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ScienceDirect

Theoretical Computer Science 00 (2019) 1–9

Journal Logo

# On the Decision Problem for MELL

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**Abstract**

In this short paper I will exhibit several mistakes in the recent attempt by Bimbó [3] to prove the decidability of the multiplicative exponential fragment of linear logic (MELL). In fact, the main mistake is so serious that there is no obvious fix, and therefore the decidability of MELL remains an open problem. As a side effect, this paper contains a complete (syntactic) proof of the decidability of the relevant version of MELL (called RMELL in this paper), that is the logic obtained from MELL by replacing the linear logic contraction rule by a general unrestricted version of the contraction rule. This proof can also be found (with a small error) in [3], and a semantic proof can be found in [35].

## 1. Introduction

Since the beginning of linear logic [7], the complexity of the decision problem of its various fragments has been studied by many researchers. For example, its multiplicative fragment (MLL) is NP-complete [16], and its multiplicative-additive fragment (MALL) is PSPACE-complete [27]. But the complexity of its multiplicative-exponential fragment (MELL) is still unknown. In fact, it is an open problem whether that logic is decidable. Finally, the multiplicative-additive-exponential fragment, i.e., full propositional linear logic (LL) is undecidable [27, 17, 18]. However, if we add second-order propositional quantifiers, already the multiplicative fragment (MLL2) is undecidable [22]. On the other hand, if we add a self-dual non-commutative multiplicative connective, the multiplicative fragment stays NP-complete [15] (this logic is called *pomset logic* [37] or BV [11, 8]), but the multiplicative exponential fragment is undecidable [42] (this logic is called NEL [12, 43, 13]).

It was observed early on that the reachability problem for Petri nets [36] can be encoded into MELL [32]. That problem has been shown to be decidable [29, 20, 30], but its precise complexity is still unknown. It is EXPSPACE-hard [28] and the known algorithms have runtimes that are not primitive recursive [26]. It has been known for a long time that the reachability problem for Petri nets is equivalent to the reachability problem of vector addition systems with states (VASS) [38]. Furthermore, it has been shown recently that the decidability problem of MELL is equivalent to the reachability problem for branching VASS [6], for which very recently a non-elementary lower bound has been found [24]. More precisely, if MELL turns out to be decidable it will be at least TOWER-hard [24, Corollary 22].

Since all known proofs [29, 20, 30, 38, 25] of the decidability of the reachability for Petri nets and VASS are very involved—in fact, the complete proof fills a textbook [38]—there is interest in the community in an alternative proof, which could be provided by a proof-theoretical proof of the decidability of MELL. The recent proposal by Bimbó [3] gives such a proof which is surprisingly simple. All the details could be given in less than ten pages.[1]

However, an inspection of the proof by Bimbó [3] shows immediately that the same argument also works for the subexponential variant of MELL, denoted by MSELL, that is obtained by allowing not one but several pairs of

[1]Bimbó [3] does not speak about complexity, but previous work by Urquhart [44, 45] gives an EXPSPACE lower bound.

the modalities ? and !, which are subject to a partial order relation. But that logic has recently been shown to be undecidable for the case of three pairs of ? and ! by Chaudhuri in [5]. Furthermore, a minor variation of the argument in [3] (using results from [35]) would allow a proof of the decidability of LL, which known to be undecidable [27, 18], as mentioned above.[2]

This, of course, is enough to dismiss [3] as being erroneous. However, that is not helpful, neither for the author of [3], nor anyone else who would like to understand what is going on. For this reason, I will in Section 4 of this paper explain in more detail the technical mistakes of [3]. After all, the main gap in the proof is a rather subtle mistake that could easily be repeated in other contexts.

Before coming to that, let me finish this introduction with the observation that the decision problem for MELL is—no matter how it will turn out—very close to the border between the decidable and the undecidable: adding just a little bit, i.e., a third self-dual (non-commutative) multiplicative connective or the additive connectives or subexponentials, renders the problem undecidable, and removing just a little bit, i.e., stepping down from branching VASS to non-branching VASS, puts the problem in the realm of the decidable.

This might be one of the reasons why the problem receives so much attention and is, after more than 3 decades, still open.

