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Dynamic equivalence and flatness of control systems: some results and open questions

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Everything is C^ω .

Control systems

$$\dot{x} = f(x, u) \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m, 0 < m \leq n, \quad (*)$$

studied locally around some $(\bar{x}, \bar{u}) \in \mathbb{R}^{n+m}$,
or around some germ/jet of solution $t \mapsto (\bar{x}(t), \bar{u}(t))$.

“**Geometry**”: f fiber preserving map $\mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{T}\mathbb{R}^n$.

$f(\mathbb{T}\mathbb{R}^n)$ is a

sub-bundle of $\mathbb{T}\mathbb{R}^n$ if $\text{Rank } \frac{\partial f}{\partial u}$ constant,

affine sub-bundle if $f(x, u) = X_0(x) + \sum_1^m u_k X_k(0)$,

distribution if $X_0 = 0$...

Underdetermined system of ODEs: the general solution
 $t \mapsto (x(t), u(t))$ depends on m arbitrary functions of time $u(\cdot)$ and
 n arbitrary constants $x(0)$. Important object:

“Set of solutions” for under-determined ODEs

\mathcal{B} = set of all (germs of) $t \mapsto (x(t), u(t))$ solution of (*).

(behavior?) (tribute to J. Willems)

“Static” equivalence

$$\begin{aligned}(\Sigma) \quad & \dot{x} = f(x, u), \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m \\(\Sigma') \quad & \dot{z} = g(z, v), \quad z \in \mathbb{R}^{n'}, v \in \mathbb{R}^{m'}\end{aligned}$$

Definition (local static equivalence)

(Σ) and (Σ') are locally static equivalent at $(\bar{x}, \bar{u})/(\bar{z}, \bar{v})$ iff there is a diffeo $(\varphi, \psi) : \mathcal{O}_{(\bar{x}, \bar{u})} \rightarrow \mathcal{O}_{(\bar{z}, \bar{v})}$, of the form $(x, u) \mapsto (\varphi(x), \psi(x, u))$, (bundle isomorphism) that conjugates f to g .

If control affine & constant control rank, then

- local in state only,
- $\varphi : \mathcal{O}_{\bar{x}} \rightarrow \mathcal{O}_{\bar{z}}$ such that φ_* conjugates affine sub-bundles.

Alternative definition:

$(\varphi, \psi) : \mathcal{O}_{(\bar{x}, \bar{u})} \rightarrow \mathcal{O}_{(\bar{z}, \bar{v})}$ maps (by composition), germs of solutions of Σ to germs of solutions of Σ' , and vice versa.

Deciding static equivalence

It is feasible. Given Σ and Σ' , there is in principle a finite algorithm to write PDEs for φ, ψ and decide (differential elimination) whether there is an obstruction of system formally integrable, Cauchy-Kovalevska, analytic solution.

Geometric study. Invariants of distributions, affine sub-bundles. In non-affine case, affine geometry for submanifolds of $\mathbb{T}\mathbb{R}^n$.

A lot of litterature.

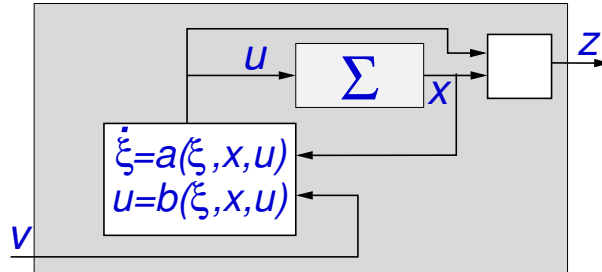
Functionnal moduli, hence equivalence is a “rare” property, or classes are very thin.

Exact linearization: characterization of static equivalence to a linear controllable system [Jakubczyk-Respondek, 80].

Dynamic feedback linearization

Since Static linearization is very restrictive, try dynamic ! (1980's)

Pre-compensator such that $v \rightarrow z$ linear controllable:



[Isidori-Moog-de Luca 86]: performing dynamic decoupling, full linearization may occur.

Decide pre-compensator, check for static feedback linearizability

[Charlet-Lévine-Marino, 91]

► Sufficient conditions.

Linear controllable systems, linearization

Linear controllable system (Σ) : $\dot{x} = Ax + Bu$.

$$\text{Rank}\{B, AB, \dots\} = n.$$

Transformation $z = Px$, $v = Kx + Qu$

with $z = (z_{1,1}, \dots, z_{1,r_1}, \dots, z_{m,1}, \dots, z_{m,r_m})$ $\sum r_k = n$

yields $z_{k,1}^{(r_k)} = v_k$, $1 \leq k \leq m$. Brunovsky canonical form.

The general solution is uniquely defined by m arbitrary functions of time $z_{1,1}, z_{2,1}, \dots, z_{m,1}$ (and no initial conditions).

