Decomposition of Kn* into k-circuits and balanced G-designs

Jean-Claude Bermond

Decomposition of K_n^* into k-circuits and Balanced G-designs

J. C. Bermond

1. INTRODUCTION

Let K_n^* be the complete directed graph with n vertices, that is, every ordered pair of vertices is joined by exactly one arc. By a k-circuit we mean a directed elementary circuit of length k.

We are interested in the following problem: For which values of n is it possible to partition the arcs of K_n^* into k-circuits ($k \leq n$). A necessary condition for such a partition is that the number of arcs is a multiple of k, that is: $n(n-1) \equiv 0 \pmod{k}$. We conjecture that this condition is sufficient except for $n = 6$, $k = 3$; $n = 4 = k$ and $n = 6 = k$. In [2], the case $k = 3$ is solved completely. In a joint paper with V. Faber [5], two methods are developed: An "oriented difference method" and a method of composition using complete directed bipartite graphs. These methods give many results for k even and, in particular, solve completely the cases $k = 4, 6, 8, 10$. By using refinements of the "difference method" we have proved in [3] that the condition is sufficient for $n = 2k$, except $k = 3$.

In this paper, we develop a "composition method" using lexicographic products. This method gives the complete solution for $k = 3$ (with a simpler proof than in [2]) and for $k = 5$.

Our problem is a special case of the problem of the existence of balanced G-designs, introduced by P. Hell and A. Rosa [10]. Let λK_n (λK_n^*) be the complete (complete directed) multigraph, with n vertices, where two vertices are joined by exactly λ edges (arcs) and let G be a graph (directed graph) with k vertices; an (n,k,λ) G-design consists of a partition of the edges (arcs) of λK_n (λK_n^*) into partial subgraphs isomorphic to G. Furthermore, if every vertex of λK_n (λK_n^*) belongs to the same number of partial sugraphs, the G-design is said to be balanced. If G is regular, that is, if every vertex of G belongs to the same number of edges (arcs) of G, then every G-design is balanced. In the case $G = K_k$ an $(n,k,\lambda)K_k$-design is nothing else than a B.I.B.D. (balanced incomplete block design) (see M. Hall [8] or H. J. Ryser . The particular case in which G is a k-circuit $\vec{C}_k$ corresponds to our original problem. Other cases have also been studied: C_k-design, C_k a k-cycle (A. Kotzig [17] and A.Rosa [19,20]), balanced P_k-design, P_k a k-chain (chain with k vertices), which is known under the name of "handcuffed design" (P.Hell and A.Rosa [10], S.H.Y.Hung and N.S.Mendelsohn [13], J.F.Lawless [15,16]), G-design, where G is a bipartite graph (C.Huang and A.Rosa [12]), G-design, where G is the transitive tournament with 3 vertices (S.H.Y.Hung and N.S.Mendelsohn [14]).

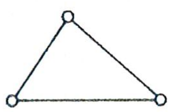
In [3] we have generalized the first two methods to the G-designs, but the composition method using bipartite graphs applies only in some cases (G an even cycle or circuit, G a chain or path). The composition method using the lexicographic product works for all the graphs G. We will first give the general lemmas, but for the applications we will restrict ourselves to the cases $G = K_k$, P_k and $\vec{C}_k$, with more emphasis to this last case. (For complements and applications of the methods to packing and covering problems see [3]. For the definitions not given here see C. Berge [1].)

2. LEXICOGRAPHIC PRODUCT AND BASIC LEMMAS

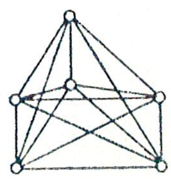
DEFINITION. Let $H_1 = (X_1,U_1)$ and $H_2 = (X_2,U_2)$ be two (directed) graphs; the lexicographic product (called also composition) of H_1 by H_2, denoted $H_1 \otimes H_2$, is the graph which has the cartesian product $X_1 \times X_2$ as its set of vertices and in which (x_1,x_2) is joined to (y_1,y_2) if and only if $x_1 y_1$ is an edge (arc) of H_1, or $x_1 = y_1$ and $x_2 y_2$ is an edge (arc) of H_2.

