On Hamiltonian Walks

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Abstract: *If G is a connected graph, of order n, a hamiltonian walk of G is a closed walk of minimum length which contains every vertex of G. The length of a hamiltonian walk is greater than or equal to n, with equality if and only if G is hamiltonian. Following J.L. Jolivet, we define $s(G)$ to be the difference of the length of a hamiltonian walk in G and n. Thus $s(G) \geq 0$ and $s(G) = 0$ if and only if G is hamiltonian. We prove the following generalizations of Ore's and Pósa's theorems (thus answering conjectures of J.L.Jolivet):*

Theorem : Let $c \leq n$; if for every pair of non-adjacent vertices x and y, $d(x) + d(y) \geq c$, then $s(G) \leq n - c$.

Theorem : Let $c \leq n$; if :

$$|\{x : d(x) \leq j\}| \leq j - 1 \quad \text{for } 0 \leq j \leq (c-1)/2$$

$$|\{x : d(x) \leq (c-1)/2\}| \leq (c-1)/2 \quad (\text{if } c \text{ is odd})$$

then $s(G) \leq n - c$.

In order to prove these theorems we need results concerning the existence of elementary cycles of length at least c in connected and 2-connected graphs.

1. INTRODUCTION

We shall follow the terminology of Berge's book [1] , in which the reader can find all the definitions not given here.

G will always denote a *connected graph of order n* .

1.1. Recall that a hamiltonian cycle (path) of G is an elementary cycle (path) of length n, that is, a cycle (path), which contains every vertex of G exactly once. Not all graphs are hamiltonian, and several weaker conditions can be considered.

1.2. One of the simplest conditions concerns the existence of a cycle of length c or of length at least c. We shall return to this problem in &3.

Another possible generalization of a hamiltonian path is a tree with at most h end vertices : existence conditions have been óbtained in this case by Las Vergnas [12] and the author [2] .

1.3. Here we are interested in another generalization introduced by Goodman and Hedetniemi [8] and Jolivet [10,11] .We define a spanning closed walk of a connected graph G to be a closed walk (called sometîmes cycle), which contains every vertex of G.(This is called in [10] a pseudo-hamiltonian cycle).The length of a spanning closed walk is greater than or equal to n ; if the length is n it is a hamiltonian cycle. Exactly we are interested in spanning closed walks of minimum length h(G), which we call hamiltonian walks such as in [8] . Following Jolivet [10,11] we define for a connected graph G, *s(G) to be the least integer k such that there exists a spanning closed walk of length n + k , that is $s(G) = h(G) - n$* . Thus $s(G) \geq 0$ and $s(G) = 0$ if and only if G is hamilto--nian. For example it is proved in [10] , that for a tree T, $s(T) = n-2$, and thus $s(G) \leq n-2$ for every graph G.

1.4. *Remark :* The parameter s(G) gives a measure of the hamiltonicity of a graph. Another parameter considered in [4,9,10,14,15] is the minimum number of edges, which must be added to G in order to make it hamiltonian. If G is not hamiltonian this number is equal to the minimum number of vertex-disjoint paths covering the vertices of G.

1.5. The following lemmas of [10,11] give a relation between the existence of elementary cycles and the parameter s(G) and will be very useful.

1.5.1. *If G contains an elementary cycle of length c, then $s(G) \leq n-c$.*

1.5.2. *Lemma : If G contains two elementary cycles of length c and c' , having at most one vertex in common, then $s(G) \leq n + 2 - (c + c')$.*

Lemma 1.5.2. was proved in [11]for two disjoint cycles , but a similar proof shows that if the two cycles have one vertex in common , then $s(G) \leq n + 1 - (c + c')$.

1.6. The aim of this paper is to prove for hamiltonian walks generaliza--tions of Ore's and Pósa's theorems, thus answering conjectures of Jolivet [10,11] . We first recall some results on the existence of hamiltonian cycles ; then we give sufficient conditions for the existence of elementary cycles of length at least c for a connected and a 2-connec--ted graph. These are used to prove the theorems.

