) \wedge X \in \mathsf{refusals}(t)\}$. *The set of stable failures of a state s with* post-divergences details obscured*, denoted* $\mathsf{failures}_\perp(s)$*, is defined as* $\mathsf{failures}(s) \cup \{(\sigma, X) \in \Sigma^* \times \mathcal{P}(\Sigma) \mid \sigma \in \mathsf{divergences}(s)\}$.

The two standard models of CSP are the *stable failures* model and the *failures-divergences* model. The refinement relations on LTSs, induced by these models, are called the *stable failures refinement* and the *failures-divergences refinement*. We remark that the LTS that is refined is commonly referred to as the *specification*, whereas the LTS that refines the specification is often referred to as the *implementation*.

Definition 6. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two LTSs. We say that $\mathcal{L}_2$ is a stable failures refinement of $\mathcal{L}_1$, denoted by $\mathcal{L}_1 \sqsubseteq_{sfr} \mathcal{L}_2$, iff* $\mathsf{traces}_w(\mathcal{L}_2) \subseteq \mathsf{traces}_w(\mathcal{L}_1)$ *and* $\mathsf{failures}(\mathcal{L}_2) \subseteq \mathsf{failures}(\mathcal{L}_1)$. *LTS $\mathcal{L}_2$ is a failures-divergences refinement of $\mathcal{L}_1$, denoted by $\mathcal{L}_1 \sqsubseteq_{fdr} \mathcal{L}_2$, iff* $\mathsf{failures}_\perp(\mathcal{L}_2) \subseteq \mathsf{failures}_\perp(\mathcal{L}_1)$ *and* $\mathsf{divergences}(\mathcal{L}_2) \subseteq \mathsf{divergences}(\mathcal{L}_1)$.

Remark 1. The notions defined above appear in different formulations in [21]. Their stable failures refinement omits the clause for weak trace inclusion, and their failures-divergences refinement replaces $\mathsf{failures}_\perp$ with $\mathsf{failures}$. This yields refinement relations different from the standard ones and neither relation seems to appear in the literature [8].

We finish this section with a small example, illustrating the difference between failures-divergences refinement and stable failures refinement.

Example 1. Consider the two transition systems (named after their initial states s_0 and t_0) depicted below.

[1] We remark that [21] states the following, non-standard, definition: $\mathsf{refusals}(s) = \{X \mid \exists s' \in S : s \overset{\epsilon}{\rightsquigarrow} s' \wedge \mathsf{stable}(s') \wedge X \subseteq \Sigma \setminus \mathsf{enabled}(s')\}$, suggesting that refusals are also defined for unstable states. As we discuss in Section 4, this has consequences for the performance of the algorithms for deciding the various refinement relations.

Observe that we have $s_0 \sqsubseteq_{\text{fdr}} t_0$, but not $s_0 \sqsubseteq_{\text{sfr}} t_0$. The latter fails because aa is a trace of t_0, but not of s_0; the same goes for the stable failure $(a, \{b\})$ of t_0. The failures-divergences refinement holds because the divergent trace a obfuscates the observations of traces of the form aa^+: since divergence is chaos, anything is permitted. We do have $t_0 \sqsubseteq_{\text{sfr}} s_0$ but not $t_0 \sqsubseteq_{\text{fdr}} s_0$. The latter fails because of the divergent trace a not being present in t_0. Stable failures refinement holds because all traces and stable failure pairs of s_0 are included in those of t_0; in particular, the instability of state s_1 causes s_1 not to contribute to the stable failures set of s_0. □

3 Refinement Checking

In general, the set of failures and divergences of an LTS can be infinite. Therefore, checking inclusion of the set of failures or divergences is not viable. In [17, 18], an algorithm to decide refinement between two labelled transition systems is sketched. As a preprocessing to this algorithm, all diverging states in both LTSs are marked. The algorithm then relies on exploring the cartesian product of the *normal form* representation of the specification, *i.e.*, the LTS that is to be refined, and the implementation. We remark that what we refer to as *cartesian product*, defined in [17], is called a *synchronous product* in [21]. For each pair in the product it checks whether it can locally decide non-refinement of the implementation state with the normal form state. A pair for which non-refinement holds is referred to as a *witness*.

Following [18, 21] and specifically the terminology of [17], we formalise the cartesian product between LTSs that is explored by the procedure.

Definition 7. *Let $\mathcal{L}_1 = (S_1, \iota_1, \Sigma, \rightarrow_1)$ and $\mathcal{L}_2 = (S_2, \iota_2, Act, \rightarrow_2)$ be LTSs. The* cartesian product *of $\mathcal{L}_1$ and $\mathcal{L}_2$ is the LTS $\mathcal{L}_1 \times \mathcal{L}_2 = (S, \iota, Act, \rightarrow)$ satisfying $S = S_1 \times S_2$; $\iota = (\iota_1, \iota_2)$; and the transition relation $\rightarrow$ is the smallest relation satisfying the following conditions for all $s_1, t_1 \in S_1$, and $s_2, t_2 \in S_2$ and $e \in \Sigma$:*

- *If $s_2 \xrightarrow{\tau}_2 t_2$ then $(s_1, s_2) \xrightarrow{\tau} (s_1, t_2)$,*
- *If $s_1 \xrightarrow{e}_1 t_1$ and $s_2 \xrightarrow{e}_2 t_2$ then $(s_1, s_2) \xrightarrow{e} (t_1, t_2)$.*

We remark that Σ is used in $\mathcal{L}_1$ to indicate that it has no transitions labelled with τ, whereas, $\mathcal{L}_2$ might contain τ-transitions. A key property of the cartesian product, provable by induction on the length of sequence σ, is the following:

Proposition 1. *Let $\mathcal{L}_1 = (S_1, \iota_1, \Sigma, \rightarrow_1)$ and $\mathcal{L}_2 = (S_2, \iota_2, Act, \rightarrow_2)$ be LTSs, and let $\mathcal{L}_1 \times \mathcal{L}_2 = (S, \iota, Act, \rightarrow)$ be their cartesian product. For all $s_1 \in S_1$, $s_2 \in S_2$ and all $\sigma \in Act^*$, $\iota \overset{\sigma}{\rightsquigarrow} (s_1, s_2)$ iff $\iota_1 \overset{\sigma}{\rightsquigarrow}_1 s_1$ and $\iota_2 \overset{\sigma}{\rightsquigarrow}_2 s_2$.*

The normal form LTS is obtained using a typical subset construction as is common when determinising a transition system. Although all states in an LTS in normal form are

stable, the states of the original LTS comprising a normal form state may not be. To avoid confusion when we wish to reason about the stability and divergences of states U in the LTS $\mathcal{L}$ underlying a normal form LTS, rather than the state of the normal form LTS, we write $[\![U]\!]_{\mathcal{L}}$ to indicate we refer to the set of states in $\mathcal{L}$. Stable failures refinement and failures-divergences refinement require different normal forms.

