Sharp stability analysis for high-order finite element discretizations of general wave propagation problems

Théophile Chaumont-Frelet, Serge Nicaise

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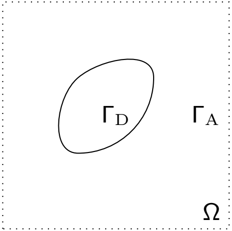
Recent advances on computational wave propagation

Model problem: the Helmholtz equation in a heterogeneous medium

Given f, we seek u such that

$$\left\{\begin{array}{rcll} -\frac{\omega^2}{\kappa}u - \nabla\cdot\left(\frac{1}{\rho}\nabla u\right) &=& f & \text{in } \Omega,\\ u &=& 0 & \text{on } \Gamma_{\mathrm{D}},\\ \frac{1}{\rho}\nabla u\cdot \boldsymbol{n} - \frac{i\omega}{\sqrt{\kappa\rho}}u &=& 0 & \text{on } \Gamma_{\mathrm{A}},\end{array}\right.$$



where:
Ω is a (sufficiently) smooth domain, $\partial\Omega = \overline{\Gamma_{\mathrm{D}}} \cup \overline{\Gamma_{\mathrm{A}}}$,
κ and ρ are (sufficiently) smooth strictly positive functions,
$\omega \geq 0$ is the angular frequency.

Variational formulation

Classically, assuming $f \in L^2(\Omega)$, we recast the original problem into:

Variational formulation

Find $u \in H^1_{\Gamma_{\mathrm{D}}}(\Omega)$ such that

$$b(u,v) = (f,v),$$

where

$$b(u,v) = -\omega^2 \left(\frac{1}{\kappa} u, v\right) - i\omega \left\langle \frac{1}{\sqrt{\kappa\rho}} u, v \right\rangle_{\Gamma_{\mathrm{A}}} + \left(\frac{1}{\rho} \nabla u, \nabla v\right).$$

We equip the space $H^1_{\Gamma_{\mathrm{D}}}(\Omega)$ with the following "energy" norm

$$\|v\|^2_{1,\omega,\Omega} = \omega^2 \|v\|^2_{0,\Omega} + |v|^2_{1,\Omega},$$

that is motivated by the coefficients in $b(\cdot,\cdot)$.

FEM discretization

We introduce a conforming finite element subspace

$$V_h = \left\{ v_h \in H^1_{\Gamma_{\mathrm{D}}}(\Omega) \mid v_h|_K \in \mathcal{P}_p(K)\ \forall K \in \mathcal{T}_h \right\} \subset H^1_{\Gamma_{\mathrm{D}}}(\Omega).$$

that is build on polynomials of degree p.

Discrete solution

Find $u_h \in V_h$ such that

$$b(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h.$$

We are interested in stability properties of FEM in the high-frequency regime.

Under which condition is the finite element method stable?
What is the influence of the polynomial degree p?

These questions are linked to the “approximation factor”

For $\phi \in L^2(\Omega)$, define $u^\star_\phi$ as the unique element of $H^1_{\Gamma_D}(\Omega)$ such that

$$b(w, u^\star_\phi) = (w, \phi) \quad \forall w \in H^1_{\Gamma_D}(\Omega).$$

Approximation factor

$$\eta = \sup_{\phi \in L^2(\Omega)\setminus\{0\}} \frac{\|u^\star_\phi - \mathcal{I}_h u^\star_\phi\|_{1,\omega,\Omega}}{\|\phi\|_{0,\Omega}}$$

It is the best constant such that

$$\|u^\star_\phi - \mathcal{I}_h u^\star_\phi\|_{1,\omega,\Omega} \leq \eta \|\phi\|_{0,\Omega} \quad \forall \phi \in L^2(\Omega).$$

η illustrate how well V_h can represent continuous solutions.
Most of this talk is devoted to a careful analysis of this quantity.

Outline

1 Relationship between approximation factor and stability of FEM
   - Warm up exercise: the zero frequency case
   - The high-frequency case

2 Upper bounds for the approximation factor and stability conditions
   - Settings
   - A naive approach
   - A regularity splitting for general media

3 Numerical experiments and sharpness of the main results
   - A methodology to illustrate the main results
   - Numerical experiments

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The zero frequency case

We set $\omega = 0$. Our problem becomes:

Variational formulation

Find $u \in H^1_{\Gamma_D}(\Omega)$ such that

$$b(u, v) = (f, v) \quad \forall v \in H^1_{\Gamma_D}(\Omega),$$

with

$$b(u, v) = \left(\frac{1}{\rho} \nabla u, \nabla v \right).$$

This is a text-book problem, often called the Poisson problem.

The sesquilinear form is continuous and coercive

$$|b(u, v)| \lesssim \|u\|_{1,\omega,\Omega} \|v\|_{1,\omega,\Omega} \qquad \operatorname{Re} b(v, v) \gtrsim \|v\|^2_{1,\omega,\Omega}.$$

Céa's lemma: the FEM is stable!