($H_1 \otimes H_2$ can be considered as formed by replacing each vertex of H_1 by a copy of H_2).

EXAMPLE:

$H_1 = C_3 =$ 

$H_2 = P_2 =$ 

$H_1 \otimes H_2 =$ 

NOTATIONS. We say that the graph H can be decomposed into partial subgraphs isomorphic to G, if we can partition the edges (arcs) of H into partial subgraphs isomorphic to G. S_n will denote a graph formed by n isolated (independent) vertices.

LEMMA 1. *If H_1, H_2 and $G \otimes S_n$ (where n is the number of vertices of H_2) can be decomposed into partial subgraphs isomorphic to G, then $H_1 \otimes H_2$ can also be decomposed into partial subgraphs isomorphic to G.*

PROOF. Let m be the number of vertices of H_1 and n that of H_2. Then $H_1 \otimes H_2$ is the edge (arc) disjoint union of $H_1 \otimes S_n$ and $S_m \otimes H_2$. $S_m \otimes H_2$ is the edge (arc) disjoint union of m copies of H_2 and can thus be decomposed into partial subgraphs isomorphic to G if H_2 can be. If H_1 can be decomposed into partial subgraphs isomorphic to G, then $H_1 \otimes S_n$ can be decomposed into partial subgraphs isomorphic to $G \otimes S_n$ and the lemma follows from the hypothesis of the decomposition of $G \otimes S_n$.

PROPOSITION 1. *If there exist an (m,k,λ) G-design, an (n,k,λ) G-design and if $G \otimes S_n$ can be decomposed into partial subgraphs isomorphic to G, then there exists an (mn,k,λ) G-design.*

PROOF. This result follows from Lemma 1 with $H_1 = \lambda K_m$ or λK_m^* and $H_2 = \lambda K_n$ or λK_n^*.

In order to apply Proposition 1, we will study the decomposition of $G \otimes S_n$ for some particular graphs G.

DEFINITION. (M.Hall [8], chap. 13). *An orthogonal array $OA(n,k)$ is a matrix with k rows, n^2 columns, whose elements, denoted $a_{i,j}$ $(i=1,\ldots,k;\ j=1,\ldots,n^2)$, belong to a set E with n elements and such that:*

$$\forall_{i,i'} \{(a_{i,j}, a_{i',j}) \text{ with } j=1,\ldots,n^2\} = E \times E.$$

It is known that an orthogonal array $OA(n,k)$ exists if and only if there exist $k-2$ mutually orthogonal Latin squares of order n.

LEMMA 2. *$K_k \otimes S_n$ can be decomposed into partial subgraphs isomorphic to K_k if and only if there exists an orthogonal array $OA(n,k)$.*

PROOF. (We will only give a sketch of the proof. The reader can refer to [3] or to the equivalence of transversal designs and orthogonal arrays (R.M.Wilson [21,22] or R.C.Bose and S.S.Shrikharde [6]), the blocks of the transversal design playing the role of the subgraphs of the decomposition). Let us identify the set E of the vertices of $OA(n,k)$ and the set of the vertices of S_n; thus, the vertices of $K_k \otimes S_n$ are labelled (i,e) with $i = 1, \ldots, k$ and $e \in E$. The equivalence of the lemma can be obtained by associating to each column j of $OA(n,k)$ the complete subgraph G_j, isomorphic to K_k, containing the vertices $(i,a_{i,j})$ and conversely. Orthogonality is equivalent to the fact that each edge of $K_k \otimes S_n$ belongs to exactly one G_j.

PROPOSITION 2. (R.C.Bose and S.S.Shrikhande [7]): If there exist an (m,k,λ) B.I.B.D., an (n,k,λ) B.I.B.D. and an orthogonal array $OA(n,k)$, then there exists an (mn,k,λ) B.I.B.D.

PROOF. Follows from Proposition 1 and Lemma 2.