Definition 8. *Let $\mathcal{L} = (S, \iota, Act, \rightarrow)$ be a labelled transition system. Set $S' = \mathcal{P}(S)$, $\iota' = \{s \in S \mid \iota \stackrel{\epsilon}{\rightsquigarrow} s\}$. The* stable failures refinement *normal form of $\mathcal{L}$ is the LTS $\mathsf{norm}_{sfr}(\mathcal{L}) = (S', \iota', \Sigma, \rightarrow')$, where $\rightarrow'$ is defined as $U \stackrel{e}{\rightarrow'} V$ if and only if $V = \{t \in S \mid \exists s \in U : s \stackrel{e}{\rightsquigarrow} t\}$ for all $U, V \subseteq S$ and $e \in \Sigma$. The* failures-divergences refinement *normal form of $\mathcal{L}$ is the LTS $\mathsf{norm}_{fdr}(\mathcal{L}) = (S', \iota', \Sigma, \rightarrow'')$ where $\rightarrow''$ is defined as $U \stackrel{e}{\rightarrow''} V$ if and only if $U \stackrel{e}{\rightarrow'} V$ and not $\mathsf{div}([\![U]\!]_{\mathcal{L}})$.*

We remark that we deliberately permit the empty set to be a state in a normal form LTS. Clearly, a normal form LTS satisfies $\emptyset \stackrel{e}{\rightarrow} \emptyset$ for all actions e. Moreover, note that a normal form LTS is *deterministic*; in particular, for all σ, and states U, T, V of a normal form LTS $U \stackrel{\sigma}{\twoheadrightarrow} T$ and $U \stackrel{\sigma}{\twoheadrightarrow} V$ implies $T = V$.

The structure explored by the refinement checking procedure of [17, 18] is essentially the cartesian product $\mathsf{norm}_{\mathrm{sfr}}(\mathcal{L}_1) \times \mathcal{L}_2$ in case of stable failures refinement, or $\mathsf{norm}_{\mathrm{fdr}}(\mathcal{L}_1) \times \mathcal{L}_2$ in case of failures-divergences refinement. For these structures the related witnesses, where the reachability of such a witness indicates non-refinement, are then as follows:

Definition 9. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be LTSs.*

- *A state (U, s) in $\mathsf{norm}_{sfr}(\mathcal{L}_1) \times \mathcal{L}_2$ is called an* SFR*-witness iff $U = \emptyset$; or $\mathsf{stable}(s)$ and not $\mathsf{refusals}(s) \subseteq \mathsf{refusals}([\![U]\!]_{\mathcal{L}_1})$,*
- *a state (U, s) in $\mathsf{norm}_{fdr}(\mathcal{L}_1) \times \mathcal{L}_2$ is called an* FDR*-witness iff not $\mathsf{div}([\![U]\!]_{\mathcal{L}_1})$, and either $\mathsf{div}(s)$; or $U = \emptyset$; or $\mathsf{stable}(s)$ and not $\mathsf{refusals}(s) \subseteq \mathsf{refusals}([\![U]\!]_{\mathcal{L}_1})$.*

The following statement formalises the insights of [17]; both results follow from Proposition 1 and the characteristics of the normal form LTSs.

Theorem 1. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be LTSs. Then:*

- *$\mathcal{L}_1 \sqsubseteq_{sfr} \mathcal{L}_2$ iff no SFR-witness is reachable in $\mathsf{norm}_{sfr}(\mathcal{L}_1) \times \mathcal{L}_2$,*
- *$\mathcal{L}_1 \sqsubseteq_{fdr} \mathcal{L}_2$ iff no FDR-witness is reachable in $\mathsf{norm}_{fdr}(\mathcal{L}_1) \times \mathcal{L}_2$.*

4 Antichain Algorithms for Refinement Checking

The normalisation of the specification LTS in refinement checking dominates the theoretical worst-case run time complexity of refinement checking, which itself is a PSPACE-hard problem. In practice, however, refinement checking can often be done quite effectively. Nevertheless, as observed in [21], there is room for improvement by exploiting an antichain approach to keep the size of the normal form LTS of the specification in check.

An antichain is a set $\mathcal{A} \subseteq X$ of a partially ordered set $(X, \leq)$ in which all distinct $x, y \in \mathcal{A}$ are incomparable: neither $x \leq y$ nor $y \leq x$. Given a partially ordered set $(X, \leq)$ and an antichain $\mathcal{A}$, the operation $\Subset$ checks whether $\mathcal{A}$ 'contains' an element x;

that is, $x \Subset \mathcal{A}$ holds true if and only if there is some $y \in \mathcal{A}$ such that $y \leq x$. We write $Y \Subset^{\forall} \mathcal{A}$ iff $y \Subset \mathcal{A}$ for all $y \in Y$. Antichain $\mathcal{A}$ can be extended by inserting an element $x \in X$, denoted $\mathcal{A} \Cup x$, which is defined as the set $\{y \mid y = x \vee (y \in \mathcal{A} \wedge x \not\leq y)\}$. Note that this operation only yields an antichain whenever $x \notin \mathcal{A}$.

As [21, 1] suggest, the state space of the cartesian product $(S, \iota, Act, \rightarrow)$ between the normal form of LTS $\mathcal{L}_1$ and the LTS $\mathcal{L}_2$ induces a partially ordered set as follows. For $(U, s), (V, t) \in S$, define $(U, s) \leq (V, t)$ iff $s = t$ and $[\![U]\!]_{\mathcal{L}_1} \subseteq [\![V]\!]_{\mathcal{L}_1}$. Then the set $(S, \leq)$ is a partially ordered set. The fundamental property underlying the reason why an antichain approach to refinement checking works is expressed by the following proposition, stating that the traces of any state (V, s) in the cartesian product can be executed from all states smaller than (V, s). We remark that this is due to including the empty set as a state in the normal form LTS.