We immediatly obtain quasi-optimality (Céa's lemma):

$$
\begin{aligned}
\|u - u_h\|^2_{1,\omega,\Omega} &\lesssim b(u - u_h, u - u_h) && \text{(Coercivity)} \\
&= b(u - u_h, u - \mathcal{I}_h u) && \text{(Galerkin's orthogonality)} \\
&\lesssim \|u - u_h\|_{1,\omega,\Omega} \|u - \mathcal{I}_h u\|_{1,\omega,\Omega}. && \text{(Continuity)}
\end{aligned}
$$

Quasi-optimality

$$\|u - u_h\|_{1,\omega,\Omega} \lesssim \|u - \mathcal{I}_h u\|_{1,\omega,\Omega}.$$

Illustration by a toy example

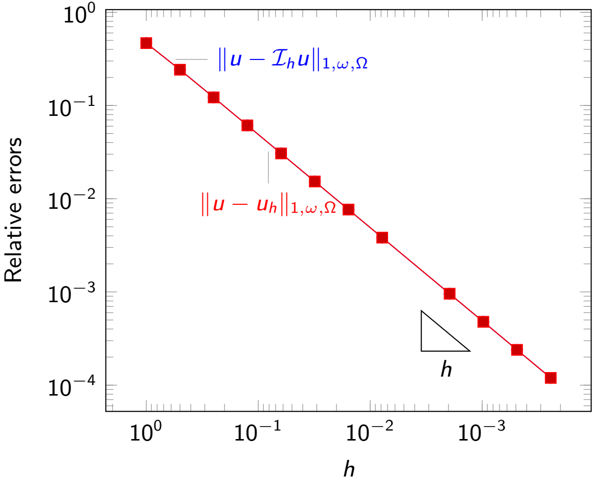
We consider the following 1D toy problem

$$\left\{\begin{array}{rcl} -u''(x) & = & (\pi/2)^2 \sin(\pi x/2) \\ u(0) & = & 0, \\ u'(1) & = & 0, \end{array}\right.$$

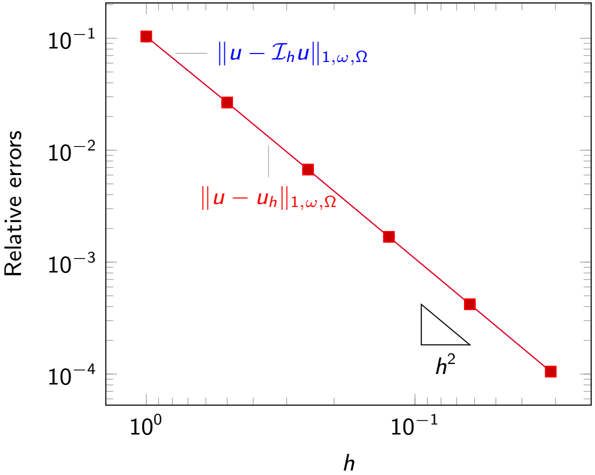
whose solution is given by

$$u(x) = \sin\left(\frac{\pi x}{2}\right).$$

Convergence of the $\mathcal{P}_1$ FEM



Convergence of the $\mathcal{P}_2$ FEM



Quick comments on the zero-frequency case

The discrete solution is quasi-optimal for arbitrary meshes.

The FEM is stable on arbitrary meshes:
up to constant, we get the best possible discrete representation.

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Let's see how the FEM is doing!

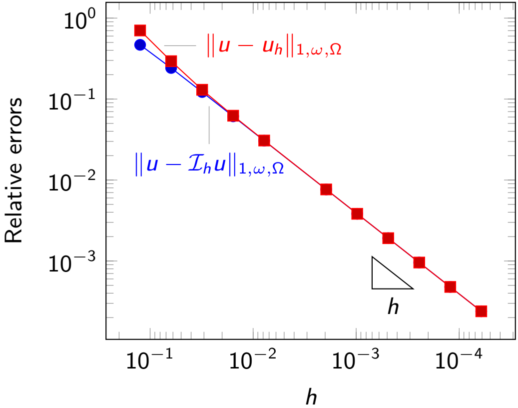
To do so, let us consider the following 1D toy problem

$$\left\{ \begin{array}{rcll} -\omega^2 u(x) - u''(x) & = & 1, & x \in (0,1), \\ u(0) & = & 0, & \\ u'(1) - iku(1) & = & 0, & \end{array} \right.$$

whose solution is given by

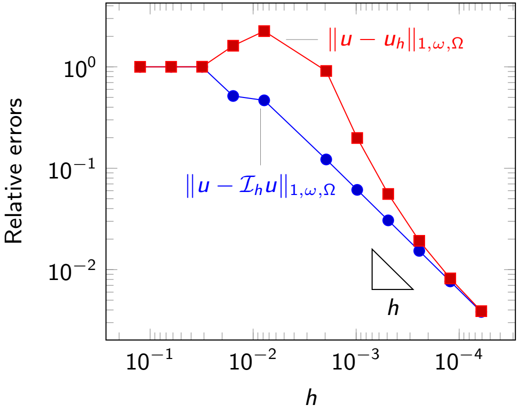
$$u(x) = \frac{1}{\omega^2}(1 - \cos(\omega x)).$$

A low frequency case with $\mathcal{P}_1$ elements: $\omega = 4\pi$



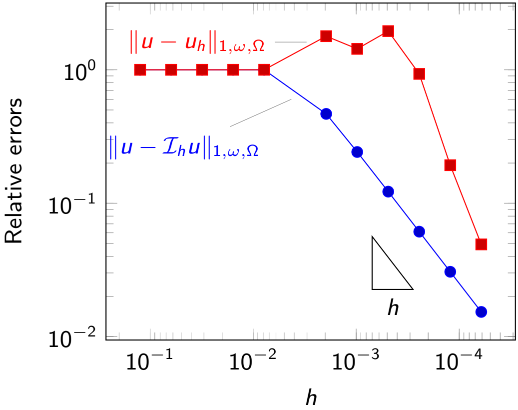
The behaviour is comparable to the coercive case.

