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October 20, 2018

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Financial support: ANR-13-BS02-0007 STINT.

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Abstract

For a given 2-partition (V_1, V_2) of the vertices of a (di)graph G, we study properties of the spanning bipartite subdigraph $B_G(V_1, V_2)$ of G induced by those arcs/edges that have one end in each V_i, $i \in \{1, 2\}$. We determine, for all pairs of non-negative integers k_1, k_2, the complexity of deciding whether G has a 2-partition (V_1, V_2) such that each vertex in V_i (for $i \in \{1, 2\}$) has at least k_i (out-)neighbours in V_{3-i}. We prove that it is $\mathcal{NP}$-complete to decide whether a digraph D has a 2-partition (V_1, V_2) such that each vertex in V_1 has an out-neighbour in V_2 and each vertex in V_2 has an in-neighbour in V_1. The problem becomes polynomially solvable if we require D to be strongly connected. We give a characterisation of the structure of $\mathcal{NP}$-complete instances in terms of their strong component digraph. When we want higher in-degree or out-degree to/from the other set the problem becomes $\mathcal{NP}$-complete even for strong digraphs. A further result is that it is $\mathcal{NP}$-complete to decide whether a given digraph D has a 2-partition (V_1, V_2) such that $B_D(V_1, V_2)$ is strongly connected. This holds even if we require the input to be a highly connected eulerian digraph.

Keywords: 2-partition, minimum out-degree, spanning bipartite subdigraph, eulerian, strong spanning subdigraph.

1 Introduction

A **2-partition** of a graph or digraph G is a vertex partition (V_1, V_2) of its vertex set $V(G)$. If (V_1, V_2) is a 2-partition of a graph (resp. digraph) G, the **bipartite graph** (resp. **digraph**) **induced by** (V_1, V_2), denoted by $B_G(V_1, V_2)$, is the spanning bipartite graph (resp. digraph) induced by the edges (resp. arcs) having one end in each set of the partition.

The following result is well-known.

Proposition 1. *Every undirected graph G admits a 2-partition (V_1, V_2) such that $d_{B_G(V_1,V_2)}(v) \geq d_G(v)/2$ for every vertex v of G.*

This proposition implies that every graph with no isolated vertex has a 2-partition (V_1, V_2) such that $\delta(B_G(V_1, V_2)) \geq 1$. Consequently, one can decide in polynomial time whether a graph has a partition such that $d_{B_G(V_1,V_2)}(v) \geq 1$ for all $v :$ if the graph has an isolated vertex, the answer is 'No', otherwise it is 'Yes'. We first study the existence of 2-partition with some higher degree constraints on the vertices in the bipartite graph induced by it. More precisely, we are interested in the following decision problem for some fixed positive integers k_1 and k_2.

Problem 2 (($\delta \geq k_1, \delta \geq k_2$)-BIPARTITE-PARTITION).
Input: A graph G.
Question: Does G admit a 2-partition (V_1, V_2) such that $d_{B_G(V_1,V_2)}(v_i) \geq k_i$ for all $v_i \in V_i$, $i \in \{1, 2\}$?

As noted above, ($\delta \geq 1, \delta \geq 1$)-BIPARTITE-PARTITION is polynomial-time solvable. We prove in Section 3, that ($\delta \geq 1, \delta \geq 2$)-BIPARTITE-PARTITION is also polynomial-time solvable, and that ($\delta \geq k_1, \delta \geq k_2$)-BIPARTITE-PARTITION is $\mathcal{NP}$-complete when $k_1 + k_2 \geq 4$.

We then consider directed analogues to Problem 2. Many other 2-partition problems have already been studied. The papers [4, 6] determined the complexity of a large number of 2-partition problems where we seek a 2-partition (V_1, V_2) with specified properties for the digraphs $D\langle V_i\rangle$ induced by this partition. In [6] the authors asked whether there exists a polynomial-time algorithm to decide whether a given digraph has a 2-partition (V_1, V_2) with $\Delta^+(D\langle V_i\rangle) < \Delta^+(D)$ for $i \in \{1, 2\}$. This was answered affirmatively in [3] where also the complexity of deciding whether a digraph D has a 2-partition (V_1, V_2) so that $\Delta^+(D\langle V_i\rangle) \leq k_i$ was determined for all non-negative integers k_1, k_2.

Thomassen [12] constructed an infinite class of strongly connected digraphs $\mathcal{T} = T_1, T_2, \ldots, T_k, \ldots$ with the property that for each k, T_k is k-out-regular and has no even directed cycle. As remarked by Alon in [1] this implies that we cannot expect any directed analogues of Proposition 1.

Proposition 3. *For every $k \geq 1$, for every 2-partition (V_1, V_2) of T_k, some vertex v has all its k out-neighbours in the same part as itself, so $d^+_{B_D(V_1,V_2)}(v) = 0$.*

The first directed analogue to Problem 2 that we study is the following.

Problem 4 (($\delta^+ \geq k_1, \delta^+ \geq k_2$)-BIPARTITE-PARTITION).
Input: A digraph D.
Question: Does D admit a 2-partition (V_1, V_2) such that $d^+_{B_D(V_1,V_2)}(v_i) \geq k_i$ for all $v_i \in V_i$, $i \in \{1, 2\}$?

Observe that, when $k_1 = 0$ (or $k_2 = 0$), the problem is pointless since the partition $(V(D), \emptyset)$ is the desired partition. We start in Section 4 by using the result of [3] mentioned to prove that Problem 4 is polynomial-time solvable when $k_1 = k_2 = 1$ and $\mathcal{NP}$-complete whenever $k_1 + k_2 \geq 3$. Then we study the following problem.

Problem 5 (($\delta^+ \geq k_1, \delta^- \geq k_2$)-BIPARTITE-PARTITION).
Input: A digraph D.
Question: Does D admit a 2-partition (V_1, V_2) such that every vertex in V_1 has at least k_1 out-neighbours in V_2 and every vertex in V_2 has at least k_2 in-neighbours in V_1?

We show in Section 5 that ($\delta^+ \geq 1, \delta^- \geq 1$)-BIPARTITE-PARTITION is $\mathcal{NP}$-complete but becomes polynomial-time solvable when the input must be a strong digraph. We also characterise the $\mathcal{NP}$-complete instances in terms of their strong component digraph. Next, in Section 6, we show that for any pair of positive integers (k_1, k_2) such that $k_1 + k_2 \geq 3$, ($\delta^+ \geq k_1, \delta^- \geq k_2$)-BIPARTITE-PARTITION is $\mathcal{NP}$-complete even when restricted to strong digraphs.

It is a simple matter to show that a connected graph G has a 2-partition inducing a connected bipartite graph. Indeed, just consider a spanning tree and its bipartition. One can even show that Theorem 6 below holds[1] (just consider a 2-partition maximizing the number of edges between the two sets). Recall that $\lambda(G)$ is the **edge-connectivity** of G, that is, the minimum number of edges whose removal from G results in a disconnected graph.

Theorem 6. *Every graph G has a 2-partition (V_1, V_2) such that $\lambda(B_G(V_1, V_2)) \geq \lfloor \lambda(G)/2 \rfloor$.*

We thus study the directed analogues, called **strong 2-partitions**, which are 2-partitions (V_1, V_2) of a digraph D such that $B_D(V_1, V_2)$ is strong. It is a well-known phenomenon that results on edge-connectivity for undirected graphs often have a counterpart for eulerian digraphs. Unfortunately, we show that it is not the case for Theorem 6 : for every $r > 0$, there exists an r-strong eulerian digraph D which has no strong 2-partition (Theorem 28). We then show that the following problem is $\mathcal{NP}$-compete even when restricted to r-strong digraphs (for some $r > 0$).

[1] Stephan Thomassé private communication, Lyon 2015.

Problem 7 (Strong 2-partition).
Input: A digraph D.
Question: Does D admit a 2-partition (V_1, V_2) such that $B_D(V_1, V_2)$ is strong?

We conclude the paper with a section presenting some remarks and open problems.

2 Notation

Notation follows [5]. We use the shorthand notation $[k]$ for the set $\{1, 2, \ldots, k\}$. Let $D = (V, A)$ be a digraph with vertex set V and arc set A.

Given an arc $uv \in A$, we say that u **dominates** v and v is **dominated** by u. If uv or vu (or both) are arcs of D, then u and v are **adjacent**. If neither uv or vu exist in D, then u and v are **non-adjacent**. The **underlying graph** of a digraph D, denoted by $UG(D)$, is obtained from D by suppressing the orientation of each arc and deleting multiple copies of the same edge (coming from directed 2-cycles). A digraph D is **connected** if $UG(D)$ is a connected graph, and the **connected components** of D are those of $UG(D)$.

The **subdigraph induced** by a set of vertices X in a digraph D, denoted by $D\langle X\rangle$, is the digraph with vertex set X and which contains those arcs from D that have both end-vertices in X. When X is a subset of the vertices of D, we denote by $D - X$ the subdigraph $D\langle V \setminus X\rangle$. If D' is a subdigraph of D, for convenience we abbreviate $D - V(D')$ to $D - D'$. The **contracted digraph** D/X is obtained from $D - X$ by adding a 'new' vertex x not in V and by adding for every $u \in D - X$ the arc ux (resp. xu) if u has an out-neighbour (resp. in-neighbour) in X (in D).