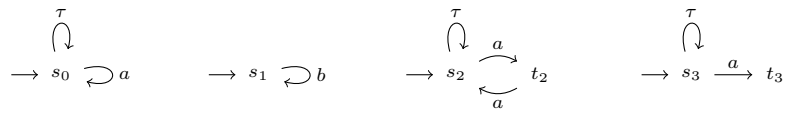
Proposition 2. *For all $(U, s) \leq (V, s)$ of a normal form LTS $\mathsf{norm}_{sfr}(\mathcal{L}_1) \times \mathcal{L}_2$ or $\mathsf{norm}_{fdr}(\mathcal{L}_1) \times \mathcal{L}_2$ and for every sequence $\sigma \in Act^*$ such that $(V, s) \stackrel{\sigma}{\twoheadrightarrow} (V', t)$, there is some (U', t) such that $(U, s) \stackrel{\sigma}{\twoheadrightarrow} (U', t)$ and $(U', t) \leq (V', t)$.*

The proof of this proposition proceeds by induction on the length of σ and is routine.

The main idea of the antichain algorithm is now as follows: the set of states of the cartesian product explored is recorded in an antichain. Whenever a new state of the cartesian product is found that is already included in the antichain (w.r.t. the membership test $\Subset$), further exploration of that state is unnecessary, thereby pruning the state space of the cartesian product. Note that it is not immediate that doing so is also 'safe' for refusals and divergences. Algorithm 1 is the pseudocode for checking stable failures refinement and failures-divergences refinement as presented in [21, Algorithm 2 and 3]; we remark that we combined these algorithms, as their check for failures-divergences refinement only requires an additional check for divergences (enabled by the Boolean $CheckDiv$).

Let us first stress that the algorithm correctly decides stable failures refinement but it fails to correctly decide failures-divergences refinement. Second, the algorithm also fails to decide the non-standard relations used in [21], see also Remark 1. All three issues are illustrated by the example below.

Example 2. Consider the four transition systems depicted below.



Let us first observe that the algorithm correctly decides that $s_1 \sqsubseteq_{\text{sfr}} s_0$ does not hold, which follows from a violation of $\mathsf{traces}_w(s_0) \subseteq \mathsf{traces}_w(s_1)$. Next, observe that we have $s_0 \sqsubseteq_{\text{fdr}} s_1$, since the divergence of the root state s_0 implies chaotic behaviour of s_0 and, hence, any system refines such a system. However, Algorithm 1 returns *false* when *CheckDiv* holds.

With respect to the refinement relations defined in [21], we observe the following. Since s_0 is not stable, we have $\mathsf{failures}(s_0) = \emptyset$ and hence $\mathsf{failures}(s_0) \subseteq \mathsf{failures}(s_1)$. Consequently, stable failures refinement as defined in [21] should hold, but as we already

Algorithm 1 Antichain-based refinement checking algorithm from [21]. The algorithm is claimed to return *true* iff LTS $\mathcal{L}_1 = (S_1, \iota_1, Act_1, \rightarrow_1)$ is refined by $\mathcal{L}_2 = (S_2, \iota_2, Act_2, \rightarrow_2)$. The refinement check conducted checks for stable failures refinement when $CheckDiv$ is *false* and failures-divergences refinement otherwise.

```
 1: procedure REFINES(L1, L2, CheckDiv)
 2:     let working be a stack containing a pair ({s | ι1 ⇝ε1 s}, ι2)
 3:     let antichain := ∅
 4:     while working ≠ ∅ do
 5:         pop (spec, impl) from working
 6:         antichain := antichain ⩔ (spec, impl)
 7:         if CheckDiv and div(impl) then
 8:             if not div(spec) then
 9:                 return false
10:         else
11:             if refusals(impl) ⊈ refusals(spec) then
12:                 return false
13:             for impl -a->2 impl' do
14:                 if a = τ then
15:                     spec' := spec
16:                 else
17:                     spec' := {s' | ∃s ∈ spec : s ⇝a1 s'}
18:                 if spec' = ∅ then
19:                     return false
20:                 if (spec', impl') ∉ antichain is not true then
21:                     push (spec', impl') into working
22:     return true
```

concluded above, the algorithm returns *false* when checking for $s_1 \sqsubseteq_{\text{sfr}} s_0$. Next, observe that the algorithm returns *true* when checking for $s_2 \sqsubseteq_{\text{fdr}} s_3$. The reason is that for the pair $(\{s_2\}, s_3)$, it detects that state s_3 diverges and concludes that since also the normal form state of the specification $\{s_2\}$ diverges, it can terminate the iteration and return *true*. This is a consequence of splitting the divergence tests over two **if**-statements in lines 7 and 8. According to the failures-divergences refinement of [21], however, the algorithm should return *false*, since $\mathsf{failures}(s_3) \subseteq \mathsf{failures}(s_2)$ fails to hold: we have $(a, \{a\}) \in \mathsf{failures}(s_3)$ but not $(a, \{a\}) \in \mathsf{failures}(s_2)$. □

We note that the algorithm explores the cartesian product between the normal form of a specification, and an implementation in a depth-first, on-the-fly manner. While depth-first search is typically used for detecting divergences, [17] states a number of reasons for running the refinement check in a breadth-first manner. A compelling argument in favour of using a breadth-first search is conciseness of the counterexample in case of a non-refinement.

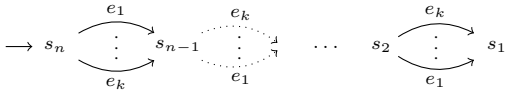
Algorithm 1 can be made to run in a breadth-first fashion simply by using a FIFO *queue* rather than a stack as the data structure for $working$. However, our implementation of this algorithm suffers from a severely degraded performance. We can trace this back to

the following three additional problems in the original algorithm, which also are present (albeit less pronounced in practice) when utilising a depth-first exploration:

1. The refusal check on line 11 is also performed for unstable states, which, combined with the definition of refusals in [21] (see also our remark in Footnote 1 on page 4), results in a repeated, potentially expensive, search for stable states;
2. Adding the pair $(\mathit{spec}, \mathit{impl})$ that is taken from $\mathit{working}$ to $\mathit{antichain}$ might result in duplicate pairs in $\mathit{working}$ since $\mathit{working}$ is filled with all successors of that pair in line 21, regardless of whether these pairs are already scheduled for exploration, *i.e.*, included in $\mathit{working}$, or not;
3. Contrary to the explicit claim in [21, Section 2.2], the set $\mathit{antichain}$ is *not* guaranteed to be an antichain.