## 2. MELL and RMELL

To make this paper self-contained, I give the sequent calculus for MELL (called $\mathsf{CLL}_{\mathsf{int}}$ in [3]) and its relevant version RMELL (called $\mathsf{RLL}_{\mathsf{int}}$ in [3]) below. *Formulas* (denoted by capital Roman letters $A, B, C, \ldots$) are generated from propositional variables $\{a, b, c, \ldots\}$ and their duals $\{a^\perp, b^\perp, c^\perp, \ldots\}$ via the grammar:

$$A ::= a \mid a^\perp \mid A \parr A \mid A \otimes A \mid ?A \mid !A$$

As in [3], I work here in the unit-free fragment. *Sequents* (denoted by capital Greek letters $\Gamma, \Delta, \ldots$) are (possibly empty) finite multisets of formulas, written as lists separated by commas, with a preceding turnstile: $\vdash A_1, A_2, \ldots, A_n$ . The inference rules for MELL are the following[3]:

$$\mathsf{id}\,\frac{}{\vdash a, a^\perp} \qquad \parr\,\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B} \qquad \otimes\,\frac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta}$$
$$!\,\frac{\vdash A, ?\Delta}{\vdash !A, ?\Delta} \qquad ?\mathsf{d}\,\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A} \qquad ?\mathsf{w}\,\frac{\vdash \Gamma}{\vdash \Gamma, ?A} \qquad ?\mathsf{c}\,\frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A} \tag{1}$$

where $?\Delta$ stands for a sequent in which every formula is of shape $?B$ for some $B$.

The relevant version RMELL, is obtained from MELL by adding a general contraction rule

$$\mathsf{c}\,\frac{\vdash \Gamma, A, A}{\vdash \Gamma, A} \tag{2}$$

Of course, the standard linear logic contraction rule ?c is a special case of this and can therefore be omitted from the system.

Note that in order to save paper, ink, and the patience of the reader, I use here the one-sided presentation of the sequent calculus, whereas [3] uses a two-sided presentation. Everything I present here also works in the two-sided systems, we only have to replace the one-sided axiom by a negation rule and a two-sided identity axiom:

$$\mathsf{id}\,\frac{}{\vdash a, a^\perp} \qquad \to \qquad (\cdot)^\perp\,\frac{\mathsf{id}\,\frac{}{a \vdash a}}{\vdash a, a^\perp}$$

[2] As pointed out by one referee, such a 'result' has indeed been announced at the 2015 North American Annual Meeting of the Association for Symbolic Logic [2, p.358].

[3] For the inference rules, I am using here the standard names from the linear logic literature.

## 3. MSELL and RMSELL

For making some arguments in this paper clearer, I will also introduce the subexponential version of MELL, also called *multiplicative subexponential linear logic* [33, 34], or MSELL. However, for the main points of this paper, subexponentials are not needed, and the reader can safely skip over this section on first reading. Formulas of MSELL are generated from propositional variables and their duals (see previous section) via the grammar:

$$A ::= a \mid a^\perp \mid A \parr A \mid A \otimes A \mid ?^v A \mid !^v A$$

where (as before) we omit the units, and where exponentials are indexed by elements from a (countable) set $V$ of *labels* which comes equipped with a partial order $\leq \subseteq V \times V$ and a subset $U \subseteq V$ of *unbounded labels*. The inference rules for MSELL are the ones in the first line of (1) above together with

$$!^v \frac{\vdash !^v A, ?^{w_1} B_1, \ldots ?^{w_n} B_n}{\vdash !^v A, ?^{w_1} B_1, \ldots ?^{w_n} B_n} \; v \leq w_i \text{ for all } i \in \{1, \ldots, n\}$$

$$?^v\mathsf{d}\, \frac{\vdash \Gamma, A}{\vdash \Gamma, ?^v A} \qquad ?^u\mathsf{w}\, \frac{\vdash \Gamma}{\vdash \Gamma, ?^u A} \; u \in U \qquad ?^u\mathsf{c}\, \frac{\vdash \Gamma, ?^u A, ?^u A}{\vdash \Gamma, ?^u A} \; u \in U \tag{3}$$

Note that contraction and weakening are only allowed if the label of the ? is among the unbounded labels, and promotion on a ! is only allowed if its label is less than or equal to all the labels of the ? in the context. Only the dereliction rule $?^v\mathsf{d}$ can be applied without restriction.