Let T_k denote any tournament with k vertices; then by taking an orientation of K_k that gives T_k, we obtain the same result as Lemma 2 or Proposition 2 for T_k-designs:

PROPOSITION 2′. If there exists an (m,k,λ) T_k-design, an (n,k,λ) T_k-design and an $OA(n,k)$ then there exists an (mn,k,λ) T_k--design.

LEMMA 3. $\vec{C}_k \otimes S_n$ can be decomposed into k-circuits.

PROOF. This lemma has been proved in [4]. We give here another proof. As the number of arcs of $\vec{C}_k \times S_n$ is kn^2, it suffices to exhibit n^2 mutually arc-disjoint k-circuits. Let the vertices of $\vec{C}_k$ be the elements of Z_k and those of S_n the elements of Z_n, where Z_p is the additive group of the integers modulo p. The following n^2 k-circuits:

$$(0,j) \ldots (i,j+i\alpha) \ldots (k-2,j+(k-2)\alpha)\ (k-1,\alpha)\ (0,j),$$

where $j \in Z_n$ and $\alpha \in Z_n$, are arc-disjoint. Indeed, suppose for example that:

$$(i,j+i\alpha)\ (i+1,\ j+(i+1)\alpha) = (i,\ j'+i\alpha')\ (i+1,\ j'+(i+1)\alpha');$$

then $j + i\alpha = j' + i\alpha'$ and $j + (i+1)\alpha = j' + (i+1)\alpha'$ and by subtraction $\alpha = \alpha'$ and $j = j'$.

From Lemma 1, Proposition 1 and Lemma 3 we deduce:

PROPOSITION 3. <u>If H_1 and H_2 can be decomposed into k-circuits, then also $H_1 \otimes H_2$. In particular, if there exist an (m,k,λ) $\vec{C}_k$-design and an (n,k,λ) $\vec{C}_k$-design then there exists an (mn,k,λ) $\vec{C}_k$-design.</u>

Similar results can be obtained for the k-cycle C_k instead of the k-circuit $\vec{C}_k$.

LEMMA 4. <u>$P_k \otimes S_n$ can be decomposed into k-chains.</u> (Recall that a k-chain P_k is an elementary chain with k vertices.)

PROOF. Let the vertices of P_k (resp. S_n) be the elements of Z_k (resp. Z_n). Then we have a partition of the edges of $P_k \otimes S_n$ into the following n^2 edge-disjoint k-chains: $(0,j) \ldots (i, j+i\alpha) \ldots \ldots (k-1, j+(k-1)\alpha)$ where $j \in Z_n$ and $\alpha \in Z_n$.

PROPOSITION 4. <u>If there exist an (m,k,λ) P_k-design and an (n,k,λ) P_k-design, then there exists an (mn,k,λ) P_k-design.</u>

REMARK. In the proof of Lemma 4, we notice that the end-vertices of the chains of the decomposition of $P_k \otimes S_n$ are the vertices $(0,j)$ and $(k-1,j)$, $j \in Z_n$ and 0 and $k-1$ are the end-vertices of P_k. Thus we obtain:

PROPOSITION 4'. (P.Hell and A.Rosa [10], J.F.Lawless [16].) <u>If there exist a balanced (m,k,λ) P_k-design and a balanced (n,k,λ) P_k-design, then there exists a balanced (mn,k,λ) P_k-design.</u>

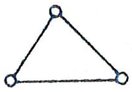
Similar results can also be obtained for k-paths $\vec{P}_k$.

3. <u>GENERALIZATION</u>

We shall now consider a generalization of the lexicographic product which we have introduced in [4]. It is obtained by identifying, in $H_1 \otimes H_2$, h points of each copy of H_2. The exact definition is:

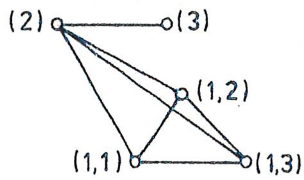
DEFINITION. Let m, n, h be three integers. Let $H_1 = (X_1, U_1)$ be a graph (directed or not) and $H_2 = (X_2' \cup X_2'', U_2)$ be a graph (directed or not) with $|X_2'| = n$ and $|X_2''| = h$. The vertices of X_2' are labelled from 1 to n and those of X_2'' from $n+1$ to $n+h$. Let H_2' be the subgraph generated by X_2' and H_2'' the subgraph generated by X_2''. We define a new graph $H_1^+(H_2,H_2')$ as follows: Its vertex set is $(X_1 \times X_2') \cup X_2''$; the vertices are labelled (i_1,i_2) with $i_1 = 1, \ldots, m$; $i_2 = 1, \ldots, n$ if they belong to $X_1' \times X_2'$ and (k) with $k = n+1, \ldots, n+h$ if they belong to X_2''. Two vertices are joined if and only if they are of the form:

- (i_1,i_2) and (j_1,j_2) with $i_1 j_1$ an edge (arc) of H_1 or $i_1 = j_1$ and $i_2 j_2$ an edge (arc) of H_2';
- (i_1,i_2) and (k), (i_2,k) being an edge (arc) of H_2;
- (k) and (k'), kk' being an edge (arc) of H_2''.

EXAMPLE: $H_1 = C_3 =$ 

$H_2 = P_3 =$ 

$H_2' = \circ$ $H_2'' =$ 



$H_1^+(H_2,H_2')$

REMARKS. In the case $h = 0$, we obtain the lexicographic product $H_1 \otimes H_2$. In general, $H_1^+(H_2,H_2')$ is the edge (arc) disjoint union of $H_1 \otimes S_n$, H_2'' and p copies of $H_2 - H_2''$ (graph obtained from H_2 by deleting the edges (arcs) of H_2''). In the particular case $h = 1$, that is H_2'' being reduced to one vertex, $H_1^+(H_2,H_2')$ is the edge (arc)-disjoint union of $H_1 \otimes S_n$ and p copies of H_2 (because there is no arc in H_2''). From this remark, we obtain exactly as in Lemma 1 the following lemma:

LEMMA 5. If H_1, H_2'', $G \otimes S_n$ (where n is the number of vertices of H_2') and $H_2 - H_2''$ (the graph obtained from H_2 by deleting the arcs of H_2'') can be decomposed into partial subgraphs isomorphic to G, then $H_1^+(H_2,H_2')$ can also be decomposed into partial subgraphs isomorphic to G.

LEMMA 5'. If H_1, H_2, $G \otimes S_n$ ($n+1$ being the number of vertices of H_2) can be decomposed into partial subgraphs isomorphic to G, then the same is true for $H_1^+(H_2,\{x\})$.

By taking $H_1 = K_m$ or K_m^*, $H_2 = K_{n+h}$ or K_{n+h}^*, $H_2' = K_n$ or K_n^* and thus $H_2'' = K_h$ or K_h^*, we obtain:

PROPOSITION 5. If there exists an (m,k,λ) G-design, an (h,k,λ) G-design and if $G \otimes S_n$ and $K_{n+h} - K_n$ or $K^*_{n+h} - K^*_n$ can be decomposed into partial subgraphs isomorphic to G, then there exists an $(mn + h, k, \lambda)$ G-design.

PROPOSITION 5'. If there exists an (m,k,λ) G-design, an $(n+1,k,\lambda)$ G-design and if $G \otimes S_n$ can be decomposed into partial subgraphs isomorphic to G, then there exists an $(mn+1,k,\lambda)$ G-design.

In the case $G = K_k$, by using Lemma 2 we obtain:

PROPOSITION 6. If there exists an (m,k,λ) B.I.B.D., an (h,k,λ) B.I.B.D., an orthogonal array $OA(n,k)$ and if $K_{n+h} - K_h$ can be decomposed into partial subgraphs isomorphic to K_k, then there exists an $(mn+h,k,\lambda)$ B.I.B.D.

PROPOSITION 6'. (R.C.Bose and S.S.Shrikhande [7].) If there exist an (m,k,λ) B.I.B.D., an $(n+1,k,\lambda)$ B.I.B.D. and an orthogonal array $OA(n,k)$, then there exists an $(mn+1,k,\lambda)$ B.I.B.D.