The first problem is readily seen to lead to undesirable overhead. The second and third problem are more subtle. We first illustrate the second problem: the following pathological example shows that the algorithm stores an excessive number of pairs in $\mathit{working}$.

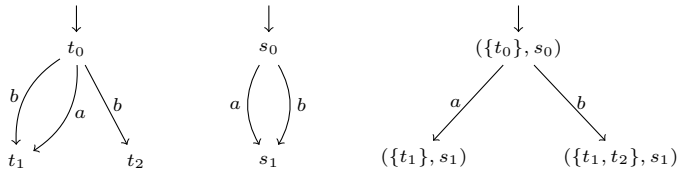
Example 3. Consider the family of LTSs $\mathcal{L}_n^k = (S_n, \iota_n, \mathit{Act}_k, \rightarrow_n)$ with states $S_n = \{s_1, \ldots, s_n\}$, event labels $\mathit{Act}_k = \{e_1, \ldots, e_k\}$ and transitions $s_i \xrightarrow{e_j}_n s_{i-1}$ for all $1 \leq j \leq k$, $1 < i \leq n$ and $\iota_n = s_n$; see also the transition system depicted below. Note that each LTS that belongs to this family is completely deterministic.



Suppose one checks for refinement between an implementation and specification both of which are given by $\mathcal{L}_n^k$; *i.e.*, we test for $\mathcal{L}_n^k \sqsubseteq_{\text{sfr}} \mathcal{L}_n^k$. Then the stack $\mathit{working}$ will contain exactly $i{\cdot}(k{-}1){+}1$ pairs at the end of the *i-th* iteration (when $i \leq n$), resulting in a working stack size of $\mathcal{O}(n{\cdot}k)$ entries. At the end of the *n-th* iteration $\mathit{antichain}$ contains all reachable pairs of the cartesian product, *i.e.*, $\mathit{antichain} = \{(\{s_j\}, s_j) \mid 1 \leq j \leq n\}$ at that point. Emptying $\mathit{working}$ after the *n-th* iteration will involve k antichain membership tests per entry. Consequently, $\mathcal{O}(n{\cdot}k^2)$ antichain membership tests are required. A breadth-first search strategy requires even more antichain membership tests, *viz.*, $\mathcal{O}(k^n)$. □

The example below illustrates the third problem of the algorithm, *viz.*, the violation of the antichain property.

Example 4. Consider the two left-most labelled transition systems depicted below, along with the (normal form) cartesian product (the LTS on the right).



Algorithm 1 starts with $working$ containing pair $(\{t_0\}, s_0)$ and $antichain = \emptyset$. Inside the loop, the pair $(\{t_0\}, s_0)$ is popped from $working$ and added to $antichain$. The successors of the pair $(\{t_0\}, s_0)$ are the pairs $(\{t_1\}, s_1)$ and $(\{t_1, t_2\}, s_1)$. Since $antichain$ contains neither of these, both successors are added to $working$ in line 21. Next, popping $(\{t_1\}, s_1)$ from $working$ and adding this pair to $antichain$ results in $antichain$ consisting of the set $\{(\{t_0\}, s_0), (\{t_1\}, s_1)\}$. In the final iteration of the algorithm, the pair $(\{t_1, t_2\}, s_1)$ is popped from $working$ and added to $antichain$, resulting in the set $\{(\{t_0\}, s_0), (\{t_1\}, s_1), (\{t_1, t_2\}, s_1)\}$. Clearly, since $(\{t_1\}, s_1) \leq (\{t_1, t_2\}, s_1)$, the set $antichain$ no longer is a proper antichain. □

5 A Correct and Improved Antichain Algorithm

We address the identified performance problems by rearranging the computations that are conducted. Note that in order to solve the first performance problem, we only perform the check to compare the refusals of the implementation and the normal form state of the specification in case the implementation state is stable.

Solving the second performance problem is more involved. The essential observation here is that in order for the information in $antichain$ to be most effective, states of the cartesian product must be added to $antichain$ as soon as these are discovered, even if these have not yet been fully explored. This is achieved by maintaining, as an invariant, that $working \Subset^{\forall} antichain$; the states in $working$ then essentially compose the *frontier* of the exploration. We achieve this by initialising $working$ and $antichain$ to consist of exactly the initial state of the cartesian product, and by extending $antichain$ with all (not already discovered) successors for the state $(spec, impl)$ that is popped from $working$. As a side effect, this also resolves the third issue, as now both $working$ and $antichain$ are only extended with states that have not yet been discovered, *i.e.*, for which the membership test in $antichain$ fails, and for which insertion with such states does not invalidate the antichain property.

Algorithm 2 shows the corrected antichain procedure for checking stable failures refinement and failures-divergences refinement. Since the algorithm fundamentally differs (in the relations that it computes) from the one in [21], we cannot reuse their arguments in our proof of correctness, which are based on invariants that do not hold in our case.

In the remainder of this section, we sketch the proof of correctness of Algorithm 2 as claimed below by Theorem 2. We focus on the proof of correctness w.r.t. failures-divergences refinement; for stable failures refinement the proof is almost the same except that it does not have to consider divergences.

Theorem 2. *Let $\mathcal{L}_i = (S_i, \iota_i, Act_i, \rightarrow_i)$ where $i \in \{1, 2\}$ be two labelled transition systems. Then:*

- REFINES$_{new}(\mathcal{L}_1, \mathcal{L}_2, false)$ *returns* true *if and only if $\mathcal{L}_1 \sqsubseteq_{sfr} \mathcal{L}_2$;*
- REFINES$_{new}(\mathcal{L}_1, \mathcal{L}_2, true)$ *returns* true *if and only if $\mathcal{L}_1 \sqsubseteq_{fdr} \mathcal{L}_2$;*

For the remainder of this section we fix the two LTSs $\mathcal{L}_i = (S_i, \iota_i, Act_i, \rightarrow_i)$ where $i \in \{1, 2\}$. First we show termination of Algorithm 2. A crucial observation of the antichain operations is that adding elements to an antichain does not affect the membership test of elements already included; see the lemma below.