[Fliess-Lévine-Martin-Rouchon 91]: system $\dot{x} = f(x, u)$ is “**flat**”

(at $\bar{x}, \bar{u}, \dots, \bar{u}^{(j)}$) iff there is a formula giving the general solution a function of arbitrary y_1, \dots, y_m and time-derivatives **and** y_1, \dots, y_m may also be recovered from x, u and derivatives.

(y_1, \dots, y_m) is called a **flat output**.

This yields the dynamic precompensator.

Equivalence

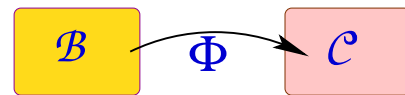
[É. Cartan, “sur l'équivalence absolue...”, 1914]:

La première idée qui vient à l'esprit, et qu'il s'agira de préciser, est la suivante : *deux systèmes seront dits « absolument équivalents » lorsqu'on pourra établir une correspondance univoque (au moins dans un champ fonctionnel suffisamment petit) entre les solutions de ces deux systèmes.*

$$\begin{aligned}
 (\Sigma) \quad \dot{x} &= f(x, u), & x \in \mathbb{R}^n, & u \in \mathbb{R}^m, & \mathcal{B} &= \{\text{solutions}\} \\
 (\Sigma') \quad \dot{z} &= g(z, v), & z \in \mathbb{R}^{n'}, & v \in \mathbb{R}^{m'}, & \mathcal{C} &= \{\text{solutions}\}
 \end{aligned}$$

Definition

Two systems are “**equivalent**” iff their (germs of) solutions are in **one-to-one correspondence**.



The nature of Φ matters a lot !!

Dynamic Feedback Transformations

$$\begin{array}{ccc}
 \Sigma & & \Sigma' \\
 \dot{x} = f(x, u) & \begin{array}{c} \left(\begin{array}{c} x \\ u \\ \dot{u} \\ \ddot{u} \\ \vdots \end{array} \right) & \begin{array}{c} (z, v) = \phi(x, u, \dots, u^{(K)}) \\ \longrightarrow \\ \longleftarrow \\ (x, u) = \psi(z, v, \dots, v^{(K')}) \end{array} & \left(\begin{array}{c} z \\ v \\ \dot{v} \\ \ddot{v} \\ \vdots \end{array} \right) \\
 & & & & \dot{z} = g(z, v)
 \end{array}$$

(Local) dynamic equivalence: $\exists \varphi : \mathcal{O}_{(\bar{x}, \bar{u}, \bar{\dot{u}}, \dots, \bar{u}^{(j)})} \rightarrow \mathbb{R}^{n'} \times \mathbb{R}^{m'}$, $\psi : \mathcal{O}_{(\bar{z}, \bar{v}, \bar{\dot{v}}, \dots, \bar{v}^{(j')})} \rightarrow \mathbb{R}^n \times \mathbb{R}^m$ such that the above induces a univocal transformation on solutions.

Flatness

Flatness: Σ is flat if this holds with Σ' trivial:

$$\Sigma \quad \begin{pmatrix} x \\ u \\ \dot{u} \\ \vdots \end{pmatrix} \quad \begin{matrix} v = \phi(x, u, \dots, u^{(K)}) \\ \longrightarrow \\ \longleftarrow \\ (x, u) = \psi(v, \dots, v^{(K')}) \end{matrix} \quad \begin{pmatrix} v \\ \dot{v} \\ \ddot{v} \\ \vdots \end{pmatrix} \quad \Sigma' \quad \text{no relation}$$

Initially, [Fliess & al] stated this in terms of differential fields extensions. Obviously more restrictive (ϕ, ψ should be algebraic).

Equivalence can however be translated into isomorphism of differential algebras or conjugation of a vector field on a (infinite) jet manifold.

Further characterizations

Differential algebra: $\mathcal{A}_\Sigma = \{(\text{germs of}) C^\omega \text{ functions of a finite number of variables among } x, u, \dot{u}, \dots\}$,

dérivation: $F = f(x, u) \frac{\partial}{\partial x} + \dot{u} \frac{\partial}{\partial u} + \ddot{u} \frac{\partial}{\partial \dot{u}} + \dots$

[Jakubczyk 1993]: Σ is equivalent to Σ' if and only \mathcal{A}_Σ and $\mathcal{A}_{\Sigma'}$ are isomorphic diff. algebras.

[Fliess & al.], [JBP] 1992: it also translates into a diffeomorphism between “manifolds” where coordinates are x, u, \dot{u}, \dots and z, v, \dot{v}, \dots , that conjugates F to G .

Similar to Lie-Bäcklund transformations.

Checkable conditions ?

How to decide flatness of a system Σ ?

or equivalence of Σ to Σ' ?

Invariants ?

- ① In principle, if K and K' are fixed, one may decide upon existence of ϕ and ψ , that depend on a fixed number of variables. (see static feedback)
- ② ... **but no known a priori bound on K, K' !!**
- ③ However, many physical systems are flat and this is useful.
- ④ A lot of current work on all possible choices of flat outputs (they are far from unique), respecting symmetries, etc... see [Respondek, Nicolau et al], [Murray et al.], [Rouchon, Martin et al.] ...