Proposition 6 seems to be new and may yield new B.I.B.D.'s. An interesting case of application is the case $h = k$; there always exists a (k,k,λ) B.I.B.D. and the existence of a decomposition of $K_{n+k} - K_k$ into K_k is equivalent to the existence of an $(n+k,k,\lambda)$ B.I.B.D. (it suffices to take as K_k one of the blocks of the B.I.B.D.). Thus we have

COROLLARY. If there exists an (m,k,λ) B.I.B.D., an $(n+k,k,\lambda)$ B.I.B.D., an orthogonal array $OA(n,k)$, then there exists an $(mn+k,k,\lambda)$ B.I.B.D.

As an example we have the following B.I.B.D.'s that cannot be obtained in [7] (they are known from the work of H.Hanani [9]:
$(285,5,1)$ B.I.B.D. by taking $m = 5$, $k = 5$, $n = 56$, $n + k = 61$,
$(232,4,1)$ B.I.B.D. by taking $m = 4$, $k = 4$, $n = 57$, $n + k = 61$.
In the case $G = \vec{C}_k$, by using Lemma 3 we obtain:

PROPOSITION 7. If K^*_m, K^*_h and $K^*_{n+h} - K^*_h$ can be decomposed into k-circuits, then K^*_{mn+h} can be decomposed into k-circuits.

PROPOSITION 7'. If K^*_m and K^*_{n+1} can be decomposed into k-circuits, then K^*_{mn+1} can be decomposed into k-circuits.

We can also obtain similar theorems with $G = C_k$, P_k, $\vec{P}_k$ instead of $\vec{C}_k$. We shall apply Propositions 3, 7 and 7′ to solve completely the case $k = 3$ and 5 of the decomposition of K_n^* into k-circuits.

4. DECOMPOSITION OF K_n^* INTO 3-CIRCUITS

We shall prove the following theorem which solves completely the problem of the existence of a decomposition of K_n^* into 3-circuits (case $k = 3$ of the conjecture of the introduction). This problem was solved in [2] as well as in N.S.Mendelsohn [18], but the proof given here is simpler.

THEOREM 1. Let n be an integer, $n \equiv 0$ or $1 \pmod 3$, $n \neq 4,6$; then there exists a decomposition of K_n^* into 3-circuits, containing two opposite 3-circuits (that is a partial subgraph isomorphic to K_3^*).

PROOF. We prove the theorem by induction on n. The theorem is true for $n = 3$. For $n = 4$ there exists a decomposition of K_4^* into 3-circuits, but no decomposition containing a K_3^*. For $n = 6$ there exists no decomposition of K_6^* into 3-circuits (see [2]). For $n = 13$, 16, 18 we can find a decomposition containing a K_3^*: For $n = 13$ by using the (13,3,1) B.I.B.D.; for $n = 16$ and 18 by applying the method of [2] with a solution of Kirkman's schoolgirl problem for only $n = 15$. Suppose that n_o is an integer, $n_o \equiv 0$ or $1 \pmod 3$ and that the theorem is true for all $n < n_o$ $(n \neq 4,6)$ with $n \equiv 0$ or $1 \pmod 3$. We break the proof in four cases depending on the congruences modulo 9.

Case 1: $n_o = 3t$ with $t \equiv 0$ or $1 \pmod 3$.
$K_{n_o}^*$ can be decomposed by Proposition 3, with $m = 3$, $n = t$ and by noticing that K_{mn}^* contains a K_3^* if at least K_m^* (or K_n^*) contains a K_3^*. The case $t = 6$, $n_o = 18$ is solved above.

Case 2: $n_o = 3t + 1$ with $t \equiv 0$ or $1 \pmod 3$.
$K_{n_o}^*$ can be decomposed into 3-circuits by Proposition 7′ with $m = t$, $n = 3$ $(n + 1 = 4)$. This works except if $t = 4$ or 6. If $t = 4$, $n_o = 13$ has been solved above. If $t = 6$, $n_o = 19$ and the theorem follows from Proposition 7′ with $m = 9$, $n = 2$.