In [5] it has been shown that provability in MSELL is undecidable for the case $V = \{\alpha, \beta, \gamma\}$ with $\leq$ being the reflexive closure of $\leq_0$ with $\alpha \leq_0 \gamma$ and $\beta \leq_0 \gamma$ and $U = \{\gamma\}$, by a straightforward encoding of two-counter machines [31, 23].

Clearly, we can also define a relevant version of MSELL, that I call here RMSELL, and that is obtained from MSELL by adding the general contraction rule $\mathsf{c}$ in (2). As before in that case, the restricted contraction rule $?^u\mathsf{c}$ can be omitted.

## 4. Technical flaws in Bimbó's decidability proof

There are three main step in Bimbó's proof [3] of the decidability of MELL:

1. Cut admissibility for MELL and RMELL and modified systems that restrict the application of contraction.
2. Decidability for RMELL by showing that proof search in the modified system terminates.
3. Decidability of MELL by deriving an upper bound for the proof search trees from the decision procedure for RMELL.

In the paper [3] all three steps have technical flaws. For the first and the second, these are easily fixable, but for the third this is not the case, and therefore the decision problem for MELL remains open. Below I explain the mistakes in each of the three steps.

### *4.1. Cut admissibility*

The cut rule for all logical systems presented so far can be given as follows:

$$\mathsf{cut}\, \frac{\vdash \Gamma, A \qquad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta} \tag{4}$$

where $A^\perp$ is the De Morgan dual of $A$. The cut admissibility result for a logical system S says that if a formula or sequent can be derived in S + cut then it can also be derived in S without the cut-rule. All four logical systems presented here have this property. For MELL and MSELL this has been proved in [7] and [33], respectively. For their relevant version, it can be shown by similar methods. But since we defined the logics without cut, the cut admissibility theorem is technically not needed for the decidability proof (for RMELL and RMSELL).

Nonetheless, [3] provides a cut admissibility proof for MELL and claims that this is a new proof. However, the proof method in [3] is standard: permute the cut upwards, start with the topmost one, and make sure that some measure decreases at each step. The measure used in [3] consists of the tuple $\langle \rho, \mu, \delta \rangle$ where $\rho$ is the the number of rule instances above the cut having the cut formula in the conclusion, $\mu$ is the number of ?c-instances that are applied to ancestors of the cut formula, and $\delta$ is the size of the cut formula, and where the order is lexicographic. But one case in [3] is problematic, namely when the derivation on the left below is reduced to the one on the right below:

$$\mathsf{cut}\,\dfrac{!\,\dfrac{\vdash ?\Delta, A}{\vdash ?\Delta, !A} \qquad ?\mathsf{c}\,\dfrac{\vdash ?A^\perp, ?A^\perp, \Gamma}{\vdash ?A^\perp, \Gamma}}{\vdash ?\Delta, \Gamma} \qquad \to \qquad ?\mathsf{c}\,\dfrac{\mathsf{cut}\,\dfrac{!\,\dfrac{\vdash ?\Delta, A}{\vdash ?\Delta, !A} \qquad \mathsf{cut}\,\dfrac{!\,\dfrac{\vdash ?\Delta, A}{\vdash ?\Delta, !A} \qquad \vdash ?A^\perp, ?A^\perp, \Gamma}{\vdash ?\Delta, ?A^\perp, \Gamma}}{\vdash ?\Delta, ?\Delta, \Gamma}}{\vdash ?\Delta, \Gamma}$$

where the dotted line stands for several instances of the ?c-rule. On first sight one might be tempted to agree with the argument that the two new cuts have a lower $\mu$ than the original one, and therefore the induction hypothesis can be applied to both cuts. However, the elimination process for the first cut increases the $\mu$-value for the second cut when the cut-formula $!A$ has a subformula $?B$ to which at some point in the original proof an instance of the ?c-rule is applied. Eventually the cut reduction of the first cut reaches this subformula, and then the second $?A^\perp$ is in the context of this cut. Then the reduction of this cut creates an additional ?c that is applied to the second $?A^\perp$. Therefore, the induction hypothesis cannot be applied.

In order to make this kind of argument work, it is not enough to just count how often the cut formula is duplicated in a contraction. One also has to find a way to take into account the instances of ?c that are newly created in the cut elimination process. This is best achieved through a notion "flow graph", as done for classical logic in [4] using Buss' *logical flowgraphs* or in [9, 10] using *atomic flows*. For MELL, these flow graphs are studied in [41] and [43] in the setting of the calculus of structures. Then, for ensuring termination, it has to be shown that there no cycle in the flow graph, or that the cycles can be eliminated.