A high frequency case with $\mathcal{P}_1$ elements: $\omega = 64\pi$



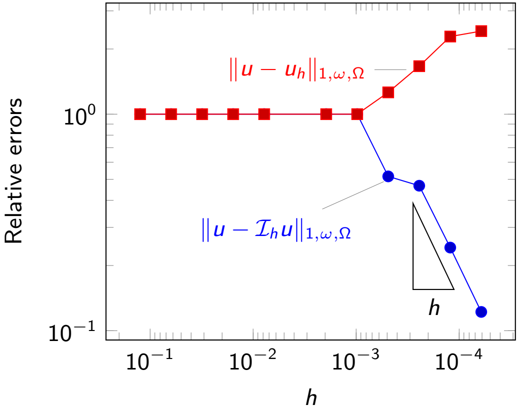
We observe a “gap” between the FEM and the interpolation error.

A high frequency case with $\mathcal{P}_1$ elements: $\omega = 256\pi$



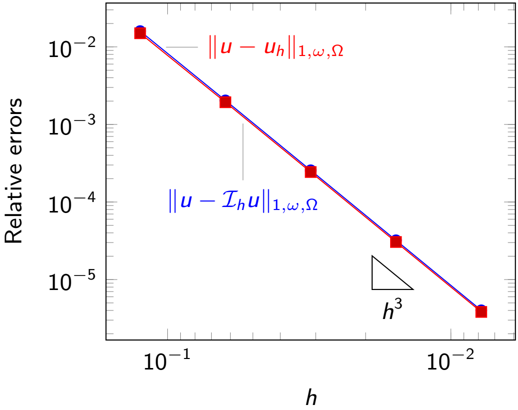
The "gap" is more important for this higher frequency.

A high frequency case with $\mathcal{P}_1$ elements: $\omega = 2048\pi$



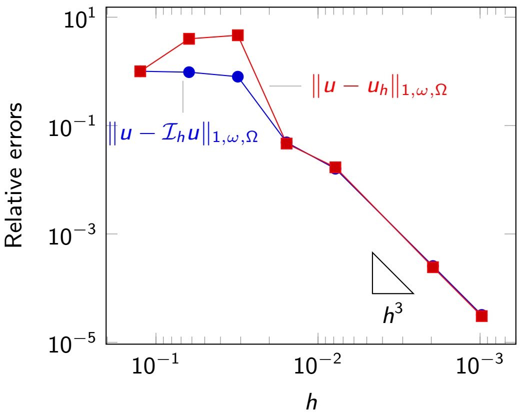
It looks like the FEM is not even converging!

A low frequency case with $\mathcal{P}_3$ elements: $\omega = 4\pi$



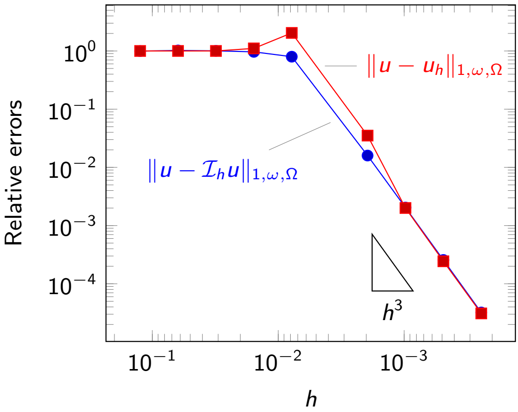
The behaviour is comparable to the coercive case.

A high frequency case with $\mathcal{P}_3$ elements: $\omega = 64\pi$



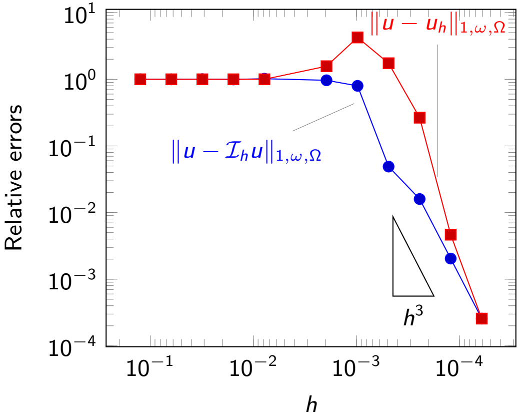
The FEM look stable.

A high frequency case with $\mathcal{P}_3$ elements: $\omega = 256\pi$



The FEM still look stable.

A high frequency case with $\mathcal{P}_3$ elements: $\omega = 2048\pi$



We finally see the gap.

Some observations

The FEM is not stable for all meshes:
there is a “gap” between the interpolation and finite element errors.

This gap increases with the frequency,
but seems less important for $p = 3$ than $p = 1$.

It is often called “pollution effect” in the literature.
The remaining of this talk is devoted to a precise analysis of this phenomenon.

The sesquilinear form reads

$$b(u,v) = -\omega^2 \left(\frac{1}{\kappa} u, v \right) - i\omega \left\langle \frac{1}{\sqrt{\kappa\rho}} u, v \right\rangle_{\Gamma_{\mathrm{A}}} + \left(\frac{1}{\rho} \nabla u, \nabla v \right),$$

where ω is large.