The results given here are part of the author's thesis [3] and we shall generally indicate only sketches of the proofs.

1.7. *Notations : In what follows cycle will always mean elementary cycle; E will denote the set of edges of G ; d(x) means the degree of the vertex x ; $\{x_1,x_2,\dots,x_n\}$ is an arbitrary ordering of the vertices of G ; the sequence of the degrees of the vertices in a non-decreasing order will be denoted by* $d_1,d_2,\dots,d_n$.

2. HAMILTONIAN CYCLES

The following results are known. The reader can find proofs, references and more results in Bondy and Chvátal [6].

G has an hamiltonian cycle if :

2.1. Dirac : $d(x) \geq n/2$ for every vertex x .

2.2. Ore : $d(x) + d(y) \geq n$ if $\{x,y\} \notin E$.

2.3. Pósa : $|\{x : d(x) \leq j\}| < j$ for $j < (n-1)/2$ and

$|\{x : d(x) \leq (n-1)/2\}| \leq (n-1)/2$ for n odd .

2.4. Bondy : $j < k , d_j \leq j , d_k \leq k-1 \Rightarrow d_j + d_k \geq n$.

2.5. Chvátal : $d_j \leq j < n/2 \Rightarrow d_{n-j} \geq n-j$.

2.6. Las Vergnas : $\left.\begin{array}{l} j < k,\ k \geq n-j,\ \{x_j,x_k\} \notin E \\ d(x_j) \leq j\ ,\ d(x_k) \leq k-1 \end{array}\right\} \Rightarrow d(x_j) + d(x_k) \geq n$.

3. ELEMENTARY CYCLES OF LENGTH AT LEAST C.

Case 1 : G connected.

3.1. *Theorem : Let $\{x_1, x_2, \ldots, x_n\}$ be any ordering of the vertices of G and let $c \leq n$. Suppose that :*

$$\left.\begin{array}{l} j < k \ , \ \{x_j, x_k\} \notin E \\ d(x_j) \leq j \ , \ d(x_k) \leq k-1 \end{array}\right\} \Rightarrow \max\{d(x_j) \ ; \ d(x_k)\} \geq c \ .$$

Then G contains a cycle of length at least c+1.

Proof : Suppose the theorem false for some value of n and let G = (X,E) be a graph with n vertices, which satisfies the hypothesis, but not the conclusion, and has the maximum possible number of edges. Clearly G is not complete. Let x_j and x_k be two non adjacent vertices of G with $j < k$, and such that : i) there exists between x_j and x_k a path having the maximum possible length p .

ii) j+k is as large as possible subject to i).

Then the graph obtained by adding to G the edge $\{x_j, x_k\}$ satisfies the hypothesis of the theorem and by the maximality of E, contains a cycle of length at least c+1. Thus the length p of the path between x_j and x_k satisfies $p \geq c$. Let $P(x_j, x_k) = \{x_j = y_1, y_2, \ldots, y_i, \ldots, y_p = x_k\}$ be the path between x_j and x_k and let :