Algorithm 2 The corrected antichain-based refinement checking algorithm. The algorithm returns *true* iff LTS $\mathcal{L}_1 = (S_1, \iota_1, Act_1, \rightarrow_1)$ is refined by $\mathcal{L}_2 = (S_2, \iota_2, Act_2, \rightarrow_2)$. The refinement check conducted checks for stable failures refinement when $CheckDiv$ is *false* and failures-divergences refinement otherwise.

```
 1: procedure REFINES_new(L1, L2, CheckDiv)
 2:     let working be a stack containing a pair ({s | ι1 ~ε~>1 s}, ι2)
 3:     let antichain := ∅ ⩏ ({s | ι1 ~ε~>1 s}, ι2)
 4:     while working ≠ ∅ do
 5:         pop (spec, impl) from working
 6:         if not div(spec) or not CheckDiv then
 7:             if CheckDiv and div(impl) then
 8:                 return false
 9:             else
10:                 if stable(impl) then
11:                     if refusals(impl) ⊈ refusals(spec) then
12:                         return false
13:                 for impl -a->2 impl' do
14:                     if a = τ then
15:                         spec' := spec
16:                     else
17:                         spec' := {s' | ∃s ∈ spec : s ~a~>1 s'}
18:                     if spec' = ∅ then
19:                         return false
20:                     if (spec', impl') ⩷ antichain is not true then
21:                         antichain := antichain ⩏ (spec', impl')
22:                         push (spec', impl') into working
23:     return true
```

Lemma 1. *Let $(X, \leq)$ be a partially ordered set, $\mathcal{A} \subseteq X$ an antichain, and let $x, y \in X$. If $x \Subset \mathcal{A}$ and $y \notin \mathcal{A}$ then $x \Subset (\mathcal{A} \Cup y)$.*

Termination now follows from the observation that all states of the cartesian product that have been processed occur in $antichain$ and do not get added back to $working$; for this we rely on Lemma 1. To reason formally about the states that have been processed, we introduce a ghost variable *done*; *i.e.*, *done* is intialised as the empty set and each pair (*spec*, *impl*) that is popped from $working$ in line 5 is added to *done* after line 22. We have the following invariants.

Lemma 2. *Let $\mathcal{L}_n = \mathsf{norm}_{fdr}(\mathcal{L}_1)$ if CheckDiv holds and $\mathcal{L}_n = \mathsf{norm}_{sfr}(\mathcal{L}_1)$ otherwise. The while loop (lines 4-22) of Algorithm 2 satisfies the following invariants: done $\cup$ working $\subseteq \mathsf{reachable}(\mathcal{L}_n \times \mathcal{L}_2)$, done $\cap$ working $= \emptyset$, done $\cup$ working $\Subset^{\forall}$ antichain and working contains no duplicates.*

Theorem 3. *Algorithm 2 terminates for finite state, finitely branching LTSs.*

Proof. The total number of pairs present in $\mathsf{norm}_{\text{sfr}}(\mathcal{L}_1) \times \mathcal{L}_2$ and $\mathsf{norm}_{\text{fdr}}(\mathcal{L}_1) \times \mathcal{L}_2$ are finite since $\mathcal{L}_1$ and $\mathcal{L}_2$ are finite state. By Lemma 2 we find that, when not

CheckDiv, $\mathit{working} \cup \mathit{done} \subseteq \mathsf{reachable}(\mathsf{norm}_{\text{sfr}}(\mathcal{L}_1) \times \mathcal{L}_2)$. Likewise, we conclude $\mathit{working} \cup \mathit{done} \subseteq \mathsf{reachable}(\mathsf{norm}_{\text{fdr}}(\mathcal{L}_1) \times \mathcal{L}_2)$ when *CheckDiv*. Furthermore, as $\mathit{done} \cap \mathit{working} = \emptyset$, *done* strictly increases in size each iteration and so only a finite number of iterations of the outer for-loop are possible. Termination of the inner for-loop follows from the assumption that $\mathcal{L}_1$ and $\mathcal{L}_2$ are finitely branching. □

The correctness of the algorithm requires a lemma that shows anti-monotonicity of witnesses (*cf.* Def. 9); see below. Combined with Proposition 2 (see page 7) this allows us to conclude that the distance (defined below) from a state in the cartesian product to a witness is at least the distance to a witness from smaller states.

Lemma 3. *Let $(U, s), (V, s)$ be states of $\mathsf{norm}_{sfr}(\mathcal{L}_1) \times \mathcal{L}_2$ satisfying $(U, s) \leq (V, s)$. If (V, s) is an SFR-witness then (U, s) is an SFR-witness. Likewise, if (V, s) is an FDR-witness in $\mathsf{norm}_{fdr}(\mathcal{L}_1) \times \mathcal{L}_2$ and $(U, s) \leq (V, s)$ then (U, s) is an FDR-witness.*

For a set of states $\mathcal{U}$ in the cartesian product, let *SFR*($\mathcal{U}$) be a predicate that is true if and only if $\mathcal{U}$ contains an SFR-witness; likewise, *FDR*($\mathcal{U}$) holds if and only if $\mathcal{U}$ contains an FDR-witness. We denote the set of all reachable SFR-witnesses in the cartesian product $\mathsf{norm}_{\text{sfr}}(\mathcal{L}_1) \times \mathcal{L}_2$ by $\mathcal{S}$, and the set of all reachable FDR-witnesses in $\mathsf{norm}_{\text{fdr}}(\mathcal{L}_1) \times \mathcal{L}_2$ by $\mathcal{F}$. For a state (U, s) in the cartesian product, we define the *distance* to a set $\mathcal{U}$ of the cartesian product by $\mathit{Dist}_{\mathcal{U}}(U, s)$ as the shortest distance from state (U, s) to a state in $\mathcal{U}$. If $\mathcal{U}$ is unreachable, the distance is set to infinity. Formally, $\mathit{Dist}_{\mathcal{U}}(U, s) = \min\{|\sigma| \mid \exists (V, t) \in \mathcal{U} : (U, s) \xrightarrow{\sigma} (V, t)\}$. We generalise this to a set of states $\mathcal{V}$ as follows: $\mathit{Dist}_{\mathcal{U}}(\mathcal{V}) = \min\{\mathit{Dist}_{\mathcal{U}}(U, s) \mid (U, s) \in \mathcal{V}\}$.