Hence, it is continuous

$$|b(u,v)| \lesssim \|u\|_{1,\omega,\Omega} \|v\|_{1,\omega,\Omega},$$

but not coercive

$$\operatorname{Re} b(v,v) \gtrsim \|v\|_{1,\omega,\Omega}^2 - \omega^2 \|v\|_{0,\Omega}^2.$$

The Schatz argument

We would like to employ Céa's Lemma, but we only have a Gårdling inequality:

$$|b(u-u_h, u-u_h)| \gtrsim \|u-u_h\|^2_{1,\omega,\Omega} - \omega^2 \|u-u_h\|^2_{0,\Omega}.$$

We cannot employ Céa's Lemma directly, as we are missing coercivity.

We can tackle this problem using the "Schatz argument".

A.H. Schatz, 1974
An observation concerning Ritz-Galerkin methods with indefinite bilinear forms.
J. Douglas, J.E. Santos, D. Sheen, L.S. Bennethum, 1993,
Frequency domain treatment of one-dimensional scalar waves.
F. Ihlenburg, I. Babuška, 1995,
Finite element solution of the Helmholtz equation with high wave number. Part I: The h-version of the FEM.

The Aubin-Nitsche trick

Set $\phi = u - u_h \in L^2(\Omega)$, and define $u_\phi^\star \in H^1_{\Gamma_{\rm D}}(\Omega)$ such that

$$b(w, u_\phi^\star) = (w, \phi) = (w, u - u_h) \quad \forall w \in H^1_{\Gamma_{\rm D}}(\Omega),$$

Then, picking the test function $w = u - u_h$, we have

$$\begin{aligned} \|u - u_h\|^2_{0,\Omega} = b(u - u_h, u_\phi^\star) = b(u - u_h, u_\phi^\star - \mathcal{I}_h u_\phi^\star) \\ \lesssim \|u - u_h\|_{1,\boldsymbol{\omega},\Omega} \|u_\phi^\star - \mathcal{I}_h u_\phi^\star\|_{1,\boldsymbol{\omega},\Omega}. \end{aligned}$$

On the other hand by definition of $\boldsymbol{\eta}$:

$$\|u_\phi^\star - \mathcal{I}_h u_\phi^\star\|_{1,\boldsymbol{\omega},\Omega} \leq \boldsymbol{\eta} \|\phi\|_{0,\Omega} = \boldsymbol{\eta} \|u - u_h\|_{0,\Omega}.$$

Aubin-Nitsche trick

$$\|u - u_h\|_{0,\Omega} \lesssim \boldsymbol{\eta} \|u - u_h\|_{1,\boldsymbol{\omega},\Omega}.$$

Céa's Lemma

We have established that $\|u - u_h\|_{0,\Omega} \leq C\boldsymbol{\eta}\|u - u_h\|_{1,\boldsymbol{\omega},\Omega}$. Thus:

$$\begin{aligned} b(u - u_h, u - u_h) &\gtrsim \|u - u_h\|^2_{1,\boldsymbol{\omega},\Omega} - \boldsymbol{\omega}^2\|u - u_h\|^2_{0,\Omega} \\ &\gtrsim \left(1 - C\boldsymbol{\omega}^2\boldsymbol{\eta}^2\right)\|u - u_h\|^2_{1,\boldsymbol{\omega},\Omega}. \end{aligned}$$

Assuming that $\boldsymbol{\omega\eta}$ is "sufficiently small", we have

$$b(u - u_h, u - u_h) \gtrsim \|u - u_h\|^2_{1,\boldsymbol{\omega},\Omega},$$

and we can employ Céa's Lemma as before!

An asymptotic stability result in the high-frequency case

Asymptotic quasi-optimality

Under the assumption that $\omega\eta$ is sufficiently small, we have

$$\|u - u_h\|_{1,\omega,\Omega} \lesssim \|u - \mathcal{I}_h u\|_{1,\omega,\Omega}.$$

This result is not really satisfactory, as we don't know what "$\omega\eta$ is sufficiently small" means!

It motivates careful analysis to develop sharp upper bounds on η. Specifically, explicit estimates in terms of ω, h and p are required.

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Settings

For the sake of simplicity, we consider a non-trapping problem.

Non-trapping problem

For all $\phi \in L^2(\Omega)$, for all $\omega \geq 0$ there exists a unique $u_\phi^\star \in H^1_{\Gamma_{\mathrm{D}}}(\Omega)$ such that

$$b(w, u_\phi^\star) = (w, \phi) \quad \forall v \in H^1_{\Gamma_{\mathrm{D}}}(\Omega).$$

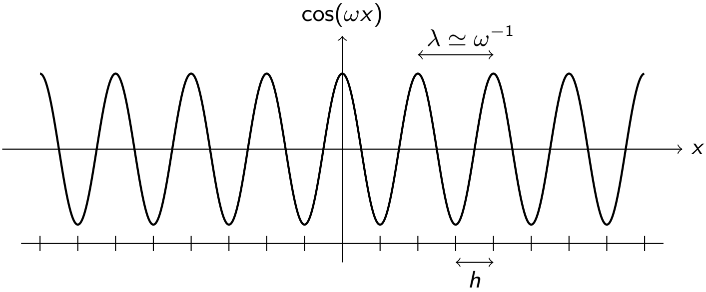
In addition, it holds that

$$\|u_\phi^\star\|_{1,\omega,\Omega} \lesssim \|\phi\|_{0,\Omega},$$

uniformly in ω.