Case 3: $n_o = 9k + 7 = 3(3k + 2) + 1$.
The theorem results from Proposition 7′ with $m = 3$, $n = 3k + 2$,

$n + 1 = 3k + 3$; except if $k = 1$, but in this case $n_o = 16$ (solved above).

Case 4: $n_o = 9k + 6 = 3(3k + 1) + 3$ with $k \neq 0$.
The theorem follows from Proposition 7 with $m = 3$, $h = 3$, $n = 3k + 1$, $n + h = 3k + 4$ and by the induction hypothesis that $K^*_{3k+4} - K^*_3$ can be decomposed into 3-circuits.

REMARKS. The theorem gives a new proof of the existence of an $(n,3,2)$ B.I.B.D. If we want to prove the existence of an $(n,3,2)$ B.I.B.D. or an $(n,3,1)$ TT_3-design (called directed triple systems by S.H.Y.Hung and N.S.Mendelsohn [14]), the proof is quite the same as the proof of Theorem 1 but simpler because there exists a solution for $n = 6$. Indeed, we do not need as induction hypothesis that the decomposition contains a K^*_3: this was used only in Case 4. In this case if $k = 2p$, that is $n_o = 18p + 6 = 6(3p + 1)$, the result follows from Propositions 2′ or 3 with $m = 6$ and $n = 3p + 1$. If $k = 2p + 1$, that is $n_o = 18p + 15 = 2(9p + 7) + 1$, the result follows from Proposition 5′ with $m = 9p + 7$, $n = 2$, $n + 1 = 3$. And we need only to start the induction to have the decomposition for $n = 3, 4, 6$. Thus we have:

THEOREM 2. If $n \equiv 0$ or $1 \pmod 3$ there exists an $(n,3,2)$ B.I.B.D. and an $(n,3,1)$ TT_3-design [14].

The same method has been used by A.J.W.Hilton [11] to obtain a proof of the existence of an $(n,3,1)$ B.I.B.D. or Steiner triple systems. The method can be used also to have the existence of an $(n,3,\lambda)$ B.I.B.D. for any λ, or other G-designs where G has 3 vertices.

5. DECOMPOSITION OF K^*_n INTO 5-CIRCUITS

We shall prove the following theorem:

THEOREM 3. K^*_n can be decomposed into 5-circuits if and only if $n \equiv 0$ or $1 \pmod 5$.

PROOF. To prove the theorem we shall use the Propositions 3, 7″, 7 called here respectively A, B, C and some lemmas D_i, whose proofs use the difference method, but are long (the reader can find all these proofs in [3]).

A: If K^*_m and K^*_n can be decomposed into 5-circuits, then K^*_{mn} can be decomposed as well.

B: If K^*_m and K^*_{n+1} can be decomposed into 5-circuits, then K^*_{mn+1} can be decomposed as well.

C: If K^*_m, K^*_n and $K^*_{n+h} - K^*_h$ can be decomposed into 5-circuits, then K^*_{mn+h} can be decomposed as well.

D: There exists a decomposition into 5-circuits of:

D_1: K^*_5; D_2: K^*_6; D_3: $K^*_{10}-K^*_5$; D_4: $K^*_{10}-K^*_6$.
D_5: $K^*_{11}-K^*_5$; D_6: $K^*_{11}-K^*_6$; D_7: K^*_{15}; D_8: $K^*_{16}-K^*_{10}$.
D_9: $K^*_{16}-K^*_6$; D_{10}: $K^*_{20}-K^*_{10}$; D_{11}: $K^*_{40}-K^*_{20}$; D_{12}: $K^*_{45}-K^*_{15}$.

(Recall that $K^*_{n+h} - K^*_h$ means the graph obtained from K^*_{n+h} by deleting the arcs of a K^*_h).

With all these lemmas we can prove the theorem by induction. We shall indicate which propositions or lemmas are used, with the corresponding values.

The theorem is true for
$n = 5$ (D_1); $n = 6$ (D_2); $n = 10$ (D_3 or D_4); $n = 11$ (D_5 or D_6); $n = 15$ (D_7); $n = 16$ (D_8 or D_9) and $n = 20$ (D_{10}).