The **in-degree** (resp. **out-degree**) of v, denoted by $d_D^-(v)$ (resp. $d_D^+(v)$), is the number of arcs from $V \setminus \{v\}$ to v (resp. v to $V \setminus \{v\}$). A **sink** is a vertex with out-degree 0 and a **source** is a vertex with in-degree 0. The **degree** of v, denoted by $d_D(v)$, is given by $d_D(v) = d_D^+(v) + d_D^-(v)$. Finally the **minimum out-degree**, respectively **minimum in-degree** and **minimum degree** is denoted by $\delta^+(D)$, respectively $\delta^-(D)$ and $\delta(D)$. The **minimum semi-degree** of D, denoted by $\delta^0(D)$, is defined as $\delta^0(D) = \min\{\delta^+(D), \delta^-(D)\}$. A vertex is **isolated** if it has degree 0.
A digraph is **k-out-regular** if all its vertices have out-degree k.

A **(u, v)-path** is a directed path from u to v. A digraph is **strongly connected** (or **strong**) if it contains a (u, v)-path for every ordered pair of distinct vertices u, v. A digraph D is **k-strong** if for every set S of fewer than k vertices the digraph $D - S$ is strong. A **strong component** of a digraph D is a maximal subdigraph of D which is strong. A strong component is **trivial**, if it has order 1. An **initial** (resp. **terminal**) strong component of D is a strong component X with no arcs entering (resp. leaving) X in D.

Let D be a strongly connected digraph. If S is a strong subdigraph of D, then an **S-handle** H of D is a directed walk $(s, v_1, \ldots, v_\ell, t)$ such that:

- the v_i are distinct and in $V(D - S)$, and
- $s, t \in V(S)$ (with possibly $s = t$ and in this case H is a directed cycle, otherwise it is a directed path).

The **length** of a handle is the number of its arcs, here $\ell + 1$. A handle of length one is said to be **trivial**.

An **out-tree** rooted at the vertex s, also called an **s-out-tree**, is a connected digraph T_s^+ such that $d^-_{T_s^+}(s) = 0$ and $d^-_{T_s^+}(v) = 1$ for every vertex v different from s. Equivalently, for every $v \in V(T_s^+)$ there is a unique (s, v)-path in T_s^+. The directional dual notion is the one of an **s-in-tree**, that is, a connected digraph T_s^- such that $d^+_{T_s^-}(s) = 0$ and $d^+_{T_s^-}(v) = 1$ for every vertex v different from s.

An **s-out-branching** (resp. **s-in-branching**) is a spanning s-out-tree (resp. s-in-tree). We use the notation B_s^+ (resp. B_s^-) to denote an s-out-branching (resp. an s-in-branching).

In our $\mathcal{NP}$-completeness proofs we use reductions from the well-known 3-SAT problem, and two variants, Not-All-Equal-3-SAT and Monotone Not-all-equal-3-SAT. In the latter the Boolean formula $\mathcal{F}$ consists of clauses all of whose literals are non-negated variables. In 3-SAT, we want to decide whether there is a truth assignment that **satisfies** $\mathcal{F}$ that is such that every clause has a true literal. In NOT-ALL-EQUAL-3-SAT and MONOTONE NOT-ALL-EQUAL-3-SAT, we want to decide whether there is a **NAE truth assignment** , that is a truth assigment such that every clause has a true literal and a false literal. Those two problems are $\mathcal{NP}$-complete [11].

Let $\mathcal{P}_1, \mathcal{P}_2$ be properties of vertices in a digraph (e.g. out-degree at least 1). Then a **$(\mathcal{P}_1, \mathcal{P}_2)$-bipartite-partition** of a graph D is a 2-partition (V_1, V_2) such that the vertices of V_i have property $\mathcal{P}_i$ in $B_D(V_1, V_2)$. For example, a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition is a 2-partition (V_1, V_2) so that in $B_D(V_1, V_2)$ the vertices of V_1 have out-degree at least 1 and the vertices of V_2 have in-degree at least 1. We also use the same definition for (undirected) graphs. The **2-colouring associated to** a 2-partition (V_1, V_2) is the 2-colouring c defined by $c(x) = i$ if $x \in V_i$. A **$(\mathcal{P}_1, \mathcal{P}_2)$-colouring** is a 2-colouring associated to a $(\mathcal{P}_1, \mathcal{P}_2)$-bipartite-partition.

3 $(\delta \geq k_1, \delta \geq k_2)$-bipartite-partitions

In this section we give a complete characterisation of the complexity of the $(\delta \geq k_1, \delta \geq k_2)$-bipartite-partition problem for undirected graphs. We first list an easy consequence of Proposition 1.

Corollary 8. *Every graph G with $\delta(G) \geq 1$ has a $(\delta \geq 1, \delta \geq 1)$-bipartite-partition and every graph G with at least one edge has a $(\delta \geq 0, \delta \geq 1)$-bipartite-partition.*

This statement can be generalized to $(\delta \geq 1, \delta \geq k)$-bipartite-partitions.

Theorem 9. *If G is a graph with $\delta(G) \geq k$, then G has a $(\delta \geq 1, \delta \geq k)$-bipartite-partition, which can be found in polynomial time.*

Proof. Let V_2 be any maximal stable set in G. That is, for every $x \notin V_2$ the set $V_2 \cup \{x\}$ is not stable. This implies that every vertex not in V_2 has an edge to a vertex in V_2 and as V_2 is stable and $\delta(G) \geq k$ every vertex in V_2 has at least k neighbours not in V_2. Therefore $(V(G) \setminus V_2, V_2)$ is the desired partition. As a maximal stable set may be computed greedily, the partition $(V(G) \setminus V_2, V_2)$ can be found in polynomial time. □

3.1 Solving $(\delta \geq 1, \delta \geq 2)$-BIPARTITE-PARTITION in polynomial time

Theorem 10. *Let G be a graph with $\delta(G) = 1$. Let S_1 be the set of vertices of degree 1 in G. Then G has a $(\delta \geq 1, \delta \geq 2)$-bipartite-partition if and only if S_1 is a stable set and every vertex in $N(S_1)$ has either two neighbours in S_1 or at least one neighbour in $V(G) \setminus (S_1 \cup N(S_1))$.*

Proof. Suppose that G has a $(\delta \geq 1, \delta \geq 2)$-bipartite-partition (V_1, V_2). Necessarily, $S_1 \subseteq V_1$. Moreover, for each $v \in S_1$, its unique neighbour is in V_2. Hence S_1 is a stable set and $N(S_1) \subseteq V_2$. Now every vertex in $N(S_1)$ has at least two neighbours in V_1. Hence either two neighbours are in S_1 or at least one neighbour is in $V_1 \setminus S_1$ which is a subset of $V(G) \setminus (S_1 \cup N(S_1))$.

Reciprocally, assume that S_1 is a stable set and that every vertex in $N(S_1)$ has either two neighbours in S_1 or one neighbour in $V(G) \setminus (S_1 \cup N(S_1))$. For every $i > 1$, let S_i be the set of vertices not in $\bigcup_{j=1}^{i-1} S_j$ which are adjacent to a vertex in S_{i-1}. Note that $S_2 = N(S_1)$. Moreover, for every vertex v in S_i, its **predecessors** (resp. **peers**, **successors**) are its neighbours in S_{i-1} (resp. S_i, S_{i+1}). By definition of the S_i, every vertex in $V(G) \setminus S_1$ has at least one predecessor.

We initially colour the vertices as follows: if $v \in S_i$ and i is odd, then v is coloured 1, otherwise it is coloured 2. Observe that a vertex has a colour different from that of its predecessors and successors. Now as long as there is a vertex w coloured 2 with exactly one neighbour coloured 1, we recolour w with 1. Let w be a recoloured vertex. As it is originally coloured 2, it must be in S_i with i even. Now w has exactly one predecessor and no successor. In particular, it is not in S_2, by our assumption on $N(S_1) = S_2$. Furthermore, it has degree at least 2 (since vertices of S_1 are coloured 1), so its has at least one peer which must be coloured 2, and will never be recoloured because it now has at least two neighbours (a peer and a predecessor) coloured 1.

Let V_1 (resp. V_2) be the set of vertices coloured 1 (resp. 2). We claim that (V_1, V_2) is a $(\delta \geq 1, \delta \geq 2)$-bipartite-partition.

Consider a vertex v_1 in V_1. Assume v_1 is originally coloured 1. Either it is in S_1 and its neighbour is in S_2 and thus in V_2 because no vertex of S_2 is recoloured, or it has a predecessor which must be in V_2 because only vertices with no successors are recoloured. If v_1 has been recoloured, then as observed above it has a peer originally coloured 2 that is not recoloured.
Consider now a vertex $v_2 \in V_2$. It was originally coloured 2 and has not been recoloured. Hence v_2 has at least two neighbours coloured 1. □

We note that any graph with an isolated vertex does not contain a $(\delta \geq 1, \delta \geq k)$-bipartite-partition for any $k \geq 1$. In Theorem 10 we consider graphs with $\delta(G) = 1$. The following easy result handles the cases when $\delta(G) \geq 2$ as a special case (when $k = 2$).

Corollary 11. *One can decide in polynomial time whether a given graph has a $(\delta \geq 1, \delta \geq 2)$-bipartite-partition. Moreover if such a partition exists, it can be found in polynomial time .*

Proof. Let G be a graph and let S_1 be the set of vertices with degree 1 in G. If G has an isolated vertex then no such partition exists, and if $S_1 = \emptyset$ then the result follows from Theorem 9, so assume that $S_1 \neq \emptyset$ and G does not contain any isolated vertices. According to Theorem 10, deciding whether a graph G has a $(\delta \geq 1, \delta \geq 2)$-bipartite-partition, we need to check that S_1 is a stable set, and that every vertex in $N(S_1)$ has either two neighbours in S_1 or one neighbour in $V(G) \setminus (S_1 \cup N(S_1))$. This can easily be done in polynomial time.