$J = \{y_i \ : \ y_i \in P(x_j, x_k) \text{ and } \{x_j, y_{i+1}\} \in E\}$

$K = \{y_i \ : \ y_i \in P(x_j, x_k) \text{ and } \{y_{i-1}, x_k\} \in E\}$

We have $|J| = d(x_j)$: indeed, x_j cannot be joined to a vertex z not in $P(x_j, x_k)$, (otherwise we would have between z and x_k a path of length greater than p. Also $\{z, x_k\} \notin E$, otherwise we would have a cycle of length greater than $p+1 \geq c+1$, contradicting i)). Furthermore if $y_i \in J$, $\{y_i, y_{i-1}, \ldots, y_1, y_{i+1}, \ldots, y_{p-1}, y_p\}$ is a path of length p ; thus $\{y_i, x_k\} \notin E$ (otherwise we would have a cycle of length $p+1 \geq c+1$), and therefore by the maximality of j+k (condition ii)) the label of y_i is at most j. Then we have |J| points with labels at most j . Thus $|J| = d(x_j) \leq j$. Similarly $|K| = d(x_k)$ and the points of K and x_j have labels at most k. Hence $|K| + 1 = d(x_k) + 1 \leq k$. Thus the conditions of the hypothesis are satisfied and either $d(x_j)$ or $d(x_k)$ is at least c . Suppose, without loss of generality, that $d(x_j) \geq c$; since x_j is only joined to vertices in $P(x_j, x_k)$, we obtain a cycle of length at least c+1, contrary to our assumption. □

3.2. *Corollaries : Let $c \leq n$. Then G contains a cycle of length at least c+1 if any one of the following conditions is satisfied :*

a) Pósa [13] : $|\{x \ : \ d(x) \leq j\}| \leq j$ for $0 \leq j \leq c-1$ $(c \geq 2)$.

b) $\left.\begin{array}{l} j < k, \ \{x_j, x_k\} \notin E \\ d(x_j) \leq j \ , \ d(x_k) \leq k-1 \end{array}\right\} \Rightarrow d(x_j) + d(x_k) \geq \min\{2c-1 \ ; \ n\}$.

c) Woodall [16] : $d(x) + d(y) \geq \min\{2c-1 \ ; \ n\}$ if $\{x,y\} \notin E$.

Proofs : In order to prove b) and c), we need the results on hamiltonian cycles (2.6. and 2.2.), when $2c-1 \geq n$.

3.3. *Conjectures : We conjecture that each of the following conditions is sufficient to ensure the existence of a cycle of length at least c+1.*

a) $\left.\begin{array}{l} j < k \ , \ k \geq \min\{2c-1 \ ; \ n\} - j \\ \{x_j, x_k\} \notin E \ , \ d(x_j) \leq j \ , \ d(x_k) \leq k-1 \end{array}\right\} \Rightarrow \max\{d(x_j) \ ; \ d(x_k)\} \geq c$.

b) $\left.\begin{array}{l} j < k \ , \ \{x_j, x_k\} \notin E \\ d(x_j) \leq j-1 \ , \ d(x_k) \leq k-2 \end{array}\right\} \Rightarrow d(x_j) + d(x_k) \geq \min\{2c-1 \ ; \ n\}$.

3.4. *Remark* . Other generalizations are possible ; some of them are false : for example we cannot replace in a) the condition $k \geq \min\{2c-1 \ ; \ n\} - j$ by $k \geq n-j$ or the conditions $d(x_j) \leq j$ and $d(x_k) \leq k-1$ by $d(x_j) \leq j-n+c+1$ or $d(x_k) \leq k-n+c$.(see [3])

Case 2 : G 2-connected.

3.5. *Theorem : Let G be a 2-connected graph and $c \leq n$. Suppose that :*

$$\left.\begin{array}{l} j < k \ , \ \{x_j, x_k\} \notin E \\ d(x_j) \leq j \ , \ d(x_k) \leq k-1 \end{array}\right\} \Rightarrow d(x_j) + d(x_k) \geq c \ .$$

Then G contains a cycle of length at least c.

Proof : The proof is very similar to Bondy's proof in [5] of the result given below (2.6.a) and thus we omit it .(for a complete proof see [3]).

3.6. *Corollaries : Let G be 2-connected and $c \leq n$. Then G contains a cycle of length at least c if any one of the following conditions is satisfied :*

a) Bondy [5] : $d_j \leq j$, $d_k \leq k$ $(j \neq k)$ $\Rightarrow$ $d_j + d_k \geq c$.

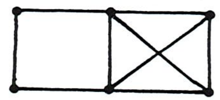
b) Pósa [13] : $|\{x \ : \ d(x) \leq j\}| \leq j-1$ for $1 \leq j \leq [(c-1)/2]$

c) $d(x) + d(y) \geq c$ if $\{x,y\} \notin E$.

d) Dirac [7] : $d(x) \geq [(c+1)/2]$ for every vertex x .