Suppose that $n_o \equiv 0$ or $1 \pmod 5$ and that the theorem is true for all $n < n_o$ with $n \equiv 0$ or $1 \pmod 5$. We break the proof in many cases depending on the congruences modulo 300.

Case 1: $n_o = 10t + 1$, $t \geq 2$.

1.a	$t=2p$ $(p \geq 1)$,	$n_o=20p+1$,	B:	$m=5p$,	$n=4$.	
1.b	$t=3p$ $(p \geq 1)$,	$n_o=30p+1$,	B:	$m=6$,	$n=5p$.	
1.c	$t=3p+1$ $(p \geq 1)$,	$n_o=30p+11$,	C,D_5:	$m=5p+1$,	$n=6$,	$h=5$.
1.d	it remains $t \equiv 5 \pmod 6$					
	$t=30k+5$,	$n_o=300k+51$,	B:	$m=60k+10$,	$n=5$.	
	$t=30k+11$,	$n_o=300k+111$,	B:	$m=30k+11$,	$n=10$.	
	$t=30k+17$,	$n_o=300k+171$,	B:	$m=5$,	$n=60k+34$.	
	$t=30k+23$,	$n_o=300k+231$,	B:	$m=60k+46$,	$n=5$.	
	$t=30k+29$,	$n_o=300k+291$,	B:	$m=10$,	$n=30k+29$.	

Case 2: $n_o = 10t + 5$, $t \geq 2$.

2.a	$t=2p$ $(p \geq 1)$,	$n_o=20p+5$,	B:	$m=5p+1$,	$n=4$.	
2.b	$t=3p$ $(p \geq 1)$,	$n_o=30p+5$,	C,D_5:	$m=5p$,	$n=6$,	$h=5$.
2.c	$t=3p+2$,	$n_o=30p+25$,	B:	$m=6$,	$n=5p+4$.	
2.d	it remains $t \equiv 1 \pmod 6$					

	$t=30k+1$,	$n_0=300k+15$,	A:	$m=15$,	$n=20k+1$.	
	$t=30k+7$,	$n_0=300k+75$,	A:	$m=15$,	$n=20k+5$.	
	$t=30k+13$.	$n_0=300k+135$,	C,D_3:	$m=60k+26$,	$n=5$,	$h=5$.
	$t=30k+19$,	$n_0=300k+195$,	C,D_{12}:	$m=10k+6$,	$n=30$,	$h=15$.
	$t=30k+25$,	$n_0=300k+255$,	A:	$m=5$,	$n=60k+51$.	

<u>Case 3:</u> $n_0 = 10t + 6, \quad t \geqq 2.$

3.a	$t=2p$ $(p \geqq 1)$,	$n_0=20p+6$,	C,D_4:	$m=5p$,	$n=4$,	$h=6$.
3.b	$t=3p$ $(p \geqq 1)$,	$n_0=30p+6$,	A:	$m=6$,	$n=5p+1$.	
3.c	$t=3p+1$ $(p \geqq 1)$,	$n_0=30p+16$,	C,D_8:	$m=5p+1$,	$n=6$,	$h=10$.
3.d	it remains $t \equiv 5 \pmod 6$					
	$t=30k+5$,	$n_0=300k+56$,	B:	$m=60k+11$,	$n=5$.	
	$t=30k+11$,	$n_0=300k+116$,	C,D_9:	$m=30k+11$,	$n=10$,	$h=6$.
	$t=30k+17$,	$n_0=300k+176$,	B:	$m=60k+35$,	$n=5$.	
	$t=30k+23$,	$n_0=300k+236$,	C,D_6:	$m=60k+46$,	$n=5$,	$h=6$.
	$t=30k+29$,	$n_0=300k+296$,	B:	$m=5$,	$n=60k+59$.	