Some vocabulary: the number of dofs per wavelength

$N_\lambda = \lambda/h \simeq (\boldsymbol{\omega} h)^{-1}$ is a measure of the number of dofs per wavelength.



In the above picture, $\lambda = 2h$: there are two dofdofs per wavelength.

In the following, we asume that $\boldsymbol{\omega} \gtrsim 1$ and $N_\lambda \gtrsim 1$ for the sake of simplicity.

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A naïve approach

Recall that

$$\eta = \sup_{\phi \in L^2(\Omega) \setminus \{0\}} \frac{\|u_\phi^\star - \mathcal{I}_h u_\phi^\star\|_{1,\omega,\Omega}}{\|\phi\|_{0,\Omega}}.$$

Consider an arbitrary $\phi \in L^2(\Omega)$. There exists a unique $u_\phi^\star \in H^1_{\Gamma_D}(\Omega)$ such that

$$b(w, u_\phi^\star) = (w, \phi) \quad \forall w \in H^1_{\Gamma_D}(\Omega).$$

We want to estimate the interpolation error $u_\phi^\star - \mathcal{I}_h u_\phi^\star$.
To do so, we are going to show that $u_\phi^\star \in H^2(\Omega)$.

A naïve approach: $H^2(\Omega)$ regularity

In strong form, we have

$$\left\{\begin{array}{rcll} -\frac{\omega^2}{\kappa}u_\phi^\star - \nabla\cdot\left(\frac{1}{\rho}\nabla u_\phi^\star\right) & = & \phi & \text{in } \Omega, \\ u_\phi^\star & = & 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \frac{1}{\rho}\nabla u_\phi^\star\cdot\boldsymbol{n} + \frac{i\omega}{\sqrt{\kappa\rho}}u_\phi^\star & = & 0 & \text{on } \Gamma_{\mathrm{A}}, \end{array}\right.$$

and we can write

$$\left\{\begin{array}{rcll} -\nabla\cdot\left(\frac{1}{\rho}\nabla u_\phi^\star\right) & := & F = \phi + \frac{\omega^2}{\kappa}u_\phi^\star & \text{in } \Omega, \\ u_\phi^\star & = & 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \frac{1}{\rho}\nabla u_\phi^\star\cdot\boldsymbol{n} & := & G = -\frac{i\omega}{\sqrt{\kappa\rho}}u_\phi^\star & \text{on } \Gamma_{\mathrm{A}}. \end{array}\right.$$

Then, standard elliptic regularity results show that

$$|u_\phi^\star|_{2,\Omega} \lesssim \|F\|_{0,\Omega} + \|G\|_{1/2,\Gamma_{\mathrm{A}}}.$$

A naïve approach: $H^2(\Omega)$-norm estimate

Since the problem is non-traping $\|u_\phi\|_{1,\omega,\Omega} \lesssim \|\phi\|_{0,\Omega}$, and we have

$$\|F\|_{0,\Omega} = \left\| \phi + \frac{\omega^2}{\kappa} u_\phi^\star \right\|_{0,\Omega} \lesssim \|\phi\|_{0,\Omega} + \omega \|u_\phi\|_{1,\omega,\Omega} \lesssim \omega \|\phi\|_{0,\Omega},$$

and

$$\|G\|_{0,\Omega} = \left\| \frac{-i\omega}{\sqrt{\kappa\rho}} u_\phi^\star \right\|_{1/2,\Gamma_{\mathrm{A}}} \lesssim \omega \|u_\phi^\star\|_{1/2,\Gamma_{\mathrm{A}}} \lesssim \omega \|u_\phi^\star\|_{1,\omega,\Omega} \lesssim \omega \|\phi\|_{0,\Omega}.$$

$H^2(\Omega)$-norm estimate

$$|u_\phi^\star|_{2,\Omega} \lesssim \omega \|\phi\|_{0,\Omega}.$$

A naïve approach: upper bound for the approximation factor

At that point, standard interpolation theory shows that

$$\|u_\phi^\star - \mathcal{I}_h u_\phi^\star\|_{1,\omega,\Omega} \lesssim h|u_\phi^\star|_{2,\Omega} \lesssim \omega h \|\phi\|_{0,\Omega}$$

Recalling that

$$\eta = \sup_{\phi \in L^2(\Omega)\setminus\{0\}} \frac{\|u_\phi^\star - \mathcal{I}_h u_\phi^\star\|_{1,\omega,\Omega}}{\|\phi\|_{0,\Omega}},$$

we obtain:

Upper bound for the approximation factor

$$\eta \lesssim \omega h.$$

A naïve approach: stability condition

We have $\eta \lesssim \omega h$. Then, the condition "$\omega\eta$ small", means $\omega^2 h \lesssim 1$.

As $\omega^2 h = (N_\lambda)^{-1}\omega$, the condition

A stability condition

$$N_\lambda \gtrsim \omega$$

guaranties stability of the FEM for all frequencies.

A naïve approach: the problem

We derived a stability condition of the form $N_\lambda \gtrsim \omega$.

This stability condition is valid for any polynomial degree p.
However, we do not see improvements when p increases,
which is not in agreement with numerical experiments.

Actually, this stability condition is optimal if $p = 1$, but not when $p > 1$.

It seems difficult to obtain an improvement with p,
since we only have $u_\phi^\star \in H^2(\Omega)$ as $\phi \in L^2(\Omega)$ only.