In any case, cut elimination for MELL is a well-established result with several different published proofs, so that there is no need to go into further details here.

### 4.2. *Decidability of* RMELL

Okada and Terui have shown in [35] via a semantic argument that the relevant version of full propositional linear logic, denoted by RLL[4], is decidable. Therefore, RMELL is also decidable.[5]

On the other hand, for a reader familiar with the syntactic proof of the the decidability of various fragments of *relevant logic*, attributed to Kripke [21] and first written up in all detail by by Belnap and Wallace in [14] for the logic of *entailment with negation*, it should be clear that the decidability of RMELL can also be shown by almost literally the same proof. This proof is presented in [3], but contains a mistake.

More precisely, Kripke's theorem (Theorem 18) is stated wrongly in [3]. That theorem is needed for cognate sequents in general, and not just modally cognate sequents. I will explain this in further detail now, because for understanding the problem with the decidability proof for MELL, one needs to understand how the decidability proof for RMELL works. So, in order to make this paper self-contained, I will give this proof.

For this, we need some definitions: Two sequents $\Gamma$ and $\Delta$ are *cognate* if they contain the same formulas, i.e., they only differ in the number of occurrences of the formulas in the sequent. They are *modally cognate*, if additionally every formula that is not of the shape $?A$ has the same number of occurrences in both sequents. A set of sequents that are cognate to each other are called a *cognation class*. In the example below, all four sequents are in the same cognation class, but only the first two are modally cognate:

$$\vdash a, b, b, ?a \qquad \vdash a, b, b, ?a, ?a, ?a \qquad \vdash a, a, a, b, ?a, ?a \qquad \vdash a, a, a, a, a, b, b, b, ?a, ?a, ?a \tag{5}$$

---

[4]That is LL extended with the general contraction rule (2). This logic is called CLL in [35], but the C is more often used for "classical". For this reason I use the R for "relevant" throughout this paper.

[5]More recently, it has been shown that RLL is ACKERMANN-complete [24, Corollary 25], and that RMELL is in 2EXP [40, Theorem 6.1].

In the following, we use the notation $\Gamma_1 \succ \Gamma_2$ if there is a derivation with premise $\vdash \Gamma_1$ and conclusion $\vdash \Gamma_2$, using only the c-rule, and we write $\Gamma_1 \succcurlyeq \Gamma_2$ (or equivalently $\Gamma_2 \preccurlyeq \Gamma_1$) iff $\Gamma_1 \succ \Gamma_2$ or $\Gamma_1 = \Gamma_2$ (where = stands for multiset equality).

A (finite or infinite) sequence $\Gamma_1, \Gamma_2, \ldots$ of sequents is *irredundant* if for all $i < j$, we have $\Gamma_i \not\preccurlyeq \Gamma_j$. We can now state Kripke's lemma:

**Theorem 4.1** (Kripke [21])**.** *If a sequence of cognate sequents is irredundant, then it is finite.*

A proof can be found for example in [14, p.289] or [1, p.139]. Note that for this theorem it is irrelevant what the inference rules are and what the language of the formulas is.[6]

We can now use Theorem 4.1 to exhibit a terminating complete proof search procedure for RMELL, from which decidability follows. This is done by using the following variant of the proof system, called (|RMELL|):[7]

$$
\mathsf{id}\;\frac{}{\vdash a, a^\perp} \qquad (|\parr|)\;\frac{\vdash \Gamma, A, B}{\vdash (|\Gamma, A \parr B|)} \qquad (|\otimes|)\;\frac{\vdash \Gamma, A \qquad \vdash B, \Delta}{\vdash (|\Gamma, A \otimes B, \Delta|)}
$$

$$
!\;\frac{\vdash A, ?\Delta}{\vdash !A, ?\Delta} \qquad (|?\mathsf{d}|)\;\frac{\vdash \Gamma, A}{\vdash (|\Gamma, ?A|)} \qquad ?\mathsf{w}\;\frac{\vdash \Gamma}{\vdash \Gamma, ?A} \tag{6}
$$

where the rules (|⅋|), (|⊗|), and (|?d|) are variants of the rules ⅋, ⊗, and ?d, respectively, which have the contraction rule built in, i.e., a sequent $\vdash (|\Sigma|)$ is obtained from $\vdash \Sigma$ by application of the c-rule in (2), such that the following additional conditions are satisfied:[8]

(|⅋|): If the formula $A \parr B$ occurs $n$ times in $\Gamma$, then it occurs at least $\max(n, 1)$ times in $(|\Gamma, A \parr B|)$. Any other formula occurs as often in $\Gamma$ as in $(|\Gamma, A \parr B|)$.