Moreover, since the proof of Theorem 10 is constructive, one can find in polynomial time a $(\delta \geq 1, \delta \geq 2)$-bipartite-partition, if one exists. □

3.2 $\mathcal{NP}$-completeness of $(\delta \geq k_1, \delta \geq k_2)$-BIPARTITE-PARTITION when $k_1 + k_2 \geq 4$

Theorem 12. *Let k_1, k_2 be integers such that $2 \leq k_1 \leq k_2$. It is $\mathcal{NP}$-complete to decide whether a graph G has a $(\delta \geq k_1, \delta \geq k_2)$-bipartite-partition.*

Proof. We reduce NOT-ALL-EQUAL-3-SAT to the problem of deciding whether a graph has a $(\delta \geq k_1, \delta \geq k_2)$-colouring (which is equivalent to $(\delta \geq k_1, \delta \geq k_2)$-BIPARTITE-PARTITION.

A 2-colouring is **good for** X if every vertex of X coloured i has k_i neighbours coloured $3 - i$. In particular, a colouring good for $V(G)$ is a $(\delta \geq k_1, \delta \geq k_2)$-colouring of G.

First we define some gadgets and then we show how to combine them to produce the desired result.

Let X' be the graph whose vertex set is the disjoint union of the sets $\{v, z, x, \bar{x}\}, X_1, X_2, X_3, X_4$, where $|X_1| = |X_4| = k_1 - 1$ and $|X_2| = |X_3| = k_2 - 1$. The graph X' has the following edges (when we write 'all edges', we mean all possible edges between the two sets): All edges between v and X_1, all edges between X_1 and X_2, all edges between X_2 and $\{x, \bar{x}\}$, all edges between $\{x, \bar{x}\}$ and X_4, all edges between X_4 and X_3, all edges between X_3 and z and finally the edge vz. Let X be obtained from X' by adding the edge $x\bar{x}$. It is easy to verify that X has a $(\delta \geq k_1, \delta \geq k_2)$-colouring and in every such colouring of X the vertices x and $\bar{x}$ must get different colours and both colourings are possible. Indeed once we fix the colour of v, which must be 1 if $k_1 < k_2$ and could be 1 or 2 if $k_1 = k_2$, then every other colour is fixed except for x and $\bar{x}$. Moreover, this property remains no matter what edges we add to $\{x, \bar{x}\}$.

Let Y be the graph that we obtain from a copy of X' by adding three new vertices ℓ_1, ℓ_2, ℓ_3 and all possible edges between these and the set $\{x, \bar{x}\}$. As previously it is easy to verify that Y has a 2-colouring good for all vertices except ℓ_1, ℓ_2, ℓ_3 (they do not have enough neighbours in Y but will get these in the graph we construct) and for every such colouring at least one of ℓ_1, ℓ_2, ℓ_3 is coloured i for $i \in [2]$. Furthermore, every colouring of $\{\ell_1, \ell_2, \ell_3\}$ where both colours are used can be extended to a full 2-colouring which is good for $V(Y) \setminus \{\ell_1, \ell_2, \ell_3\}$

Now let $\mathcal{F}$ be an instance of NOT-ALL-EQUAL-3-SAT with variables $v_1, v_2, \ldots, v_n$ and clauses $C_1, C_2, \ldots, C_m$. Form a graph $G = G(\mathcal{F})$ from $\mathcal{F}$ as follows: make n disjoint copies $X_1, X_2, \ldots, X_n$ of X and denote the copies of $x, \bar{x}$ in X_i by $x_i, \bar{x}_i$. Then make m disjoint copies $Y_1, Y_2, \ldots, Y_m$ of Y, where the jth copy will correspond to the clause C_j. Denote the copies of ℓ_1, ℓ_2, ℓ_3 in Y_j by $\ell_{j,1}, \ell_{j,2}, \ell_{j,3}$. Now, for each $j \in [m]$ identify the vertices $\ell_{j,1}, \ell_{j,2}, \ell_{j,3}$ with those vertices from $Z = \{x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n\}$

that correspond to the literals of C_j. E.g. if $C_j = (v_1 \vee \bar{v}_3 \vee v_7)$ then we identify $\ell_{j,1}$ with x_1, $\ell_{j,2}$ with $\bar{x}_3$ and $\ell_{j,3}$ with x_7. Note that each vertex from Z may be identified with many vertices in this way.

We claim that G has a $(\delta \geq k_1, \delta \geq k_2)$-colouring if and only if there is NAE truth assignment for $\mathcal{F}$. Suppose first that c is a $(\delta \geq k_1, \delta \geq k_2)$-colouring of G. Let ϕ be the truth assignment which sets v_i true precisely when $c(x_i) = 1$. By the property of the vertices ℓ_1, ℓ_2, ℓ_3 (which is inherited in all the subgraphs $Y_1, \ldots, Y_m$) and the fact that c is a $(\delta \geq k_1, \delta \geq k_2)$-colouring implies that for each j the number of vertices from $\{\ell_{j,1}, \ell_{j,2}, \ell_{j,3}\}$ that have colour 1 is either one or two. Moreover by the property of X for every $i = 1, \ldots, n$ the vertices x_i and $\bar{x}_i$ receive different colours by c. So ϕ is a NAE truth assignment.
Conversely, if ϕ is a NAE truth assignment, then we first colour each x_i by 1 and $\bar{x}_i$ by 2 if $\phi(v_i)$ is true and do the opposite otherwise. If is easy to check from the definition of X, Y that we can extend this partial 2-colouring to a $(\delta \geq k_1, \delta \geq k_2)$-colouring of G. □

Recall Theorem 9 which states that if G is a graph with $\delta(G) \geq k$ then G has a $(\delta \geq 1, \delta \geq k)$-bipartite-partition. In contrast to this result we prove the following result.

Theorem 13. *For all integers $k \geq 3$ it is $\mathcal{NP}$-complete to decide whether a graph G has a $(\delta \geq 1, \delta \geq k)$-bipartite-partition. In fact the problem remains $\mathcal{NP}$-complete even for graphs G with $\delta(G) = k-1$.*

Proof. Reduction from 3-SAT. $(\delta \geq 1, \delta \geq k)$-bipartite-partition.

First suppose that $k = 3$. Let the gadget G^* contain the vertices $\{a_1, a_2, x, \bar{x}, y_1, y_2, b_1, b_2\}$ and all edges from $A = \{a_1, a_2\}$ to $X = \{x, \bar{x}\}$ and all edges from X to $Y = \{y_1, y_2\}$ and all edges from Y to $B = \{b_1, b_2\}$. See Figure 1.

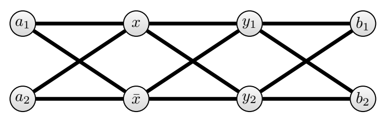

Figure 1: The gadget G^*.

No matter what edges we later add to X we note that the vertices in $A \cup B$ must receive colour 1 (as they have degree 2) in any $(\delta \geq 1, \delta \geq 3)$-colouring. Furthermore at least one vertex in Y must get colour 2 (as the vertices in B needs a neighbour of colour 2). Without loss of generality, assume y_1 has colour 2. Due to the vertices in A one vertex in X must be coloured 2 and due to y_1 one vertex in X must be coloured 1. So the vertices in X must receive different colours in any $(\delta \geq 1, \delta \geq 3)$-colouring. Furthermore if we do colour exactly one vertex from X and one vertex from Y and all vertices in $A \cup B$ with the colour 1 then we get a $(\delta \geq 1, \delta \geq 3)$-colouring of the gadget G^*.

Let $\mathcal{F}$ be an instance of 3-SAT with variables $v_1, v_2, \ldots, v_n$ and clauses $C_1, C_2, \ldots, C_m$. Form a graph $G = G(\mathcal{F})$ from $\mathcal{F}$ as follows: make n disjoint copies $X_1^*, X_2^*, \ldots, X_n^*$ of G^* and denote the copies of $x, \bar{x}$ in X_i^* by $x_i, \bar{x}_i$. Then add m disjoint copies of 3-cycles with vertex sets y_j, y_j', y_j'' for $j \in [m]$. Now, for each $j \in [m]$ add an edge from y_j to those vertices from $Z = \{x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n\}$ that correspond to the literals of C_j. E.g. if $C_j = (v_1 \vee \bar{v}_3 \vee v_7)$, then we add an edge from y_j to x_1, $\bar{x}_3$ and x_7. As in a $(\delta \geq 1, \delta \geq 3)$-colouring y_j' and y_j'' must be given colour 1 we note that there is a $(\delta \geq 1, \delta \geq 3)$-colouring of G if each y_j (which has to be coloured 2) has a neighbour in Z of colour 1. It is now not difficult to see that G has a $(\delta \geq 1, \delta \geq 3)$-colouring if and only if $\mathcal{F}$ is satisfied (where the vertex x_i is given colour 1 if the variable v_i is true and otherwise $\bar{x}_i$ is given colour 1). Furthermore we note that $\delta(G') = 2 = k-1$ by construction.