3.7. *Remark :* In contrast with the hamiltonian case, when $c \leq n$, there exist graphs satisfying the hypothesis of 2.6.c) but not 2.6.b) : for example consider the graph containing a complete graph on j vertices , a complete graph on $n-j$ vertices and a matching of cardinality j between these two subgraphs , with $j \leq [(c-1)/2]$.

Example : $j = 2$, $c = 5$, $n = 6$



3.8. *Conjecture : Let G be a 2-connected graph and let $c \leq n$. We conjecture that the following condition is sufficient for the existence of a cycle of length at least c :*

$$\left.\begin{array}{l} j < k\ ,\ k \geq c-j\ ,\ \{x_j,x_k\} \notin E \\ d(x_j) \leq j\ ,\ d(x_k) \leq k-1 \end{array}\right\} \Rightarrow d(x_j) + d(x_k) \geq c.$$

3.9. *Remark :* The reader can find other results and conjectures in Woodall [16,17] and in [3].

Case 3 : We shall give a refinement of theorem 3.1., which will be very useful in the proofs of theorems 4.3. and 4.4.

3.10. *Theorem : Let G' denote a 2-connected graph of order n+1 and let G be the graph obtained by deleting a vertex of G' and let $c \leq n$. The degrees being the degrees in the graph G ; G' contains a cycle of length at least c+2 if any one of the following conditions is satisfied :*

a) $$\left.\begin{array}{l} j < k\ ,\ \{x_j,x_k\} \notin E \\ d(x_j) \leq j\ ,\ d(x_k) \leq k-1 \end{array}\right\} \Rightarrow \max\{d(x_j)\ ;\ d(x_k)\} \geq c .$$

b) $|\{x : d(x) \leq j\}| \leq j$ for $0 \leq j \leq c+1$.

c) $d(x) + d(y) \geq 2c-1$ if $\{x,y\} \notin E$.

Proof : Cases b and c are corollaries of case a). The beginning of the proof of the case a is the same as the proof of theorem 3.1. Suppose the theorem false for some value of n and let $G' = (X \cup a, E')$ be a 2-connected graph of order $n+1$, which satisfies the condition a) , but has no cycle of length at least $c+2$ and has the maximum possible number of edges. Let G be the subgraph obtained by deleting the vertex a. G is not complete ; otherwise as G' is 2-connected, a is joined to at least two vertices of G and then G' would contain an hamiltonian cycle of length $n+1 \geq c+2$.

Let x_j and x_k be two non adjacent vertices defined as in the proof of theorem 3.1.Similarly as in the proof of this theorem we can show that $\max\{d(x_j)\ ;\ d(x_k)\} \geq c$. If $\max\{d(x_j)\ ;\ d(x_k)\} \geq c+1$, we obtain a cycle of length at least $c+2$. Thus we can suppose without loss of generality that $d(x_j) = c$ .Furthermore x_j must be joined to the vertices y_i of the path $P(x_j,x_k)$ such that $2 \leq i \leq c+1$, otherwise we would have a cycle of length at least $c+2$. As G' is 2-connected, there exist in G' two vertex-disjoint paths joining the two set of vertices : $\{y_i : 1 \leq i \leq c\}$ and $\{y_i : c+2 \leq i \leq p\}$.(Remark that , as $y_p = x_k$ is non adjacent to $y_1 = x_j$, y_p is different from x_{c+1} and thus the two sets are non empty). One of these paths does not contain the vertex y_{c+1} . Let $P'(y_r,y_s)$ be the part of this path , where the end vertices y_r and y_s satisfy $1 \leq r \leq c$ and $c+2 \leq s \leq p$ and where the internal vertices do not belong to $P(x_j,x_k)$. Then $\{y_1,y_2,\ldots,y_{r-1},P'(y_r,y_s),y_{s-1},\ldots,y_{r+1},y_1\}$ is a cycle of length greater than or equal to $c+2$, contrary to our assumption. □

4. EXISTENCE OF HAMILTONIAN WALKS.

We shall prove the generalizations of Ore's and Pósa's theorems (2.2. and 2.3.). The generalization of Dirac's theorem has been obtained by Jolivet [10,11] and is a corollary of our results.