(|⊗|): If the formula $A \otimes B$ occurs $n$ times in $\Gamma$ and $m$ times in $\Delta$, then it occurs at least $\max(n, m, 1)$ times in $(|\Gamma, A \otimes B, \Delta|)$. If another formula occurs $n$ times in $\Gamma$ and $m$ times in $\Delta$, then it occurs at least $\max(n, m)$ times in $(|\Gamma, A \otimes B, \Delta|)$.

(|?d|): If the formula $?A$ occurs $n$ times in $\Gamma$, then it occurs at least $\max(n, 1)$ times in $(|\Gamma, ?A|)$. Any other formula occurs as often in $\Gamma$ as in $(|\Gamma, ?A|)$.

We can now prove the following two theorems about (|RMELL|). The first one states the height-preserving admissibility of contraction:

**Theorem 4.2.** *If $\vdash \Gamma$ has a (|RMELL|) proof $\pi$ with height $h$, and $\Gamma \succ \Gamma'$, then $\vdash \Gamma'$ has a (|RMELL|) proof $\pi'$ with height $h' \leq h$.*

*Proof.* This is proved by a straightforward induction on $h$. □

**Theorem 4.3.** *A sequent $\vdash \Gamma$ is provable in (|RMELL|) if and only if it is provable in* RMELL.

*Proof.* Any proof in (|RMELL|) can be expanded to a proof in RMELL by adding the necessary instances of the c-rule. Conversely, every rule in RMELL, except for c, is an instance of a rule in (|RMELL|). Hence, a proof in in RMELL is already a proof in (|RMELL|) + c, and the result follows from Theorem 4.2.[9] □

With these ingredients, we can give the full proof of decidability for RMELL.

[6] In fact, it is equivalent to Dickson's Lemma in number theory which states that every set of $n$-tuples of natural numbers has finitely many minimal elements (see also [39]).

[7] The (|·|) notation is taken from [3].

[8] In [3], the formulation of these conditions is slightly ambiguous. For this reason, I took here the formulation of [14], adapted to the case of RMELL.

[9] In [3] it is stated that also cut admissibility is necessary for obtaining this theorem, but as we have shown here, it is not needed.

**Theorem 4.4.** *Provability in* RMELL *is decidable.*

*Proof.* By Theorem 4.3 we can restrict proof search to (|RMELL|). By Theorem 4.2, we can stop the search when we reach a sequent Γ such that on the current branch of the proof search tree there is an ancestor Γ′ with Γ′ ≼ Γ.

Then, we observe that any formula that occurs in an (|RMELL|) is a subformula of the endsequent and there are only finitely many such formulas. There are infinitely many sequents that can be formed from these formulas, but there are only finitely many cognation classes. Furthermore, from the previous paragraph and Theorem 4.1 it follows that from each cognation class only finitely many sequents need to be visited in a single branch in the proof search tree. Since the proof search tree is finitely branching (each inference rule has only finitely many premises and at each step there are only finitely many choices for applying an inference rule), we can conclude by König's lemma that the proof search tree is finite. □

It is important to observe that the whole argument breaks down if we assume that Theorem 4.1 only holds for sequences of modally cognate sequents. In that case it is still true that in each branch only finitely many sequents of the same cognation class are visited, but now there are infinitely many cognation classes (see example in (5)), i.e., the sequents visited in a single branch of the proof search tree can become arbitrarily large. And this is exactly the reason why the decidability proof for MELL in [3] is not correct, as we will see in the next section.

Finally, it is easy to see that we can define a system (|RMSELL|) in the same way as (|RMELL|), and that we can prove the decidability of provability in RMSELL with almost literally the same proof as for RMELL.