We now consider the case when $k \geq 4$. We will reduce from the case when $k = 3$ as follows. Let G be an instance of the case when $k = 3$ and now assume that $k \geq 4$. Let $X_1, X_2, \ldots, X_{k-3}$ be $k-3$ cliques of size k and let $x_i \in X_i$ be arbitrary for $i \in [k-3]$. Let G' be the graph obtained from G by adding $X_1, X_2, \ldots, X_{k-3}$ and the vertices $\{y_1, y_2, \ldots, y_{k-3}\}$ to G and all edges $x_i y_i$ and all edges from y_i to $V(G)$ for all $i \in [k-3]$. Note that if (V_1, V_2) is a $(\delta \geq 1, \delta \geq k)$-bipartite-partition of G' the vertices in $V(X_i) \setminus \{x_i\}$ must be in V_1 (as they have degree $k-1$) for $i \in [k-3]$ and therefore $x_1, x_2, \ldots, x_{k-3}$ must be in V_2 and $y_1, y_2, \ldots, y_{k-3}$ must be in V_1 (as $d(x_i) = k$ for $i \in [k-3]$). This implies that G admits a $(\delta \geq 1, \delta \geq 3)$-bipartite-partition if and only if G' admits a $(\delta \geq 1, \delta \geq k)$-bipartite-partition. This completes the proof as we note that $\delta(G') = k - 1$ (as $\delta(G) = 2$). □

4 $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-bipartite-partitions

We now use results from [3] to settle the complexity of the $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-bipartite-partition problem for digraphs for all pairs of positive integers k_1, k_2.

The following result was proved by the authors in [3]. Note that one can find an even cycle in a digraph that has such a cycle in polynomial time [10].

Theorem 14 ([3])**.** *A digraph D admits a $(\delta^+ \geq 1, \delta^+ \geq 1)$-bipartite-partition if and only if every non-trivial terminal strong component contains an even directed cycle. The desired 2-partition can be constructed in polynomial time when it exists.*

We now show that for all other positive values of $k_1, k_2 \geq 1$ the $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-BIPARTITE-PARTITION problem is $\mathcal{NP}$-complete. In fact, this remains true even if the input is strong and out-regular.

Theorem 15. *Let k_1, k_2 be positive integers such that $k_1 + k_2 \geq 3$. Then $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-*BIPARTITE-PARTITION *is $\mathcal{NP}$-complete. It remains $\mathcal{NP}$-complete when the input is required to be strongly connected and out-regular.*

Proof. In [3] it was shown that deciding the existence of a 2-partition (V_1, V_2) of a digraph D so that $\Delta^+(D\langle V_1\rangle) \leq a_1$ and $\Delta^+(D\langle V_2\rangle) \leq a_2$ is $\mathcal{NP}$-complete for all a_1, a_2 with $\max\{a_1, a_2\} \geq 1$ even when the input is a strong out-regular digraph. More precisely, when $a_1 = a_2$ the problem is $\mathcal{NP}$-complete for strong p-out-regular digraphs when $p \geq a_1 + 2$ and when $a_1 < a_2$, the problem is $\mathcal{NP}$-complete for strong p-out-regular digraphs when $p \geq a_2 + 1$. The first of these results implies that $(\delta^+ \geq k, \delta^+ \geq k)$-BIPARTITE-PARTITION is $\mathcal{NP}$-complete for strong $(k+2)$-out-regular digraphs. The second result implies that $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-BIPARTITE-PARTITION PROBLEM with $1 \leq k_1 < k_2$ is $\mathcal{NP}$-complete for strong $(k_2 + 1)$-out-regular digraphs. □

5 $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partitions

We now turn to the complexity of deciding whether a given digraph has a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition. We first show that this problem is $\mathcal{NP}$-complete for acyclic digraphs but polynomial-time solvable when the input is a strong digraph. Then we classify, in terms of the strong component digraph, those classes of non-strong digraphs for which the problem is $\mathcal{NP}$-complete. For all the remaining classes it turns out that a partition always exists.

Theorem 16. *It is $\mathcal{NP}$-complete to decide whether an acyclic digraph has a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition.*

Proof. Let $\mathcal{F}$ be an instance of 3-SAT with clauses $C_1, C_2, \ldots, C_m$ over the variables $v_1, v_2, \ldots, v_n$. Form a digraph $M = M(\mathcal{F})$ as follows:

$$V(M) = \bigcup_{i\in[n]} \{x_i, \bar{x}_i, y_i, z_i\} \cup \{c_j \mid j \in [m]\},$$

$$E(M) = \bigcup_{i\in[n]} \{y_i x_i, y_i \bar{x}_i, x_i z_i, \bar{x}_i z_i\} \cup \{x_i c_j \mid x_i \text{ literal of } C_j\} \cup \{\bar{x}_i c_j \mid \bar{x}_i \text{ literal of } C_j\}.$$

Observe that in M the vertices y_i are sources, the vertices z_i are sinks, and the vertices c_j are sinks too. Hence if (V_1, V_2) is a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition of M, then for each $i \in [n]$ we have $y_i \in V_1$ and $z_i \in V_2$, and for each $j \in [m]$ we have $c_j \in V_2$. Consequently, for each $i \in [n]$, exactly one of the vertices $x_i, \bar{x}_i$ belongs to V_1. Hence if we interpret $x_i \in V_1$ (resp. $x_i \in V_2$) as meaning the the variable v_i is true (resp. false), then M has a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition if and only if $\mathcal{F}$ is satisfiable. □

5.1 Solving $(\delta^+ \geq 1, \delta^- \geq 1)$-BIPARTITE-PARTITION for strong digraphs

The digraph M in the above proof is very far from being strong as it has many sources and sinks. A natural question is thus to determine the complexity of $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition problem when the input is restricted to be a strong digraph. We show below that in this case the problem becomes solvable in polynomial time.

For any digraph D we define the following reduction rule.

Reduction Rule A: If for some arc $xy \in A(D)$ we have $d^+(x) = d^-(y) = 1$ then we reduce D by deleting x and y and adding all arcs from $N^-(x)$ to $N^+(y)$ in D (if an arc uv is already present we do not add an extra copy. Similarly, if $z \in N^-(x) \cap N^+(y)$, then we do not add a loop at z.).

In this case, we say that the arc xy was **reduced**.

We call D' a **reduction of** D if D' is obtained from D by applying Reduction Rule A one or more times. We first prove the following lemma.

Lemma 17. *If D' is a reduction of D, then D has a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-colouring if and only if D' has a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-colouring. Furthermore if D is strong then so is D'.*

Proof. Clearly if suffices to prove the lemma when D' was obtained from D by applying Reduction Rule A once, since then the claim follows by induction. So let xy be the arc that was reduced in D in order to obtain D'.

First assume that D' has a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-colouring c. Now consider the following two cases.

Case 1. There exist $u \in N_D^-(x)$ and $v \in N_D^+(y)$ such that $c(u) = 1$ and $c(v) = 2$. In this case we can assign $c(x) = 2$ and $c(y) = 1$ and note that c is now a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-colouring of D.

Case 2. All vertices in $N_D^-(x)$ have colour 2 or all vertices in $N_D^+(y)$ have colour 1. In this case we can assign $c(x) = 1$ and $c(y) = 2$ and note that c is now a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-colouring of D.

Therefore if D' has a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-colouring then so does D.

Now assume that D has a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-colouring, c and consider the following two cases.

Case A. $c(x) = 2$. In this case $c(y) = 1$ as y does not have any in-neighbour of colour 1. Therefore there must be a vertex in $N_D^-(x)$ with colour 1 and a vertex in $N_D^+(y)$ with colour 2. Therefore just restricting the 2-colouring c to $V(D')$ gives us a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-colouring for D'.

Case B. $c(x) = 1$. In this case $c(y) = 2$ as x needs an out-neighbour of colour 2. Now restricting the 2-colouring c to D' gives us a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-colouring for D'.

This completes the proof of the first part of the lemma.

Assume that D is strong. We will show that D' is also strong. Let $u, v \in V(D')$ be arbitrary. As D is strong there exists a (u, v)-path, P, in D. If x or y belong to P, then the arc xy belongs to P as in D we have $d^+(x) = d^-(y) = 1$. If x^- is the predecessor of x on P and y^+ is the successor of y on P, then we obtain a (u, v)-path in D' by deleting x and y from P and adding the arc x^-y^+ (which by definition belongs to D'). As u and v were picked arbitrarily this implies that D' is strong. □

A non-trivial **out-star** (resp. **in-star**) is an out-tree (resp. in-tree) of depth 1, that is, it consists of at least two vertices and the root dominates (resp. is dominated by) all the other vertices in the tree. An **out-galaxy** (resp. **in-galaxy**) is a set of vertex-disjoint non-trivial out-stars (resp. in-stars). A **nebula** is a set of vertex-disjoint non-trivial out- or in-stars.

Every nebula has a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-colouring : colour with 1 the sinks and with 2 the sources. Consequently, if a digraph has a spanning nebula, then it also has a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition. The following result is proved in [7].

Lemma 18 ([7]). *If D is a strong digraph and $D' \subseteq D$ is a strong subdigraph of D of even order, then D has a spanning out-galaxy and therefore also a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition.*

Corollary 19. *Let D be a strong digraph and assume that there exists an arc $xy \in A(D)$ and two vertex disjoint (y, x)-paths in D. Then D has a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition.*

Proof. Let P_1 and P_2 be the two vertex-disjoint (y, x)-paths in D. If P_i $(i \in \{1, 2\})$ has an even number of vertices, then, by Lemma 18, D has a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition as $D\langle V(P_i)\rangle$ is strong (as $P_i \cup xy$ is a cycle) and of even order. So we may assume that both P_1 and P_2 have an odd number of vertices. Let $D' = D\langle V(P_1) \cup V(P_2)\rangle$ and note that D' is strong and $|V(D')| = |V(P_1)| + |V(P_2)| - 2$ is even, implying that D has a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition, by Lemma 18. □

Theorem 20. *If D is a strong digraph, then at least one of the following holds.*

(a): *D has a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition.*

(b): *D contains exactly one vertex.*

(c): *D can be reduced.*

Proof. Assume the theorem is false and that none of (a), (b) or (c) hold. That is $|V(D)| \geq 2$, D cannot be reduced and D does not have a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition.