We cannot simply assume more regularity on ϕ,
since we are employing a duality argument.

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The breaktrough idea: regularity splitting

We cannot expect more than

$$\eta \simeq \mathcal{O}(h),$$

asymptotically, as the right-hand sides are in $L^2(\Omega)$.

The breakthrough idea is to introduce a clever splitting of the solution as

$$u_\phi^\star = u_0 + \widetilde{u},$$

where $u_0 \in H^2(\Omega)$ "behaves well" at high-frequency and $\widetilde{u}$ is more regular.

Sor far, such splitting have only been obtained in homogeneous media.

J.M. Melenk and S.A. Sauter, 2010
Convergence analysis for finite element discretizations of the Helmholtz equation with Dirichlet-to-Neumann boundary conditions.
J.M. Melenk and S.A. Sauter, 2011
Wavenumber explicit convergence analysis for Galerkin discretizations of the Helmholtz equation.

A novel regularity splitting for general media: the key idea

The key idea (with a lot of hand-waving!) is the formal expansion

$$u_\phi^\star = \sum_{j \geq 0} \omega^j u_j,$$

where we hope that the iterates are independent of ω with increasing regularity.

This expansion is purely formal.
However, plugging it in the PDE problem solved by $u_\phi^\star$,
we obtain an actual definition for the u_j.

T. Chaumont-Frelet, S. Nicaise, 2019
Wavenumber explicit convergence analysis for finite element discretizations of general wave propagation problems.

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Definition of the iterates

Identifying powers of ω, we introduce the following definition

$$\left\{\begin{array}{rcll} -\boldsymbol{\nabla}\cdot\left(\dfrac{1}{\rho}\boldsymbol{\nabla} u_0\right) & = & \phi & \text{in } \Omega, \\ u_0 & = & 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \dfrac{1}{\rho}\boldsymbol{\nabla} u_0\cdot\boldsymbol{n} & = & 0 & \text{on } \Gamma_{\mathrm{A}}, \end{array}\right.$$

$$\left\{\begin{array}{rcll} -\boldsymbol{\nabla}\cdot\left(\dfrac{1}{\rho}\boldsymbol{\nabla} u_1\right) & = & 0 & \text{in } \Omega, \\ u_1 & = & 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \dfrac{1}{\rho}\boldsymbol{\nabla} u_1\cdot\boldsymbol{n} & = & \dfrac{1}{\sqrt{\kappa\rho}}u_0 & \text{on } \Gamma_{\mathrm{A}}, \end{array}\right.$$

and

$$\left\{\begin{array}{rcll} -\boldsymbol{\nabla}\cdot\left(\dfrac{1}{\rho}\boldsymbol{\nabla} u_j\right) & = & \dfrac{1}{\kappa}u_{j-2} & \text{in } \Omega, \\ u_j & = & 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \dfrac{1}{\rho}\boldsymbol{\nabla} u_j\cdot\boldsymbol{n} & = & \dfrac{1}{\sqrt{\kappa\rho}}u_{j-1} & \text{on } \Gamma_{\mathrm{A}}. \end{array}\right.$$

Regularity of the iterates

We have

$$\begin{cases} -\boldsymbol{\nabla}\cdot\left(\dfrac{1}{\rho}\boldsymbol{\nabla} u_0\right) &= \phi & \text{in } \Omega, \\ u_0 &= 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \dfrac{1}{\rho}\boldsymbol{\nabla} u_0\cdot\boldsymbol{n} &= 0 & \text{on } \Gamma_{\mathrm{A}}, \end{cases}$$

so that $u_0 \in H^2(\Omega)$ with

$$\|u_0\|_{2,\Omega} \lesssim \|\phi\|_{0,\Omega}.$$

Similarly, since

$$\begin{cases} -\boldsymbol{\nabla}\cdot\left(\dfrac{1}{\rho}\boldsymbol{\nabla} u_1\right) &= 0 & \text{in } \Omega, \\ u_1 &= 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \dfrac{1}{\rho}\boldsymbol{\nabla} u_1\cdot\boldsymbol{n} &= \dfrac{1}{\sqrt{\kappa\rho}}u_0 & \text{on } \Gamma_{\mathrm{A}}. \end{cases}$$

we have

$$\|u_1\|_{3,\Omega} \lesssim \|u_0\|_{3/2,\Gamma_{\mathrm{A}}} \lesssim \|u_0\|_{2,\Omega} \lesssim \|\phi\|_{0,\Omega}.$$

Regularity of the iterates

Finally, by induction, we have

$$\begin{cases} -\nabla \cdot \left(\dfrac{1}{\rho}\nabla u_j\right) &= \dfrac{1}{\kappa} u_{j-2} & \text{in } \Omega, \\ u_j &= 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \dfrac{1}{\rho}\nabla u_j \cdot \boldsymbol{n} &= \dfrac{1}{\sqrt{\kappa\rho}} u_{j-1} & \text{on } \Gamma_{\mathrm{A}}, \end{cases}$$

so that $u_j \in H^{j+2}(\Omega)$ with

$$\|u_j\|_{j+2,\Omega} \lesssim \|u_j\|_{j-2,\Omega} + \|u_{j-1}\|_{j-1,\Omega} \lesssim \|\phi\|_{0,\Omega}.$$

Regularity of the iterates

We have introduced a sequence u_j so that $u_j \in H^{j+2}(\Omega)$ and

$$|u_j|_{j+2,\Omega} \lesssim \|\phi\|_{0,\Omega}.$$

The definition of the sequence was motivated by the formal expansion

$$u_\phi^\star = \sum_{j \geq 0} \omega^j u_j.$$

This expansion is purely formal, and actually does not converge.
We need to thoroughly examinate the residuals.