When G is 2-connected, lemma 1.5.1. and theorem 3.5. give :

4.1. *Theorem : Let G be 2-connected and let $c \leq n$. If :*

$$\left.\begin{array}{l} j < k\ ,\ \{x_j,x_k\} \notin E \\ d(x_j) \leq j\ ,\ d(x_k) \leq k-1 \end{array}\right\} \Rightarrow d(x_j) + d(x_k) \geq c \quad ;$$

then $s(G) \leq n - c$.

4.2. *Definitions : A block* of a graph G is a maximal 2-connected subgraph of G. A *cut-vertex* in a connected graph is a vertex whose deletion disconnect G . An *end-block* is a block containing exactly one cut-vertex It is known that,if G is not 2-connected, then G contains at least two end-blocks.

4.3. *Theorem : Let* $c \leq n$. *If :*

$$|\{x : d(x) \leq j\}| \leq j-1 \quad \text{for} \quad 0 \leq j \leq (c-1)/2 \text{ and}$$
$$|\{x : d(x) \leq (c-1)/2\}| \leq (c-1)/2 \quad (\text{for } c \text{ odd}) \quad ;$$

then $s(G) \leq n - c$.

Proof : If G is 2-connected the theorem is a corollary of theorem 4.1. . Thus we suppose that G is not 2-connected and we distinguish 2 cases.

Case 1 : $c = 2p$.

i) Every end-block of G contains at least p+1 vertices : indeed let B be an end-block and let b be the only cut-vertex of G in B. Suppose that $|B| \leq p$; $|B| = j+1$ with $j \leq p-1$. If $x \in B-b$, then x is only joined to the vertices of B (B is an end-block) ; thus $d(x) = d_B(x) \leq j$ and as $|B-b| = j$ we have $|\{x : d(x) \leq j\}| \geq j$ for $j \leq p-1 < (c-1)/2$ contrary to the hypothesis.

ii) Every end-block contains a cycle of length at least p+1 : indeed if $x \in B-b$, $d_{B-b}(x) \geq d(x) - 1$ .Therefore the subgraph generated by B-b satisfies by hypothesis the condition : $|\{x : d_{B-b}(x) \leq j\}| \leq j$ for $0 \leq j \leq p-2$; then theorem 3.10.b) with $G' = B$, $G = B-b$, and $c = p-1$, implies the existence in B of a cycle of length at least p+1.

As G contains two end-blocks with at most one vertex in common, G contains two cycles of length at least p+1 having at most one vertex in common and therefore by lemma 1.5.2. $s(G) \leq n+2-2(p+1) = n-c$.

Case 2 : $c = 2p - 1$.

Similarly as in case 1, it can be shown that each end-block contains p vertices. As G satisfies the hypothesis of corrolary 3.2.a) with $c = p$, G contains a cycle C of length at least p+1. C is included in a block. Let B be an end-block which do not contain C . As $d_B(x) = d(x)$ for every vertex of B-b we have : $|\{x : d(x) \leq j\}| \leq j$ for $0 \leq j \leq p-2$ and thus by 3.2.a) B contains a cycle of length at least p . Therefore G contains two cycles having at most one vertex in common of length $d \geq p+1$ and $d' \geq p$ and by lemma 1.5.2. $s(G) \leq n + 1 - 2p = n - c$. □

4.4. *Theorem : Let* $c \leq n$. *If* $d(x) + d(y) \geq c$ *for every pair of non adjacent vertices x and y , then* $s(G) \leq n - c$.