We now build a sequence of handles as follows. Let $D_1 = H_1$ be a shortest cycle in D (which exists since D is strong and not an isolated vertex) and let $i = 1$. While $V(D_i) \neq V(D)$, let H_{i+1} be a shortest non-trivial D_i-handle and $D_{i+1} = D_i \cup H_{i+1}$. Continue this until $V(D_i) = V(D)$ (which is easy seen to be possible as D is strong).

Let $H_p = (x, v_1, v_2, \cdots, v_l, y)$ be the last handle added in the above process.

If $p = 1$, then the shortest cycle in D is a Hamilton cycle, and so D itself is this cycle. As (a) is false, we note that D is not a 2-cycle and letting xy be any arc on the cycle we note that D can be reduced (as $d^+(x) = d^-(y) = 1$), a contradiction. So we may assume that $p \geq 2$.

As all D_i $(i \in [p])$ are strong and D has no $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition, both $|V(D_{p-1})|$ and $|V(D_p)|$ are odd, by Lemma 18. As $|V(D_{p-1})| + l = |V(D_p)|$ we must therefore have that l is even. We now prove a number of claims.

Claim 20.1. *$d^+_D(v_1) = 1$ and $d^-_D(v_l) = 1$.*

Proof of Claim 20.1. For the sake of contradiction, assume that $d^+_D(v_1) > 1$ and let v_1z be any arc out of v_1 different from v_1v_2. Note that $z \notin D_{p-1}$ by the minimality of H_p. Also $z \notin \{v_3, v_4, \dots, v_l\}$ by the minimality of H_p. As z is also not v_1 or v_2, z does not exist, a contradiction. This proves that $d^+_D(v_1) = 1$. We can prove $d^-_D(v_l) = 1$ analogously. ♢

Claim 20.2. *If $d^+_D(v_i) = 1$, then $d^-_D(v_{i+1}) > 1$ for all $i \in \{1, 2 \dots, l-1\}$.*

Proof of Claim 20.2. This follows immediately from the fact that D cannot be reduced. ♢

Claim 20.3. *If $d^-_D(v_i) > 1$, then $d^+_D(v_{i+1}) = 1$ for all $i = 2, 3, \dots, l-2$.*

Proof of Claim 20.3. For the sake of contradiction, assume that $d^-_D(v_i) > 1$ and $d^+_D(v_{i+1}) > 1$ for some $i \in \{2, 3, \dots, l-2\}$. Let zv_i be any arc in D different from $v_{i-1}v_i$. Note that $z = v_j$ for some $j \in \{i+1, i+2, \dots, l\}$ as otherwise there would exist a shorter handle than H_p. Analogously let $v_{i+1}v_k$ be an arc out of v_{i+1} different from $v_{i+1}v_{i+2}$ and note that $k \in \{1, 2, \dots, i\}$. If $j = i+1$ or $k = i$ then $v_iv_{i+1}v_i$ is a 2-cycle, which is also a strong digraph of even order, a contradiction by Lemma 18. So $j > i+1$ and $k < i$ which implies that we get a contradiction to Corollary 19 (as v_iv_{i+1} is an arc and $v_{i+1}v_{i+2} \dots v_jv_i$ and $v_{i+1}v_kv_{k+1} \dots v_i$ are vertex-disjoint paths). This contradiction completes the proof of Claim 20.3. ♢

By Claim 20.1, $d^+_D(v_1) = 1$. By Claim 20.2, $d^-_D(v_2) > 1$. By Claim 20.3, $d^+_D(v_3) = 1$. By Claim 20.2, $d^-_D(v_4) > 1$. Continuing this, we note that $d^+_D(v_i) = 1$ for all odd $i < l$ and $d^-_D(v_i) > 1$ for all even $i \leq l$. However, by Claim 20.1, we note that $d^-_D(v_l) = 1$, which is the desired contradiction, as l was even. □

Corollary 21. *We can decide in polynomial time whether a strong digraph has a* $(\delta^+ \geq 1, \delta^- \geq 1)$*-bipartite-partition.*

Proof. Let D be a strong digraph. We continuously reduce the digraph, D, until it cannot be reduced any more. Let D' be the resulting digraph. By Lemma 17, D' is strong and has a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition if and only if D does. By Theorem 20, D', and therefore D, has a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition if and only if D' is not a single vertex. All the operations can be done in polynomial time, which completes the proof. □

5.2 Classification of $\mathcal{NP}$-complete instances in terms of their strong component digraph

The **strong component digraph**, denoted by $SC(D)$, of a digraph is obtained by contracting every strong component of D to a single vertex and deleting any parallel arcs obtained in the process. For any acyclic digraph H, let $\mathcal{D}^c(H)$ denote the class of all digraphs D with $SC(D) = H$.

Proposition 22. *A digraph D has a* $(\delta^+ \geq 1, \delta^- \geq 1)$*-bipartite-partition if and only if it has a spanning nebula. Furthermore, given any* $(\delta^+ \geq 1, \delta^- \geq 1)$*-bipartite-partition of D we can produce a spanning nebula in polynomial time, and vice-versa.*

Proof. If $\mathcal{F}$ is a spanning nebula of D, then we obtain a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition by colouring the root of every out-star by 1 and the leaves by 2 and the root of every in-star by 2 and its leaves by 1. Suppose conversely that (V_1, V_2) is a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition of D, where V_i denotes the vertices of colour i and let D' be the spanning subdigraph of D induced by the arcs from V_1 to V_2. Clearly it suffices to prove that D' has a spanning nebula. We prove this by induction on the number of vertices. If D' has just two vertices x, y, then this is clear so assume $|V(D')| \geq 3$. Let v_1v_2 be an arc of D' with $v_i \in V_i$, $i \in \{1, 2\}$. If $(V_1 \setminus \{v_1\}, V_2 \setminus \{v_2\})$ is a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition of $D' - \{v_1, v_2\}$, then we are done by induction, so we may assume that either v_1 is the unique in-neighbour of some non-empty set $V_2' \subseteq V_2$, or v_2 is the unique out-neighbour of some non-empty set $V_1' \subseteq V_1$. We choose V_1', V_2' to be maximal with the given property. By the assumption that $(V_1 \setminus \{v_1\}, V_2 \setminus \{v_2\})$ is not a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition of $D' - \{v_1, v_2\}$, we have $V_i' \neq \{v_i\}$ for $i = 1$ or $i = 2$. Without loss of generality, we have $V_2' \setminus \{v_2\} \neq \emptyset$. Now it is easy to see that $(V_1 \setminus \{v_1\}, V_2 \setminus V_2')$ is a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition of $D' - (\{v_1\} \cup V_2')$ and we are done by induction. The process above clearly yields a polynomial-time algorithm for producing the spanning nebula. □

Let B_r^+ be an r-out-branching. Consider the following procedure that produces an out-galaxy: Let v be a leaf at maximum depth, let v' be its parent and let $T_{v'}^+$ be the out-tree rooted at v' in B_r^+. Then $T_{v'}^+$ is an out-star. Remove this from B_r^+ and continue recursively until either no vertex remains or only the root r remains. In first case, we say that B_r^+ is **winning** and in the second case that B_r^+ is **losing**. Observe that if the root r dominates a leaf in B_r^+, then B_r^+ is winning. Similarly, an in-branching is **winning** (resp. **losing**) if its converse is winning (resp. losing). It is easy to check the following.

Proposition 23. *Let B_r^+ be an r-out-branching.*

- *If B_r^+ is winning, then it has a spanning out-galaxy, and so a* $(\delta^+ \geq 1, \delta^- \geq 1)$*-bipartite-partition.*
- *If B_r^+ is losing, then $B_r^+ - r$ has a spanning out-galaxy, and so a* $(\delta^+ \geq 1, \delta^- \geq 1)$*-bipartite-partition.*

We sometimes use this proposition without explicitly referring to it.

Theorem 24. *Let H be any connected acyclic digraph of order at least 2. The following holds.*

(a) If H has a $(\delta^+ \geq 1, \delta^- \geq 1)$*-bipartite-partition, then all digraphs in $\mathcal{D}^c(H)$ have a* $(\delta^+ \geq 1, \delta^- \geq 1)$*-bipartite-partition and we can produce such a partition in polynomial time.*

(b) If H has no $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition, then it is $\mathcal{NP}$-complete to decide whether a digraph in $\mathcal{D}^c(H)$ has a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition.

Proof. We first prove (a). Let H be an acyclic digraph which has a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition and let $\mathcal{N}$ be a spanning nebula of H (by Proposition 22). Let D be any digraph such that $SC(D) = H$. We prove by induction on the number of stars in $\mathcal{N}$ that D has a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition.