Proposition 3. *For $(U, s), (V, s)$ in $\mathsf{norm}_{sfr}(\mathcal{L}_1) \times \mathcal{L}_2$ satisfying $(U, s) \leq (V, s)$ we have $\mathit{Dist}_{\mathcal{S}}(U, s) \leq \mathit{Dist}_{\mathcal{S}}(V, s)$. Likewise, for $(U, s), (V, s)$ in $\mathsf{norm}_{fdr}(\mathcal{L}_1) \times \mathcal{L}_2$ satisfying $(U, s) \leq (V, s)$ we have $\mathit{Dist}_{\mathcal{F}}(U, s) \leq \mathit{Dist}_{\mathcal{F}}(V, s)$.*

Proof. Follows from Lemma 3 and Proposition 2. □

We conclude with a sketch of the proof of correctness of the algorithm. The full proof can be found in [14].

Proof (Theorem 2). We prove both implications, by contraposition, for the case of failures-divergences refinement. The proof of correctness for stable failures refinement proceeds along the same lines.

- Assume that Algorithm 2 returns *false*. This occurs when the pair $(\mathit{spec}, \mathit{impl})$ satisfies the conditions of an FDR-witness, as follows from lines 7, 11 and 18 of Algorithm 2. Since $\mathit{working} \subseteq \mathsf{reachable}(\mathsf{norm}_{\text{fdr}}(\mathcal{L}_1) \times \mathcal{L}_2)$ and $(\mathit{spec}, \mathit{impl}) \in \mathit{working}$, the FDR-witness is reachable. By Theorem 1 we find that $\mathcal{L}_1 \sqsubseteq_{\text{fdr}} \mathcal{L}_2$ fails to hold.
- Assume that an FDR-witness is reachable in $\mathsf{norm}_{\text{fdr}}(\mathcal{L}_1) \times \mathcal{L}_2$, *i.e.*, $\mathcal{F} \neq \emptyset$. Then the following invariant holds in the while loop (lines 4-22):

  $$\mathit{Dist}_{\mathcal{F}}(\mathit{done}) > \mathit{Dist}_{\mathcal{F}}(\mathit{working}) \text{ and } \mathit{Dist}_{\mathcal{F}}(\mathit{working}) = \mathit{Dist}_{\mathcal{F}}(\mathit{antichain}).$$

  Towards a contradiction, assume Algorithm 2 returns *true*. This can only be the case when $\mathit{working}$ is empty. Upon termination of the while loop, $\mathit{Dist}_{\mathcal{F}}(\mathit{working}) =$

Dist$_{\mathcal{F}}(\emptyset) = \infty$. By the above invariants, *Dist*$_{\mathcal{F}}(\mathit{working}) =$ *Dist*$_{\mathcal{F}}(\mathit{antichain})$. Since $\iota = (\{s \in S_1 \mid \iota_1 \overset{\epsilon}{\rightsquigarrow}_1 s\}, \iota_2) \Subset \mathit{antichain}$ and *Dist*$_{\mathcal{F}}(\iota) < \infty$, we also have *Dist*$_{\mathcal{F}}(\mathit{antichain}) < \infty$. Contradiction. □

We remark that the correctness of the algorithm is independent of the search order that is used. That is, replacing the data structure for $\mathit{working}$ with a FIFO queue results in a breadth-first search strategy and does not impair the correctness of the algorithm. As explained in [17], breadth-first search has the advantage to yield the shortest possible counterexamples. Reconstructing such counterexamples can be done efficiently by recording, for each state stored in $\mathit{working}$, its breadth-first search level. We close this section by briefly returning to Example 3.

Example 5. Reconsider the family of transition systems of Example 3. Contrary to the original algorithm, the improved algorithm will, in each iteration, only add a single successor state to $\mathit{working}$, because every other successor will already be part of $\mathit{antichain}$. This results in $\mathit{working}$ containing $\mathcal{O}(1)$ entries; $\mathit{antichain}$ will be queried $\mathcal{O}(n{\cdot}k)$ times. Compared to the original algorithm, this reduces overhead for the depth-first search strategy by a factor $n{\cdot}k$ in the working stack size and by a factor k in the number of antichain checks. For the breadth-first search strategy, the $\mathit{working}$ size is reduced by a factor k^n and the $\mathit{antichain}$ checks by a factor k^n/n. □

6 Experimental Validation

We have conducted several benchmarks to compare the run time of both algorithms to show that solving the identified issues actually improves the run time performance in practice.

For this purpose we have implemented a depth-first and breadth-first variant of both Algorithm 1 and 2 in a branch of the mCRL2[2] tool set [5] as part of the `ltscompare` tool, which is implemented in C++. The implementation of the $\mathit{working}$ and $\mathit{antichain}$ operations are the same. For the implementation of refusals in Algorithm 1 we follow the definition of [21] (see also Footnote 1 on page 4), implementing refusals for any state, whereas for Algorithm 2 we follow Definition 3. The source modifications and experiments can be obtained from the downloadable package [15].

The experiments we consider are taken from three sources. First, Example 3 for exposing the performance overhead of the original algorithm. Second, several *linearisability tests* of concurrent data structures for more practical benchmarks. These models have been taken from [16], and consist of six implementations of concurrent data types that, when *trace-refining* their specifications, are guaranteed to be linearisable. As in [21], we approximate trace-refinement by the stronger stable failures refinement. For these models, the implementation and specification pairs are based on the same descriptions; the difference between the two is that the specification uses a simple construct to guarantee that each method of the concurrent data structure executes atomically. This significantly reduces the non-determinism and the number of transitions in the specification models.