**Theorem 4.5.** *Provability in* RMSELL *is decidable.*

### *4.3. The Error in the Decidability Proof for* MELL

The proof of the decidability of MELL in [3] is based on the observation that every MELL-proof is also a RMELL-proof of the same endsequent. Thus, given a sequent Γ, if the RMELL proof search for Γ comes back with a failure, then we know that Γ is not provable in MELL. On the other hand, if there is an RMELL-proof of Γ, then we can count for each subformula occurrence $?A$ of Γ, how often the contraction rule is applied to it in any RMELL-proof of Γ.

Even though there is no bound on the proof search for RMELL, we have that (|RMELL|) proof search is bounded. The argument of [3] is now that it is enough to count for each occurrence of a subformula $?A$ of a sequent Γ, how often the subformula occurrence $A$ and its ancestors[10] are contracted in any (|RMELL|) proof of Γ. This is called the *heap number of* $?A$ [3, Def. 22], which is then used to restrict proof search in the system [MELL] shown below:

$$\mathsf{id}\,\frac{}{\vdash a, a^{\perp}} \qquad \parr\,\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B} \qquad [\otimes]\,\frac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash [\Gamma, A \otimes B, \Delta]}$$

$$!\,\frac{\vdash A, ?\Delta}{\vdash !A, ?\Delta} \qquad [?\mathsf{d}]\,\frac{\vdash \Gamma, A}{\vdash [\Gamma, ?A]} \qquad ?\mathsf{w}\,\frac{\vdash \Gamma}{\vdash \Gamma, ?A} \tag{7}$$

where [⊗] and [?d] are variants of the rules ⊗ and ?d, respectively, which have the ?c-rule built in, i.e., a sequent $\vdash [\Sigma]$ is obtained from $\vdash \Sigma$ by application of the ?c-rule, such that the following additional conditions are satisfied:

[⊗]: If a formula $?C$ occurs $n$ times in Γ and $m$ times in Δ, then it occurs at least $\max(n, m)$ times in $[\Gamma, A \otimes B, \Delta]$.

[?d]: If the formula $?A$ occurs $n$ times in Γ, then it occurs at least $\max(n, 1)$ times in $[\Gamma, ?A]$. Any other formula occurs as often in Γ as in $[\Gamma, ?A]$.

We have then for [MELL] analogous results as stated in Theorems 4.2 and 4.3 for (|RMELL|), namely the ?c is height-preserving admissible for [MELL], and a sequent is provable in [MELL] if and only if it is provable in MELL.

The decidability proof for MELL in [3, Theorem 23] now simply bounds the number of application of the [?d]-rule to (the ancestors of) an occurrence of a $?A$-formula by its heap-number.

[10] I.e., the ancestors of the auxiliary formulas of the instances of the (|?d|)-rule in which an ancestors of that $?A$ are principal.

However, the same argument would apply to prove decidability of MSELL from the decidability of RMSELL, by going via the system [MSELL] that is defined in the same way as [MELL]. But MSELL has been shown undecidable in [5].

Therefore, something must be wrong with Bimbó's argument in [3]. First, it is obvious that Bimbó's decision procedure terminates. This follows from the termination argument for (|RMELL|). However, there is no argument explaining why her decision procedure should be complete.

To understand the problem, consider for example the following MELL sequent:

$$\vdash (a^\perp \otimes a^\perp) \otimes (a^\perp \otimes a^\perp), a, ?(a^\perp \otimes (a \parr a)) \tag{8}$$

The obvious proof in (|RMELL|) removes the ?-formula with a weakening, applies the (|⊗|)-rule three times, and puts a copy of $a$ into each branch:

$$?\mathsf{w}\,\dfrac{(\!|\otimes|\!)\,\dfrac{(\!|\otimes|\!)\,\dfrac{\mathsf{id}\,\dfrac{}{a^\perp, a} \quad \mathsf{id}\,\dfrac{}{a^\perp, a}}{\vdash a^\perp \otimes a^\perp, a} \qquad (\!|\otimes|\!)\,\dfrac{\mathsf{id}\,\dfrac{}{a^\perp, a} \quad \mathsf{id}\,\dfrac{}{a^\perp, a}}{\vdash a^\perp \otimes a^\perp, a}}{\vdash (a^\perp \otimes a^\perp) \otimes (a^\perp \otimes a^\perp), a}}{\vdash (a^\perp \otimes a^\perp) \otimes (a^\perp \otimes a^\perp), a, ?(a^\perp \otimes (a \parr a))} \tag{9}$$