<u>Case 4:</u> $n_0 = 10t + 10, \quad t \geqq 2.$

4.a	$t=2p$ $(p \geqq 1)$,	$n_0=20p+10$,	C,D_4:	$m=5p+1$,	$n=4$,	$h=6$.
4.b	$t=3p$ $(p \geqq 1)$,	$n_0=30p+10$,	C,D_8:	$m=5p$,	$n=6$,	$h=10$.
4.c	$t=3p+2$,	$n_0=30p+30$,	A:	$m=6$,	$n=5p+5$.	
4.d	it remains $t \equiv 1 \pmod 6$					
	$t=30k+1$,	$n_0=300k+20$,	A:	$m=20$,	$n=15k+1$.	
	$t=30k+7$,	$n_0=300k+80$,	A:	$m=5$,	$n=60k+16$.	
	$t=30k+13$,	$n_0=300k+140$,	C,D_{11}:	$m=15k+6$,	$n=20$,	$h=20$.
	$t=30k+19$,	$n_0=300k+200$,	A:	$m=5$,	$n=60k+40$.	
	$t=30k+25$;	$n_0=300k+260$,	A:	$m=10$,	$n=30k+26$.	

REFERENCES

[1] C.Berge: Graphs and Hypergraphs, North Holland, Amsterdam, 1973.

[2] J.C.Bermond: An application of the solution of Kirkman's schoolgirl problem: the decomposition of the symmetric oriented complete graph into 3-circuits, Discrete Mathematics 8(1974), 301-304.

[3] J.C.Bermond: Thesis.

[4] J.C.Bermond: The circuit hypergraph of a tournament, to appear in Proc. Colloquium, Keszthely (June 1973).

[5] J.C.Bermond and V.Faber: Decomposition of the complete directed graph into k-circuits, submitted for publication.

[6] R.C.Bose and S.S.Shrikhande: On the construction of sets of mutually orthogonal Latin squares and the falsity of a conjecture of Euler, Trans. Amer. Math. Soc. 95(1960), 191-209.

[7] R.C.Bose and S.S.Shrikhande: On the composition of balanced incomplete block designs, Canadian Journal Math. 12(1960),177-188.

[8] M.Hall, Jr.: Combinatorial Theory, Blaisdell, Waltham, Mass., 1967.

[9] H.Hanani: The existence and construction of balanced incomplete block designs, Ann. Math. Statist. 32(1961), 361-386.

[10] P.Hell and A.Rosa: Graph decomposition, Handcuffed prisoners and balanced P-designs, Discrete Mathematics, 2(1972), 229-252.

[11] A.J.W.Hilton: A simplification of Moore´s proof of the existence of Steiner systems, J. Combinatorial Theory A,13(1972), 422-425.

[12] C.Huang and A.Rosa: On the existence of balanced bipartite designs, Utilitas Mathematica, 4(1973), 55-75.

[13] S.H.Y.Hung and N.S.Mendelsohn: Handcuffed designs, to appear in J. Combinatorial Theory A.

[14] S.H.Y.Hung and N.S.Mendelsohn: Directed triple systems, J. Combinatorial Theory A,14(1973), 310-318.

[15] J.F.Lawless: On the construction of handcuffed designs, J. Combinatorial Theory A,16(1974), 76-86.

[16] J.F.Lawless: Further results concerning the existence of handcuffed designs, to appear in Aequationes Mathematicae.

[17] A.Kotzig: On the decomposition of complete graphs into 4k-gons, (in Russian) Mat.-Fyz. Časop. 15(1965), 229-233.

[18] N.S.Mendelsohn: A natural generalization of Steiner triple systems, Computers in number Theory (A.O.Atkin and B.J.Birch eds.) Academic Press, New York, 1971, 323-339.

[19] A.Rosa: On the cyclic decompositions of the complete graph into polygons with odd number of edges, (in Slovak with English summary) Časop. pěstov. Mat. 91(1966), 53-63.

[20] A.Rosa: On cyclic decompositions of the complete graph into (4n + 2)-gons, Mat.-Fyz. Časopis 16(1966), 349-353.

[21] R.M.Wilson: Concerning the number of mutually orthogonal Latin squares, (to appear).

[22] R.M.Wilson: An existence theory for pairwise balanced designs, J.Combinatorial Theory 13(1972), 220-273.