Finally, an industrial model of a control system modelled in the Dezyne language [3] that first exposed the performance issues in practice. The industrial example is of a more

[2] www.mcrl2.org

traditional flavour in which the specification is an abstract description of the behaviours at the external interface of a control system, and the implementation is a detailed model that interacts with underlying services to implement the expected interface. For reasons of confidentiality, the industrial model cannot be made available.

All measurements have been performed on a machine with an Intel Core i7-7700HQ CPU 2.80Ghz and a 16GiB memory limit imposed by `ulimit -Sv 16777216`.

6.1 Benchmarking Example 3

Example 3 has been benchmarked for all combinations of $k, n \leq 500$, where k and n are multiples of 10, checking stable failures refinement between equivalent LTSs, *i.e.*, $\mathcal{L}_n^k \sqsubseteq_{\text{sfr}} \mathcal{L}_n^k$. Figure 1 shows the run time performance (in seconds) of the depth-first variant of Algorithm 1 on the left and Algorithm 2 on the right.

The plots match the asymptotic growth as stated in Example 3, illustrating a factor k speed-up of our algorithm compared to the original one. A comparison of the performance of breadth-first search is infeasible as the original algorithm already runs into the memory limit for small k and n, whereas for Algorithm 2 there is only little difference between the depth-first and breadth-first variants.

Note that due to the absence of τ-transitions, there is no performance difference in the computation of refusals in both algorithms. Consequently, the difference in performance is entirely due to the different way of inspecting and extending $working$.

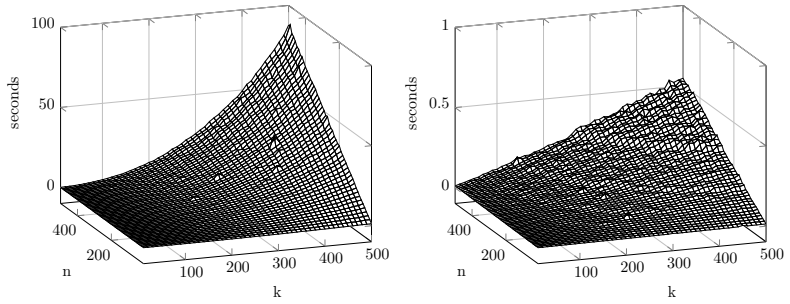


Fig. 1. The run time performance (in seconds) of Example 3 for depth-first search (left: original algorithm, right: our improved algorithm).

6.2 Benchmarking Practical Examples

Our next batch of experiments consists of more typical refinement checks, assessing whether the behaviours of the implementations are in line with the behaviours prescribed by their specifications. Characteristics of the state spaces are listed in Table 1.
The run time performance of both algorithms (both depth-first and breadth-first) can be found in Table 2. The run times we report are averages obtained from five runs. As illustrated by the figures in that table, we see small improvements of our algorithm over

Table 1. The size of the state space for each specification and associated implementation.

	Ref.	*specification*		$\sqsubseteq_{\text{sfr}}$	*implementation*	
Model		*#states*	*#transitions*		*#states*	*#transitions*
Coarse set	[11]	50 488	64 729	yes	55 444	145 043
Fine-grained set	[11]	3 720	3 305	yes	5 077	9 006
Lazy set	[11]	3 565	3 980	yes	24 496	41 431
Optimistic set	[11]	25 435	28 154	yes	234 332	389 344
Non-blocking queue	[19]	1 248	1 473	no	3 030	5 799
Treiber stack	[20]	87 389	124 740	yes	205 634	564 862
Industrial		24	45	yes	24 551	45 447

the original algorithm for depth-first search, whereas the improvements for breadth-first search are dramatic.

Table 2. Run time comparison between Algorithm 1 and Algorithm 2 using both a depth-first and breadth-first search strategy. All run times are in seconds; † indicates an out-of-memory issue indicating that the algorithm failed to complete within the imposed 16GiB memory limit.

	depth-first (sec.)		*breadth-first* (sec.)	
Model	Alg. 1	Alg. 2	Alg. 1	Alg. 2
Coarse set	9.15	8.61	†	9.06
Fine-grained set	0.37	0.32	†	0.46
Lazy set	1.19	1.02	†	1.26
Optimistic set	16.96	14.13	†	22.67
Non-blocking queue	0.03	0.02	0.17	0.09
Treiber stack	148.39	137.52	†	352.59
Industrial	1.36	0.15	296.29	0.17

To better understand the reason behind the performance gains we obtain, we report on the maximal size of $working$ and $antichain$, and the number of successful and unsuccessful antichain membership tests; see Table 3. We only report on metrics for the breadth-first search strategy; the figures for the depth-first search strategy for both algorithms are similar; see [14].

For the breadth-first search strategy, the fact that the original algorithm delays adding state pairs to the antichain induces an enormous overhead in the size of $working$ due to the many failing antichain checks. This can be seen from the large size of $working$ and the small size of $antichain$. Because of these differences in size, most antichain membership tests fail in the original algorithm. The situation is largely reversed in our improved algorithm, explaining the substantial performance improvements we observe. Since the original algorithm for failures-divergences refinement is incorrect, we only compared the performance of both algorithms for stable failures refinement. The performance of our failures-divergences refinement algorithm is comparable to our stable failures refinement algorithm; we refer to [14] for further details. In [14], we also performed additional experiments which show that further run time improvements can be obtained by applying divergence-preserving branching bisimulation [7] minimisation as a preprocessing step to refinement checking.

Table 3. Metrics for the breadth-first search strategy experiments. For the original algorithm most of these figures are *under-approximations* due to the out-of-memory issue. All values we report on are in thousands (*i.e.*, the actual number is obtained by multiplying with 10^3).

	max size $working$		max size $antichain$		$antichain$-hits		$antichain$-misses	
Model	Alg. 1	Alg. 2	Alg. 1	Alg. 2	Alg. 1	Alg. 2	Alg. 1	Alg. 2
Coarse set	4 710.3	3.4	3.6	55.4	13.9	96.2	7 807.4	60.3
Fine-grained set	6 604.5	0.4	1.9	5.1	180.7	7.2	15 547.9	9.7
Lazy set	6 726.4	1.7	4.3	24.5	130.5	24.3	14 852.8	35.2
Optimistic set	6 366.5	15.2	4.4	234.4	38.7	292.5	14 238.0	434.2
Non-blocking queue	6.3	0.3	0.3	2.7	3.1	342.6	14.6	4.0
Treiber stack	5 829.9	139.2	4.8	214.8	76.1	2 411.6	8 340.6	1 523.8
Industrial	549.2	224.3	43.1	43.1	54 591.1	36.4	12 888.4	43.1

7 Conclusions

Our study of the antichain-based algorithms for deciding stable failures refinement and failures-divergences refinement presented in [21] revealed that the failures-divergences refinement algorithm is incorrect; both algorithms perform suboptimally when implemented using a depth-first search strategy and poorly when implemented using a breadth-first search strategy. Moreover, both violate the claimed antichain property. We have proposed alternative algorithms for which we showed correctness and which utilise proper antichains. Our experiments indicate significant performance improvements for deciding stable failures refinement and a performance of deciding failures-divergences refinement that is comparable to deciding stable failures refinement.