Checkable conditions ?

Difficult point is: necessary conditions.

If no ϕ, ψ for some K , why not for $K + 1$?

Invariants:

- ① m (number of inputs) is an invariant of dynamic equivalence.
- ② If $m = 1$, dynamic equivalence is static equivalence. [Charlet et al., 1991], [JBP 1993].
- ③ Singular curves are an obstruction to flatness: if $t \mapsto (x(t), u(t))$ is singular, flatness fails at any truncated jet $(x(0), u(0), \dot{u}(0), \dots, u^{(j)}(0))$ [common knowledge ?]
- ④ Ruled manifold criterium

Ruled systems

Regularity assumption

Σ and Σ' define smooth (C^ω) sub-bundles of $T\mathbb{R}^n$ and $T\mathbb{R}^{n'}$.

E.g. $\text{Rank} \frac{\partial f}{\partial u} = m$, $\text{Rank} \frac{\partial g}{\partial v} = m'$

- $\Sigma \rightarrow \mathbb{R}^n$ sub-bundle of $T\mathbb{R}^n \rightarrow \mathbb{R}^n$.
 $\Sigma_x = \{f(x, u), u \in \mathbb{R}^m\}$ sub-manifold of the linear space $T_x\mathbb{R}^n$.
- A submanifold of an affine space is called (locally) **ruled** if it is a union of straight lines (locally of open segments).

Definition (ruled system)

Σ is ruled iff each Σ_x is a ruled submanifold of $T_x\mathbb{R}^n (\approx \mathbb{R}^n)$.

Note: This property is preserved by static equivalence.

Necessary condition for flatness

$$(\Sigma) \quad \dot{x} = f(x, u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad (\Sigma') \text{ "trivial" .}$$

Theorem ([Rouchon], [Sluis], 1992)

A flat system must be ruled

What if

$$\begin{array}{l} (\Sigma) \quad \dot{x} = f(x, u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \\ (\Sigma') \quad \dot{z} = g(z, v), \quad z \in \mathbb{R}^{n'}, \quad v \in \mathbb{R}^{m'} \end{array} \quad \text{dynamic equivalent?}$$

Necessary condition for equivalence

$$\begin{aligned}(\Sigma) \quad & \dot{x} = f(x, u), \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m \\(\Sigma') \quad & \dot{z} = g(z, v), \quad z \in \mathbb{R}^{n'}, v \in \mathbb{R}^{m'}\end{aligned}$$

Theorem (JBP, 2009)

If Σ and Σ' are dynamic equivalent and $n = n'$, then, locally,

- ① either they are **static equivalent**,
- ② or they are **both ruled**.

If $n < n'$, Σ' must be ruled.

In the C^∞ case, ① and ② may both occur for the same Σ, Σ' .

Note: the condition for flatness is a consequence (trivial system is ruled).

Flatness with 2 controls and 3 states

$$\dot{z} = f(x, y, z, \dot{x}, \dot{y})$$

[Avanessof, JBP 2007].

We look for a parameterization x, y, z function of $(u, \dot{u}, \dots, u^{(k)}, v, \dots, v^{(\ell)})$, $k \leq \ell$, such that all solutions are covered. one has

- Either no parameterization because of ruled criterion of non-controllability,
- or parameterization with $(k, \ell) = (1, 2)$,
- or there could be a parameterization, but k and ℓ have to be no smaller than 3 and the largest one no smaller than 4.

Examples

Example 1: $\dot{z} = y + (\dot{y} - z\dot{x})\dot{x}$. Parameterization of order (1, 2) around jets such that $\ddot{x} + \dot{x}^3 \neq 1$, given by :

$$x = v, \quad y = \frac{\dot{v}^2 u + \dot{v}}{\ddot{v} + \dot{v}^3 - 1}, \quad z = \frac{(1 - \ddot{v})u + \dot{v}\dot{v}}{\ddot{v} + \dot{v}^3 - 1}. \quad (*)$$

Formulas can be “inverted” by $u = -z + y\dot{x}$, $v = x$, hence flatness. Note on singularity: $\ddot{x} + \dot{x}^3 = 1$ is the equation of the singular curves. There is a parameterization of higher order at jets of order 3 such that $x^{(3)} + 3\dot{x}^2\ddot{x} \neq 0$.

Example 2: $\dot{z} = y + (\dot{y} - z\dot{x})^2\dot{x}$: if there is a parameterization, it has order at least (3, 4). Conjecture: no parameterization.

Open questions

- ① Given Σ , give an a priori bound on the number K of derivatives.
- ② Prove that at least one system for which above mentioned necessary conditions do not work is not flat, or does not admit a parameterization.
- ③ Does existence of a parameter imply flatness ? (exogenous implies exogenous, according to [Fliess et al.])

Thank you all for attention, and
thank you for your action in the community, Bronek !