Suppose first that $\mathcal{N}$ consists of one star. Since a digraph has a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition if and only if its converse (obtained by reversing all arcs) has one, we may assume w.l.o.g. that $\mathcal{N}$ consists of an out-star S_r^+ with root r and leaves $s_1, s_2, \dots, s_k$. Let $R, S_1, \dots, S_k$ be the strong components of D that correspond to these vertices. Fix an arc uv such that $u \in R, v \in S_1$. As all of the digraphs $R, S_1, S_2, \dots, S_k$ are strong, they all have an out-branching rooted at any prescribed vertex. In particular this implies that $D' = D - (S_1 - v)$ has an out-branching B' rooted at u. Since its root u is adjacent to one of its leaves v, the out-branching B' is winning. Hence, by Proposition 23, B' has a spanning out-galaxy, and so D' has a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition (V_1', V_2'). Observe that necessarily $v \in V_2'$ because it is a sink in D'. Let B'' be an in-branching of S_1 rooted at v. If B'' is losing, then $S_1 - v$ has a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition (V_1'', V_2''), and $(V_1' \cup V_1'', V_2' \cup V_2'')$ is a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition of D. If B'' is winning, then B'' has a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition (V_1'', V_2''). In addition $v \in V_2''$, because it is a sink in B''. Thus $(V_1' \cup V_1'', V_2' \cup V_2'')$ is a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition of D.

Assume now that $\mathcal{N}$ has more than one star. Let S be such a star (out- or in-). Then it follows from the proof above that the subdigraph of D induced by the vertices of those strong components that are contracted into S in $SC(D)$ has a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition. Now that partition can be combined with any $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition of the digraph induced by the remaining strong components, the existence of which follows by induction. This completes the proof of (a).

We proceed to prove (b). Let H be a connected acyclic digraph on at least two vertices which has no $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition. We construct a maximal induced subdigraph H' from H which has a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition as follows. Let S contain all sinks in H and let H' be induced by $S \cup N^-(S)$. Clearly H' has a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition (by letting the vertices in S have colour 2 and the vertices in $N^-(S)$ have colour 1). Now repeatedly add vertices or a set of vertices to H' such that H' has a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition. When no more vertices can be added we have our desired H'. Let X be the set of vertices not in H' and $X' = N^+(X)$. We fix a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-colouring c' of H'.

Now let $\mathcal{F}$ be an instance of 3-SAT with variables $x_1, x_2, \dots, x_n$ and clauses $C_1, C_2, \dots, C_m$. We may assume that $\mathcal{F}$ cannot be satisfied by setting all variables true or all variables false. Form the digraph $W = W(\mathcal{F})$ as follows: the vertex set of W is $a, b, c_1, c_2, \dots, c_m, v_1, v_2, \dots, v_n$. The arc set consists of all arcs from a to $\{c_1, c_2, \dots, c_m\}$, all arcs from $\{c_1, c_2, \dots, c_m\}$ to b, all arcs from b to $\{v_1, v_2, \dots, v_n\}$, all arcs from $\{v_1, v_2, \dots, v_n\}$ to a and the following arcs between $\{c_1, c_2, \dots, c_m\}$ and $\{v_1, v_2, \dots, v_n\}$: For each $j \in [m]$, and $i \in [n]$, if C_j contains the literal x_i we add the arc c_jv_i to W, and if C_j contains the literal $\bar{x}_i$ we add the arc v_ic_j to W.

Claim 24.1. *The digraph W has a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-colouring c where $c(a) = 2$ and $c(b) = 1$ if and only if $\mathcal{F}$ is satisfiable.*

Proof of Claim 24.1. Assume first that the digraph W has a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-colouring c where $c(a) = 2$ and $c(b) = 1$. Let ϕ be the truth assignment defined by $\phi(x_i) = \mathit{true}$ if $c(v_i) = 2$ and $\phi(x_i) = \mathit{false}$ otherwise. We claim that ϕ satisfies $\mathcal{F}$: For each $j \in [m]$ consider the vertex c_j. If $c(c_j) = 1$, then c_j has an out-neighbour v_q coloured 2, because c is a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-colouring. But by construction $x_q \in C_j$, and $\phi(x_q) = \mathit{true}$ by definition. Hence the clause C_j is satisfied. Similarly, if $c(c_j) = 2$, then it has an in-neighbour v_p coloured 1, and $\bar{x}_p \in C_j$ and $\phi(x_q) = \mathit{false}$. So the clause C_j is satisfied.

Conversely, given a truth assignment ϕ which satisfies $\mathcal{F}$, we start by colouring v_i, $i \in [n]$ by 2 if x_i is true and 1 otherwise. Since ϕ satisfies all clauses it is easy to check that we can extend this colouring to all vertices of $\{c_1, c_2, \dots, c_m\}$. As ϕ sets at least one variable true and at least one false (by our assumption on $\mathcal{F}$), we can finish the colouring by colouring a by colour 2 and b by colour 1. This gives the desired $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-colouring and completes the proof of the claim. $\Diamond$

We will now show how to form a digraph in $\mathcal{D}^c(H)$ which has a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition if and only if $\mathcal{F}$ is satisfiable. Fix a vertex $x \in X$ and an out-neighbour $y \in X'$ of x. Construct the digraph D from H and W as follows: For every vertex of $u \in X \setminus \{x\}$, we add three new (private) vertices u_1, u_2, u_3 and the arcs of the 4-cycle (u_1, u_2, u_3, u, u_1). Replace the vertex y by a copy of W where we let every arc into y in H enter the vertex a (eg. xy becomes xa) and let every arc out of y be incident with b.

Suppose first that $\mathcal{F}$ is satisfiable. By Claim 24.1, there exists a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-colouring c of W with $c(a) = 2, c(b) = 1$. This can easily be extended to a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-colouring of D by letting $c(x) = 1$, colouring each of the private 4-cycles (u_1, u_2, u_3, u, u_1) as $c(u_1) = c(u_3) = 1, c(u) = c(u_2) = 2$ and extending this colouring to the remaining vertices of $H' - y$ using c' above.

Suppose now that D has a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition and let c^* be the associated $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-colouring. By the claim above, it suffices to prove that we must have $c^*(a) = 2$ and $c^*(b) = 1$.

For the sake of contradiction assume that $c^*(b) = 2$. In this case if we restrict c^* to $V(H' - y)$ and assign $c^*(x) = 1$ and $c^*(y) = 2$ we get a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-colouring of $H' \cup \{x\}$ contradicting the fact that H' was maximal. Therefore $c^*(b) = 1$. Now, for the sake of contradiction assume that $c^*(a) = 1$. In this case if we restrict c^* to $V(H' - y) \cup \{x\}$ and assign $c^*(y) = 1$ we get a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-colouring of $H' \cup \{x\}$ contradicting the fact that H' was maximal. Therefore $c^*(a) = 2$ and $c^*(b) = 1$ and the proof is complete. □

6 $(\delta^+ \geq k_1, \delta^- \geq k_2)$-bipartite-partitions when $k_1 + k_2 \geq 3$

Theorem 25. *Let $k_1 \geq 2$ be an integer. It is $\mathcal{NP}$-complete to decide whether a given strong digraph D has a $(\delta^+ \geq k_1, \delta^- \geq 1)$-bipartite-partition.*

Proof. Reduction from 3-SAT.

Let Q be the digraph whose vertex set is the disjoint union of two sets W, Z of size $k_1 - 1$, and $\{v, \bar{v}, y\}$ and with arc set $\{yv, y\bar{v}\} \cup \bigcup_{w \in W}\{wy, yw\} \cup \bigcup_{z \in Z}\{vz, \bar{v}z, zy\}$. See Figure 2.

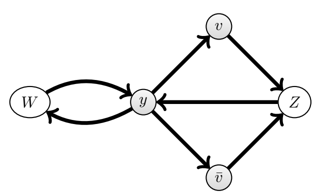

Figure 2: The gadget Q.

Let $\mathcal{F}$ be an instance of 3-SAT with variables $x_1, x_2, \dots, x_n$ and clauses $C_1, C_2, \dots, C_m$. By adding extra clauses not affecting the truth value of the original ones if necessary, we may assume that every variable x_i appears in some clause as the literal x_i and in another clause as the literal $\bar{x}_i$. Form the digraph $D = D(\mathcal{F})$ as follows: take n disjoint copies $Q_1, Q_2, \dots, Q_n$ of Q and denote the sets corresponding to W, Z in Q_i by W_i, Z_i respectively and the vertices of Q_i corresponding to $v, \bar{v}, y$ by $v_i, \bar{v}_i, y_i$ respectively. The vertices v_i and $\bar{v}_i$ will correspond to the variable x_i: v_i to the literal x_i and $\bar{v}_i$ to the literal $\bar{x}_i$. Add m vertices $c_1, c_2, \dots, c_m$, where c_j corresponds to clause C_j, $j \in [m]$. Add the arcs of a directed cycle with vertex set $\bigcup_{i \in [n]} Z_i$, and the arc $c_j y_1$ for all $j \in [m]$. Finally, for each $j \in [m]$ we add three arcs from the vertices corresponding to the literals of C_j to the vertices c_j. E.g. if $C_j = (\bar{x}_1 \vee \bar{x}_2 \vee x_3)$ then we add the arcs $\bar{v}_1 c_j, \bar{v}_2 c_j, v_3 c_j$. It is easy to check that D is strong.

Assume that (V_1, V_2) is a $(\delta^+ \geq k_1, \delta^- \geq 1)$-bipartite-partition of D. Since the out-degree of each c_j, $j \in [m]$ is 1, these vertices must belong to V_2. Similarly, $W_i \subset V_2$, for all $i \in [n]$. This implies that $y_i \in V_1$ and thus $Z_i \subset V_2$ for every $i \in [n]$. Those two facts imply that exactly one of $v_i, \bar{v}_i$ belong to V_1 and the other belongs to V_2. Let ϕ be the truth assignment defined by $\phi(x_i) = \mathit{true}$ if $v_i \in V_1$ and

$\phi(x_i) = false$ otherwise. One easily sees that ϕ satisfies $\mathcal{F}$ as every vertex c_j has an in-neighbour in V_1 which is a vertex corresponding to a literal of C_j which is then assigned *true* by ϕ.