The residuals

We thus introduce, for $p \geq 1$ the residuals

$$r_p = u_\phi^\star - \sum_{j=0}^{p-2} \omega^j u_j,$$

so that

$$u_\phi^\star = \left(\sum_{j=0}^{p-2} \omega^j u_j \right) + r_p.$$

We need to study the regularity and frequency behaviour of r_p.

Inductive definition and regularity of the residuals

We have $r_1 = u_\phi^\star$. Thus, $r_1 \in H^2(\Omega)$ with

$$|r_1|_{2,\Omega} \lesssim |u_\phi^\star|_{2,\Omega} \lesssim \boldsymbol{\omega}\|\phi\|_{0,\Omega}.$$

Then, we see that r_2 satisfies

$$\left\{\begin{array}{rcll} -\boldsymbol{\nabla}\cdot\left(\dfrac{1}{\rho}\boldsymbol{\nabla} r_2\right) & = & \dfrac{\boldsymbol{\omega}^2}{\kappa}u_\phi^\star & \text{in } \Omega, \\ r_2 & = & 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \dfrac{1}{\rho}\boldsymbol{\nabla} r_2\cdot\boldsymbol{n} & = & -\dfrac{i\boldsymbol{\omega}}{\sqrt{\kappa\rho}}r_1. & \end{array}\right.$$

Hence, $r_2 \in H^3(\Omega)$ with

$$|r_2|_{3,\Omega} \lesssim \boldsymbol{\omega}^2\|u_\phi^\star\|_{1,\Omega} + \boldsymbol{\omega}\|r_1\|_{2,\Omega} \lesssim \boldsymbol{\omega}^2\|\phi\|_{0,\Omega}.$$

Inductive definition and regularity of the residuals

More generally, we have

$$\begin{cases} -\nabla \cdot \left(\frac{1}{\rho} \nabla r_p \right) &= \frac{\omega^2}{\kappa} r_{p-2} & \text{in } \Omega, \\ r_p &= 0 & \text{on } \Gamma_{\mathrm{A}}, \\ \frac{1}{\rho} \nabla r_p &= -\frac{i\omega}{\sqrt{\kappa\rho}} r_{p-1} & \text{on } \Gamma_{\mathrm{D}}. \end{cases}$$

By induction, we show that $r_p \in H^{p+1}(\Omega)$ with

$$\|r_p\|_{p+1,\Omega} \lesssim \omega^2 \|r_{p-2}\|_{p-1,\Omega} + \omega \|r_{p-1}\|_{p,\Omega} \lesssim \omega^p \|\phi\|_{0,\Omega}.$$

Grouping up the pieces!

We have shown that for all $p \geq 1$, we have

$$u_\phi^\star = \left(\sum_{j=0}^{p-1} \omega^j u_j \right) + r_p,$$

with $u_j \in H^{j+2}(\Omega)$, $r_p \in H^{p+1}(\Omega)$

$$\|u_j\|_{j+2,\Omega} \lesssim \|\phi\|_{0,\Omega} \quad \|r_p\|_{p+1,\Omega} \lesssim \omega^p \|\phi\|_{0,\Omega}.$$

The residual r_p behaves as a solution with smooth right-hand side.
It is similar to the "regular part" of the Melenk-Sauter splitting.

The u_j have increasing regularity, and behave "nicely" at high frequencies.

Upper bound for the approximation factor

We have

$$\begin{aligned}
\|u_\phi^\star - \mathcal{I}_h u_\phi^\star\|_{1,\omega,\Omega} &\lesssim \sum_{j=0}^{p-1} \omega^j \|u_j - \mathcal{I}_h u_j\|_{1,\omega,\Omega} + \|r_p - \mathcal{I}_h r_p\|_{1,\omega,\Omega} \\
&\lesssim \sum_{j=0}^{p-1} \omega^j h^{j+1} |u_j|_{j+2,\omega,\Omega} + h^p |r_p|_{p+1,\omega,\Omega} \\
&\lesssim h \sum_{j=0}^{p-1} (\omega h)^j \|\phi\|_{0,\Omega} + \omega^p h^p \|\phi\|_{0,\Omega} \\
&\lesssim (h + \omega^p h^p) \|\phi\|_{0,\Omega}. \quad (\ (\omega h)^j = N_\lambda^{-j} \lesssim 1\ )
\end{aligned}$$

Upper bound for the approximation factor

$$\eta \lesssim h + \omega^p h^p.$$

Stability condition

We have shown that $\omega\eta \lesssim \omega h + \omega^{p+1} h^p \simeq (N_\lambda)^{-1} + \omega (N_\lambda)^{-1/p}$.

It follows that for any fixed p, the FEM is stable if

Stability condition

$$N_\lambda \gtrsim \omega^{1/p}$$

For any fixed p, N_λ must be increased to preserve stability,
but the increase rate is lower for larger p.

High order methods require less dofs per wavelength to achieve stability.