Proof : If G is 2-connected the theorem is a corollary of theorem 4.1. . Thus suppose that G is not 2-connected and that the theorem is false and let c be the smallest value for which the theorem is false . Among the graphs , which do not verify the theorem for c, let G be a graph with the minimum possible number of vertices n . As the theorem is true for $n = c$ (2.2.) we have $n > c$.

Every end-block of G is non hamiltonian : otherwise suppose that G contains a hamiltonian end-block B and let b be the cut vertex of B and let $P = B-b$. First $|B| < c$, otherwise G would contain a cycle of length at least c and thus $s(G) \leq n-c$. We claim that $s(G) \leq s(G-P) + 1$; indeed let C be a hamiltonian walk of G-P of length $|G-P| + s(G-P)$ and let C' be a hamiltonian cycle of B ; then $C \cup C'$ is a closed spanning walk of G of length $|G| + 1 + s(G-P)$ and therefore $s(G) \leq s(G-P) + 1$. Now let $|P| = k$ and thus $|B| = k+1 < c$. We have $d(x) = d_B(x) \leq k$ for every vertex x of P . As B is an end-block, if $x \in P$ and $y \in G-B$, x and y are not adjacent and then, by hypothesis, $d(x) + d(y) \geq c$. Thus : $d_{G-P}(y) = d(y) \geq c-k$. We have also $d_{G-P}(b) \geq 1$. As $c > k+1$, $2(c-k) > c+1-k$ and therefore $d_{G-P}(y) + d_{G-P}(z) \geq c+1-k$ for every pair of vertices y and z in G-P .The subgraph generated by G-P satisfies the hypothesis of the theorem for $c' = c+1-k \leq c$ and $n' = |G-P| = n-k < n$. By the choice of c and n (minimality), we have $s(G-P) \leq n'-c' \leq n-c-1$. From $s(G) \leq s(G-P) + 1$ we deduce $s(G) \leq n-c$ contrary to our assumption .

Let B be an end-block ; $P = B-b$ is not complete, otherwise B would be hamiltonian ; thus there exist two non adjacent vertices x and y , which satisfy (by hypothesis) $d(x) + d(y) \geq c$. If $x \in P$, $d_P(x) \geq d(x) - 1$ and therefore $d_P(x) + d_P(y) \geq c-2$. Let $p = [(c+1)/2]$,. We have $|P| > p+1$ otherwise , if $|P| \leq p$, $d_P(x) \leq p-2$ and $d_P(y) \leq p-2$ (x and y are not adjacent) and thus $d_P(x) + d_P(y) \leq 2p-4 < c-2$. Therefore the subgraph generated by P satisfies the hypothesis of theorem 3.10.c) with $G = P$, $G' = B$, and $c = p-1$ and thus contains a cycle of length at least p+1 . As G contains two end-blocks, G contains two cycles of length at least p+1, having at most one vertex in common and therefore $s(G) \leq n-c$. (Remark that the minimality of c and n is used only to deal with the case where B is an end block such that $P = B-b$ is a complete subgraph with $|P| < p$.) □

4.5. *Remark* : The conditions of theorems 4.3. and 4.4. are independent. As we have seen in remark 3.7, there exist graphs satisfying the hypothe-sis of 4.4. and not of 4.3. . It is known that for $c = n$ there exist graphs, which satisfy the hypothesis of 4.3. and not of 4.4. ; it is possible to exhibit such graphs for $5 < c < n$: for example take a complete graph of order $n-2$ and add two non adjacent vertices of degree 2 and 3 respectively .

Thus it will be interesting to prove the following conjecture (a proof similar to that of theorem 3.3. shows that this conjecture is a consequence of conjecture 3.3.b) .)

4.6. *Conjecture* : *Let $c \leq n$ and suppose that*

$$\left.\begin{array}{l} j < k \ , \ \{x_j, x_k\} \notin E \\ d(x_j) \leq j \ , \ d(x_k) \leq k-1 \end{array}\right\} \Rightarrow \quad d(x_j) + d(x_k) \geq c \ .$$

Then $s(G) \leq n - c$.

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