Reciprocally, assume that there is a truth assignment ϕ satisfying $\mathcal{F}$. Let (V_1, V_2) be the 2-partition of D defined by

$$V_1 = \{y_i \mid i \in [n]\} \cup \{v_i \mid \phi(x_i) = true\} \cup \{\bar{v}_i \mid \phi(x_i) = false\}, \text{and}$$
$$V_2 = \{c_j \mid j \in [m]\} \cup \bigcup_{i=1}^{n} (W_i \cup Z_i) \cup \{v_i \mid \phi(x_i) = false\} \cup \{\bar{v}_i \mid \phi(x_i) = true\}.$$

One easily checks that (V_1, V_2) is a $(\delta^+ \geq k_1, \delta^- \geq 1)$-bipartite-partition of D. In particular, since x_i and $\bar{x}_i$ belong to at least one clause, v_i and $\bar{v}_i$ have each at least one out-neighbour in $\{c_1, \ldots, c_m\}$, which is a subset of V_2. □

Theorem 26. *It is $\mathcal{NP}$-complete to decide whether a given strong digraph D has a $(\delta^+ \geq 2, \delta^- \geq 2)$-bipartite-partition.*

Proof. The proof is a reduction from 3-SAT, which is very similar to the one of Theorem 25.

Let Q' be the digraph with vertex set $\{w, y, y', v, \bar{v}, z\}$ and with arc set

$$\{y'y, yw, y'w, wy', yv, y\bar{v}, y'v, y'\bar{v}, y'z, zy, vz, \bar{v}z\}.$$

See Figure 3.

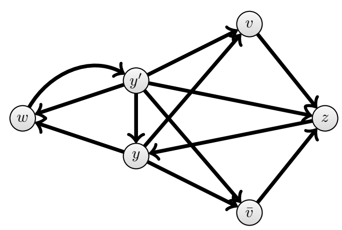

Figure 3: The gadget Q'.

Let $\mathcal{F}$ be an instance of 3-SAT with variables $x_1, x_2, \ldots, x_n$ and clauses $C_1, C_2, \ldots, C_m$. By adding extra clauses not affecting the truth value of the original ones if necessary, we may assume that every variable x_i appears in some clause as the literal x_i and in another clause as the literal $\bar{x}_i$. Form the digraph $D = D(\mathcal{F})$ as follows: take n disjoint copies $Q'_1, Q'_2, \ldots, Q'_n$ of Q' and denote the vertices of Q_i corresponding to $w, y, y', v, \bar{v}, z$ by $w_i, y_i, y'_i, v_i, \bar{v}_i, z_i$ respectively. The vertices v_i and $\bar{v}_i$ will correspond to the variable x_i: v_i to the literal x_i and $\bar{v}_i$ to the literal $\bar{x}_i$. Add m vertices $c_1, c_2, \ldots, c_m$, where c_j corresponds to clause C_j, $j \in [m]$. Add the arcs of the directed cycle $z_1 z_2 \ldots z_n z_1$, and the arc $c_j y_1$ for all $j \in [m]$. Finally, for each $j \in [m]$ we add three arcs from the vertices corresponding to the literals of C_j to the vertices c_j. E.g. if $C_j = (\bar{x}_1 \vee \bar{x}_2 \vee x_3)$ then we add the arcs $\bar{v}_1 c_j, \bar{v}_2 c_j, v_3 c_j$. It is easy to check that D is strong.

Assume that (V_1, V_2) is a $(\delta^+ \geq 2, \delta^- \geq 2)$-bipartite-partition of D. Since the out-degree of each c_j, $j \in [m]$ is 1, these vertices must belong to V_2. Similarly, $w_i \in V_2$, for all $i \in [n]$. This implies that $\{y_i, y'_i\} \subset V_1$ and thus $z_i \in V_2$ for every $i \in [n]$. Those two facts imply that exactly one of $v_i, \bar{v}_i$ belong to V_1 and the other belongs to V_2. Let ϕ be the truth assignment defined by $\phi(x_i) = true$ if $v_i \in V_1$ and $\phi(x_i) = false$ otherwise. One easily sees that ϕ satisfies $\mathcal{F}$ as every vertex c_j has an in-neighbour in V_1 which is a vertex corresponding to a literal of C_j which is then assigned *true* by ϕ.

Reciprocally, assume that there is a truth assignment ϕ satisfying $\mathcal{F}$. Let (V_1, V_2) be the 2-partition of D defined by

$$V_1 = \bigcup_{i=1}^{n} \{y_i, y'_i\} \cup \{v_i \mid \phi(x_i) = true\} \cup \{\bar{v}_i \mid \phi(x_i) = false\}, \text{and}$$

$$V_2 = \{c_j \mid j \in [m]\} \cup \bigcup_{i=1}^{n} \{w_i, z_i\} \cup \{v_i \mid \phi(x_i) = false\} \cup \{\bar{v}_i \mid \phi(x_i) = true\}.$$

One easily checks that (V_1, V_2) is a $(\delta^+ \geq 2, \delta^- \geq 2)$-bipartite-partition of D. In particular, since x_i and $\bar{x}_i$ belong to at least one clause, v_i and $\bar{v}_i$ have each at least one out-neighbour in $\{c_1, \dots, c_m\}$, which is a subset of V_2. □

Corollary 27. *Let $k_1, k_2 \geq 1$ be positive integers such that $k_1 + k_2 \geq 3$. It is $\mathcal{NP}$-complete to decide whether a given strong digraph D has a $(\delta^+ \geq k_1, \delta^- \geq k_2)$-bipartite-partition.*

Proof. Without loss of generality we may assume that $k_1 \geq k_2$ (otherwise swap k_1 and k_2 and reverse all arcs).

We prove the result by induction on $k_1 + k_2$. If $k_2 = 1$, then we have the result by Theorem 25, and if $k_1 = k_2 = 2$, we have the result by Theorem 26.

Assume now that $k_1 + k_2 \geq 5$ and $k_2 \geq 2$. We give a reduction from $(\delta^+ \geq k_1 - 1, \delta^- \geq k_2 - 1)$-BIPARTITE-PARTITION which is $\mathcal{NP}$-complete by the induction hypothesis. Let D be a digraph. We construct D' from D by adding two vertices x_1, x_2 and all arcs from $V(D)$ to x_2, all arcs from x_1 to $V(D)$ and the two arcs x_1x_2, x_2x_1. Clearly D' is strong.

For any $(\delta^+ \geq k_1, \delta^- \geq k_2)$-bipartite-partition (V'_1, V'_2) of D', x_2 is in V'_2 because it has out-degree 1, and $x_1 \in V'_1$ because it has in-degree 1. Therefore $(V'_1 \setminus \{x_1\}, V'_2 \setminus \{x_2\})$ is a $(\delta^+ \geq k_1 - 1, \delta^- \geq k_2 - 1)$-bipartite-partition of D. Reciprocally, if there is $(\delta^+ \geq k_1 - 1, \delta^- \geq k_2 - 1)$-bipartite-partition (V_1, V_2) of D, then $(V_1 \cup \{x_1\}, V_2 \cup \{x_2\})$ is a $(\delta^+ \geq k_1, \delta^- \geq k_2)$-bipartite-partition of D'.

Hence D' has a $(\delta^+ \geq k_1, \delta^- \geq k_2)$-bipartite-partition if and only if D has a $(\delta^+ \geq k_1 - 1, \delta^- \geq k_2 - 1)$-bipartite-partition. □

7 Strong 2-partitions

Recall that a strong 2-partition of a digraph D is a partition (V_1, V_2) such that $B_D(V_1, V_2)$ is strong.

Theorem 28. *For every $r > 0$, there exists an r-strong eulerian digraph D which has no strong 2-partition (that is, D has no spanning strong bipartite subdigraph).*

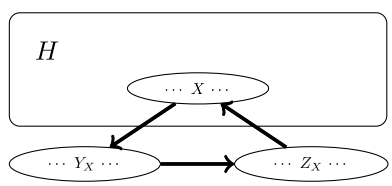

Figure 4: Adding the gadget $G_r(X)$.

Proof. Let $r > 0$ be an arbitrary integer, let H be an arbitrary digraph on at least r vertices and let $X \subseteq V(H)$ be a subset of size r. Let D be the digraph that we obtain from H and X by adding two new vertex sets Y_X and Z_X of size r to H and all arcs from X to Y_X, all arcs from Y_X to Z_X and all arcs from Z_X to X. The digraph induced by $X \cup Y_X \cup Z_X$ is the gadget $G_r(X)$ and we say that the digraph D is obtained from H by adding the gadget $G_r(X)$ to H (see Figure 4).

Now the following holds:

(1) If (V_1, V_2) is a strong 2-partition of D, then the vertices of X cannot all belong to the same set V_i.

(2) For every set S of at most $r - 1$ vertices in D, we have that $D\langle (X \cup Y_X \cup Z_X) \setminus S \rangle$ is strong.