References

1. Abdulla, P.A., Chen, Y., Holík, L., Mayr, R., Vojnar, T.: When simulation meets antichains. In: Esparza, J., Majumdar, R. (eds.) TACAS 2010. LNCS, vol. 6015, pp. 158–174. Springer (2010). https://doi.org/10.1007/978-3-642-12002-2_14
2. Bergstra, J.A., Klop, J.W., Olderog, E.: Failures without chaos: a new process semantics for fair abstraction. In: Wirsing, M. (ed.) IFIP TC 2/WG 2.2 1986. pp. 77–104. North-Holland (1987)
3. van Beusekom, R., Groote, J.F., Hoogendijk, P.F., Howe, R., Wesselink, W., Wieringa, R., Willemse, T.A.C.: Formalising the Dezyne modelling language in mCRL2. In: Petrucci, L., Seceleanu, C., Cavalcanti, A. (eds.) FMICS-AVoCS 2017. LNCS, vol. 10471, pp. 217–233. Springer (2017). https://doi.org/10.1007/978-3-319-67113-0_14
4. Brookes, S.D., Roscoe, A.W.: An improved failures model for communicating processes. In: Brookes, S.D., Roscoe, A.W., Winskel, G. (eds.) Seminar on Concurrency 1984. LNCS, vol. 197, pp. 281–305. Springer (1984). https://doi.org/10.1007/3-540-15670-4_14, https://doi.org/10.1007/3-540-15670-4_14
5. Bunte, O., Groote, J.F., Keiren, J.J.A., Laveaux, M., Neele, T., de Vink, E.P., Wesselink, W., Wijs, A., Willemse, T.A.C.: The mCRL2 toolset for analysing concurrent systems - improvements in expressivity and usability. In: Vojnar, T., Zhang, L. (eds.) TACAS (2). LNCS, vol. 11428, pp. 21–39. Springer (2019). https://doi.org/10.1007/978-3-030-17465-1_2
6. Gibson-Robinson, T., Armstrong, P.J., Boulgakov, A., Roscoe, A.W.: FDR3 - A modern refinement checker for CSP. In: Ábrahám, E., Havelund, K. (eds.) TACAS 2014. LNCS, vol. 8413, pp. 187–201. Springer (2014). https://doi.org/10.1007/978-3-642-54862-8_13

7. van Glabbeek, R.J., Luttik, B., Trcka, N.: Branching bisimilarity with explicit divergence. Fundam. Inform. **93**(4), 371–392 (2009). https://doi.org/10.3233/FI-2009-109
8. van Glabbeek, R.J.: Personal Communication, 7 January 2019
9. van Glabbeek, R.J.: A branching time model of CSP. In: Gibson-Robinson, T., Hopcroft, P.J., Lazic, R. (eds.) Concurrency, Security, and Puzzles - Essays Dedicated to Andrew William Roscoe on the Occasion of His 60th Birthday. LNCS, vol. 10160, pp. 272–293. Springer (2017). https://doi.org/10.1007/978-3-319-51046-0_14
10. Gomes, A.O., Butterfield, A.: Modelling the haemodialysis machine with circus. In: Butler, M.J., Schewe, K., Mashkoor, A., Biró, M. (eds.) ABZ 2016. LNCS, vol. 9675, pp. 409–424. Springer (2016). https://doi.org/10.1007/978-3-319-33600-8_34
11. Herlihy, M., Shavit, N.: The art of multiprocessor programming. Morgan Kaufmann (2008)
12. Hoare, C.: Communicating Sequential Processes. Prentice-Hall (1985)
13. Kanellakis, P.C., Smolka, S.A.: CCS expressions, finite state processes, and three problems of equivalence. Inf. Comput. **86**(1), 43–68 (1990). https://doi.org/10.1016/0890-5401(90)90025-D
14. Laveaux, M., Groote, J.F., Willemse, T.A.C.: Correct and efficient antichain algorithms for refinement checking. CoRR **abs/1902.09880** (2019)
15. Laveaux, M.: Downloadable sources and benchmarks for the experimental validation (2019). https://doi.org/10.5281/zenodo.2573095
16. Paval, R.: Modeling and Verifying Concurrent Data Structures. Master's thesis, Eindhoven University of Technology (2018), https://research.tue.nl/files/93882157/Thesis_Roxana_Paval.pdf
17. Roscoe, A.W.: Model-checking CSP. In: Roscoe, A. (ed.) A Classical Mind: essays in Honour of C.A.R. Hoare, chap. 21, pp. 353–378. Prentice Hall International (UK) Ltd. (1994)
18. Roscoe, A.W.: Understanding Concurrent Systems. Texts in Computer Science, Springer (2010). https://doi.org/10.1007/978-1-84882-258-0
19. Shann, C., Huang, T., Chen, C.: A practical nonblocking queue algorithm using compare-and-swap. In: ICPADS 2000. pp. 470–475. IEEE Computer Society (2000). https://doi.org/10.1109/ICPADS.2000.857731
20. Treiber, R.K.: Systems programming: Coping with parallelism. International Business Machines Incorporated, Thomas J. Watson Research (1986)
21. Wang, T., Song, S., Sun, J., Liu, Y., Dong, J.S., Wang, X., Li, S.: More anti-chain based refinement checking. In: Aoki, T., Taguchi, K. (eds.) ICFEM. LNCS, vol. 7635, pp. 364–380. Springer (2012). https://doi.org/10.1007/978-3-642-34281-3_26