Outline

1 Relationship between approximation factor and stability of FEM
- Warm up exercise: the zero frequency case
- The high-frequency case

2 Upper bounds for the approximation factor and stability conditions
- Settings
- A naive approach
- A regularity splitting for general media

3 Numerical experiments and sharpness of the main results
- A methodology to illustrate the main results
- Numerical experiments

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A methodology to illustrate the main results

Our key results state that if $N_\lambda \gtrsim \omega^{1/p}$, then

$$\|u - u_h\|_{1,\omega,\Omega} \lesssim \|u - \mathcal{I}_h u\|_{1,\omega,\Omega}.$$

To illustrate this, we would like to compute, for a fixed p,
what is the minimal value $N_\lambda^\star(\omega)$ such that the FEM is stable when $N_\lambda \geq N_\lambda^\star(\omega)$.

We consider a fixed domain and and a fixed right-hand side,
and solve the Helmholtz problem for several frequencies ω.

For each frequency, we approximate the problem for different mesh sizes h,
and record the convergence history of

$$\|u - u_h\|_{1,\omega,\Omega} \quad \text{and} \quad \|u - \mathcal{I}_h u\|_{1,\omega,\Omega}.$$

A methodology to illustrate the main results

For each frequency, we denote by $N_\lambda^\star(\omega)$ the smallest value such that

$$\|u - u_h\|_{1,\omega,\Omega} \leq 2\|u - \mathcal{I}_h u\|_{1,\omega,\Omega} \quad \forall N_\lambda \geq N_\lambda^\star(\omega),$$

where the constant 2 is chosen arbitrarily.

This $N_\lambda^\star(\omega)$ then defines a sufficient number of dofs per wavelength to ensure stability.

According to our main result, we shall observe that

$$N_\lambda^\star(\omega) \lesssim \omega^{1/p}.$$

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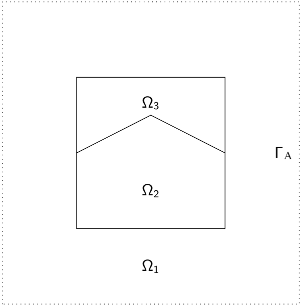
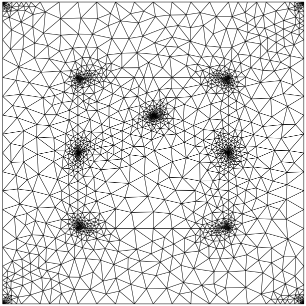


Heterogeneous domain



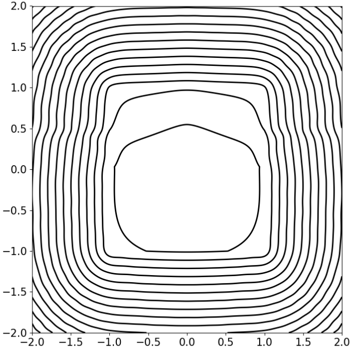
Locally refined mesh

Piecewise constant coefficients

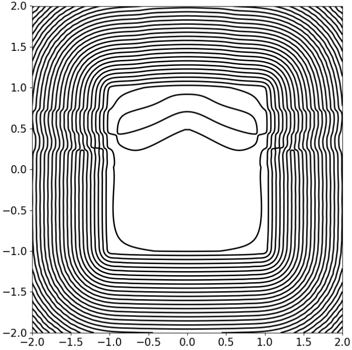
$$\kappa_1 = 1, \qquad \kappa_2 = 10, \qquad \kappa_3 = 1000,$$
$$\rho_1 = 1, \qquad \rho_2 = 0.5, \qquad \rho_3 = 0.1.$$

The right-hand side is Gaussian load term centered at the origin.

Zero-levelset curves of the real parts of solutions

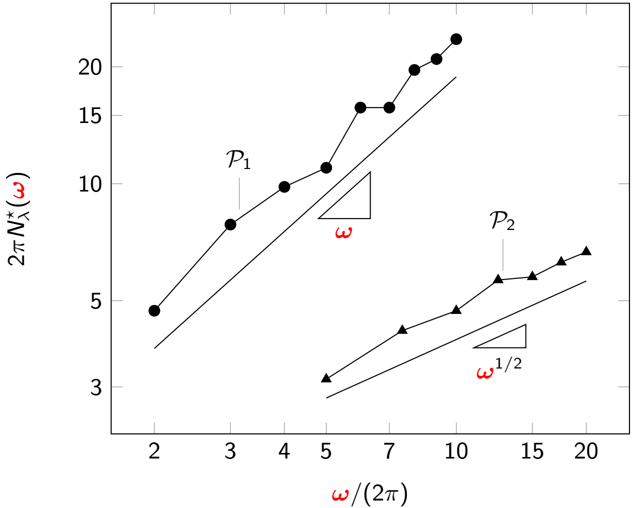


$\omega = 10\pi$



$\omega = 20\pi$

Required dofs per wavelength $N_\lambda^\star(\omega)$



We have $N_\lambda^\star(\omega) \simeq \omega^{1/p}$, which indicate that our stability condition is sharp.

Final comments

We derived a novel frequency-explicit stability condition for heterogeneous domains with smooth coefficients.

With slight modifications, we can actually take into account piecewise smooth coefficients, so that our analysis applies to a wide range of problems.

The derived stability condition is valid for any fixed polynomial degree p, and numerical experiments indicate that it is sharp.

For non-trapping domains, this stability condition is: $N_\lambda \gtrsim \omega^{1/p}$.

This analysis strongly encourages the use of high order FEM,
as they exhibit an improved stability.