Property (2) follows from the fact that the gadget $G_r(X)$ is clearly r-strong. To prove (1) assume that (V_1, V_2) is a strong 2-partition, and without loss of generality assume that $X \subseteq V_1$. In this case, $Y_X \subseteq V_2$ as if $y \in Y_X \cap V_1$ then y has no arc into it in $B_D(V_1, V_2)$, a contradiction. Analogously

$Z_Y \subseteq V_1$ (as $Y_Z \subseteq V_2$) and $Z_Y \subseteq V_2$ (as $X \subseteq V_1$), a contradiction.

Now let U be any digraph on $2r-1$ vertices such that $d^+_U(v) = d^-_U(v)$ for all vertices v of U (in particular, U could be just a stable set). Construct D' from U by adding a gadget $G_r(X)$ for each of the $\binom{2r-1}{r}$ subsets X of r vertices of U. By property (2) of the gadget $G_r(X)$, D' is r-strong and it is easy to check that is is eulerian. Furthermore, D' cannot have a strong 2-partition, because in any 2-partition (V_1, V_2) of $V(D')$ there will be a set, $X \subseteq V(U)$, of size r belonging to the same set V_i, contradicting property (1) of the gadget $G_r(X)$. □

Theorem 29. *For every fixed positive integer $r \geq 3$, it is $\mathcal{NP}$-complete to decide whether an r-strong eulerian digraph has a strong 2-partition.*

Proof. We prove the result by reduction from 2-colourability of r-uniform hypergraphs. In this problem, being given an r-uniform hypergraph, we want to colour its ground set with two colours such that no hyperedge is monochromatic. It is known that this problem is $\mathcal{NP}$-complete for $r \geq 3$ [9], even restricted to connected hypergraphs. So let $\mathcal{H}$ be a connected hypergraph on ground set V and with hyperedges $\{X_1, \dots, X_m\}$. We construct from $\mathcal{H}$ the digraph $D_{\mathcal{H}}$ by adding to V the gadget $G_r(X_i)$ for $i = 1, \dots, m$. By Property (2) of the proof of Theorem 28 and as $\mathcal{H}$ is connected, $D_{\mathcal{H}}$ is an r-strong eulerian digraph. Using Property (1) it is straightforward to check that $D_{\mathcal{H}}$ has a strong 2-partition if, and only if, $\mathcal{H}$ admits a 2-colouring. □

The above proof also works for finding spanning bipartite subgraphs with semi-degree at least 1, so we obtain the following.

Theorem 30. *For every integer $r \geq 3$, it is $\mathcal{NP}$-complete to decide whether an r-strong eulerian digraph contains a spanning bipartite digraph with minimum semi-degree at least 1.*

8 Remarks and open questions

We looked at some natural properties of the spanning bipartite subdigraphs induced by a 2-partition. We list some further results and open problems in that field.

Theorem 31 ([2])**.** *It is $\mathcal{NP}$-complete to decide whether a digraph has a cycle factor in which all cycles are even.*

Corollary 32. *It is $\mathcal{NP}$-complete to decide whether a digraph D has a 2-partition (V_1, V_2) such that the bipartite digraph $B_D(V_1, V_2)$ has a cycle-factor.*

A **total dominating set** in a graph $G = (V, E)$ is a set of vertices $X \subseteq V$ such that every vertex of V has a neighbour in X.

Theorem 33 ([8])**.** *It is $\mathcal{NP}$-complete to decide whether a graph G has a 2-partition (V_1, V_2) so that V_i is a total dominating set of G for $i \in \{1, 2\}$.*

This directly implies the following.

Corollary 34. *It is $\mathcal{NP}$-complete to decide whether a symmetric digraph D has a 2-partition (V_1, V_2) such that $\delta^+(D\langle V_i \rangle) \geq 1$ for $i \in \{1, 2\}$ and $\delta^+(B_D(V_1, V_2)) \geq 1$.*

For any digraph D, it is easy to obtain a 2-partition of D such that the bipartite digraph $B_D(V_1, V_2)$ is an eulerian digraph. Indeed the 2-partition $(V(D), \emptyset)$ produces a corresponding bipartite digraph with no arcs which is then eulerian. On the other hand, if we ask for a bipartite eulerian subdigraph with minimum semi-degree at least 1, a slight variation in the proof of Theorem 29 gives the following result.

Theorem 35. *It is $\mathcal{NP}$-complete to decide whether a digraph D has a 2-partition (V_1, V_2) such that $B_D(V_1, V_2)$ is a bipartite eulerian digraph with minimum semi-degree at least 1.*

Proof. We use the same reduction as in the proof of Theorem 29 and the same gadget as in the proof of Theorem 28. From a hypergraph $\mathcal{H}$ with hyperedges $X_1, \dots, X_m$ we construct the digraph $D_{\mathcal{H}}$. Using the same arguments as in the proof of Theorem 28, it is easy to see that if $D_{\mathcal{H}}$ admits a 2-partition (V_1, V_2) such that $B_D(V_1, V_2)$ is a bipartite eulerian subdigraph of $D_{\mathcal{H}}$ with minimum semi-degree at least 1 then no hyperedge of $\mathcal{H}$ is totally contained in a part V_i what means that $\mathcal{H}$ is 2-colourable. Conversely if $\mathcal{H}$ is 2-colourable, it is possible to obtain a partition of $D_{\mathcal{H}}$ whose arcs going across form a bipartite eulerian subdigraph of $D_{\mathcal{H}}$ with minimum semi-degree at least 1. Indeed if an hyperedge X_i contains p vertices of colour 1 and q vertices of colour 2, then we colour p vertices of both Y_{X_i} and Z_{X_i} by colour 1 and we colour the remaining q vertices of Y_{X_i} and Z_{X_i} by colour 2. It is easy to see now that this partition of $G_r(X_i)$ produces a spanning eulerian subdigraph of $G_r(X_i)$, and therefore also for $D_{\mathcal{H}}$. □

However if we just ask for a non-empty bipartite eulerian subdigraph of a digraph, we obtain the following question.

Question 36. What is the complexity of deciding whether a digraph D has a 2-partition (V_1, V_2) such that $B_D(V_1, V_2)$ is an eulerian digraph with at least one arc?

Notice that if we restrict ourselves to the eulerian instances, this latter question is equivalent to the following one.

Question 37. What is the complexity of deciding whether an eulerian digraph D has a 2-partition (V_1, V_2) such that $D\langle V_i\rangle$ is eulerian and non empty for $i \in \{1, 2\}$?

Corollary 21 asserts that we can decide in polynomial time whether a strong digraph has a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition. On the other hand a consequence of Claim 24.1 is that this problem becomes $\mathcal{NP}$-complete if we fix the colour of two vertices. A slight modification in the proof of Claim 24.1 shows that it is also the case if we only fix the colour of one vertex. More precisely we look at the following problem.

Problem 38 (($\delta^+ \geq 1, \delta^- \geq 1$)-BIPARTITE-PARTITION-WITH-A-FIXED-VERTEX).
Input: A digraph D, a vertex x of D and a colour $i \in \{1, 2\}$.
Question: Does D admit a $(\delta^+ \geq 1, \delta^- \geq 1)$-bipartite-partition (V_1, V_2) such that $x \in V_i$?

Theorem 39. ($\delta^+ \geq 1, \delta^- \geq 1$)-BIPARTITE-PARTITION-WITH-A-FIXED-VERTEX *is $\mathcal{NP}$-complete even when restricted to strong digraphs.*

Proof. We use the same reduction from 3-SAT than in the proof of Claim 24.1 and the same gadget W to encode a 3-SAT formula $\mathcal{F}$. We add two vertices c and d and the arcs bc, cd and da to W and call W' the resulting digraph. We consider W' as an instance of ($\delta^+ \geq 1, \delta^- \geq 1$)-BIPARTITE-PARTITION-WITH-A-FIXED-VERTEX where we ask that $c \in V_2$. Putting c in V_2 forces b and d to be in V_1 and a to be in V_2. So using Claim 24.1 we have that $\mathcal{F}$ is satisfiable if and only if D' is a positive instance of the problem. □

Finally if we just want a 2-partition (V_1, V_2) so that every vertex in V_1 has an out-neighbour in V_2 and every vertex in V_2 has a neighbour (can be out- or in-) in V_1, then it turns out that such a partition always exists.

Theorem 40. *Every digraph D with $\delta(D) \geq 1$ has a $(\delta^+ \geq 1, \delta \geq 1)$-bipartite-partition.*

Proof. Clearly we may assume that D is connected. Let X_1 contain one vertex from each terminal component of D (if D is strong then D itself is a terminal component and $|X_1| = 1$). Note that X_1 is a stable set. Let X_2 be all vertices not in X_1 with an arc into X_1. Let X_3 be all vertices not in $X_1 \cup X_2$ with an arc into X_2. Let X_4 be all vertices not in $X_1 \cup X_2 \cup X_3$ with an arc into X_3. continue this process until some $X_k = \emptyset$. Note that $V(D) = X_1 \cup X_2 \cup \cdots \cup X_{k-1}$ as every vertex in D has a path into a vertex from X_1. Let V_1 contain all X_i when i is even and let V_2 contain all X_i when i is odd. Note that as every vertex in X_i has an arc into X_{i-1} when $i > 1$ and every vertex in X_1 has an arc into it from X_2 as D is connected. So the partition (V_1, V_2) is a $(\delta^+ \geq 1, \delta \geq 1)$-bipartite-partition of D. □

It could be interesting to extend the previous result to other values of k_1 and k_2 or at least to determine the complexity of finding such a partition.

Question 41. For any fixed pair (k_1, k_2) of positive integers, what is the complexity of deciding whether a given digraph has a $(\delta^+ \geq k_1, \delta \geq k_2)$-bipartite-partition?

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