For proving the sequent (8) in [MELL] we need to apply the [?d]-rule three times in order to create the necessary four copies of $a$:

$$[?\mathsf{d}]\,\dfrac{[\otimes]\,\dfrac{\mathsf{id}\,\dfrac{}{a, a^\perp} \qquad \parr\,\dfrac{[\otimes]\,\dfrac{[?\mathsf{d}]\,\dfrac{[\otimes]\,\dfrac{\mathsf{id}\,\dfrac{}{a, a^\perp} \quad \parr\,\dfrac{[\otimes]\,\dfrac{\mathsf{id}\,\dfrac{}{a^\perp, a} \quad \mathsf{id}\,\dfrac{}{a^\perp, a}}{\vdash a^\perp \otimes a^\perp, a, a}}{\vdash a^\perp \otimes a^\perp, a \parr a}}{\vdash a^\perp \otimes a^\perp, a, a^\perp \otimes (a \parr a)}}{\vdash a^\perp \otimes a^\perp, a, ?(a^\perp \otimes (a \parr a))} \qquad [?\mathsf{d}]\,\dfrac{[\otimes]\,\dfrac{\mathsf{id}\,\dfrac{}{a, a^\perp} \quad \parr\,\dfrac{[\otimes]\,\dfrac{\mathsf{id}\,\dfrac{}{a^\perp, a} \quad \mathsf{id}\,\dfrac{}{a^\perp, a}}{\vdash a^\perp \otimes a^\perp, a, a}}{\vdash a^\perp \otimes a^\perp, a \parr a}}{\vdash a^\perp \otimes a^\perp, a, a^\perp \otimes (a \parr a)}}{\vdash a^\perp \otimes a^\perp, a, ?(a^\perp \otimes (a \parr a))}}{\vdash (a^\perp \otimes a^\perp) \otimes (a^\perp \otimes a^\perp), a, a, ?(a^\perp \otimes (a \parr a)) \quad *}}{\vdash (a^\perp \otimes a^\perp) \otimes (a^\perp \otimes a^\perp), a \parr a, ?(a^\perp \otimes (a \parr a))}}{\vdash (a^\perp \otimes a^\perp) \otimes (a^\perp \otimes a^\perp), a, a^\perp \otimes (a \parr a), ?(a^\perp \otimes (a \parr a))}}{\vdash (a^\perp \otimes a^\perp) \otimes (a^\perp \otimes a^\perp), a, ?(a^\perp \otimes (a \parr a))} \tag{10}$$

This proof is at the same time a correct (|RMELL|) proof, but it is not visited by the (|RMELL|) proof search described in the proof of Theorem 4.4, because the sequent marked with a $*$ in (10) is in $>$ relation with the conclusion, and therefore the (|RMELL|) proof search is aborted at that point. For this reason, we cannot assume, *a priori*, that the *heap numbers* determined by the (|RMELL|) decision procedure are high enough to ensure a complete [MELL] proof search, which leaves a large gap in the decidability proof for MELL in [3].

However, if we consider again the derivation in (10), we can see that there is a rule permutation variant that does not visit the sequent $*$: we can permute one instance of [?d] down below the [⊗] and the ⅋-rule instance and keep a copy of $?(a^\perp \otimes (a \parr a))$. Unfortunately, it is not at all clear whether such a rule permutation always exists. More generally, we can formulate the following conjecture:

**Conjecture 4.6.** *Let $\pi$ be a proof in* [MELL] *of a sequent $\Gamma$. Then there is a proof $\pi'$ in* [MELL] *with the same endsequent $\Gamma$, such that for any two sequents $\Gamma_1$ and $\Gamma_2$ occurring in $\pi'$ with $\Gamma_1$ being an ancestor of $\Gamma_2$ in the tree of $\pi'$, we have $\Gamma_1 \not\gtrsim \Gamma_2$.*

It follows immediately from [5] that Conjecture 4.6 does not hold if we replace [MELL] by [MSELL], but it might well be that it holds for the special case where the label set $V$ is a singleton. A proof of Conjecture 4.6 would indeed close the gap in Bimbó's proof of the decidability of MELL, and would provide an alternative proof for the decidability of the reachability problem for VASS and Petri nets.

Note that a counterexample to Conjecture 4.6 would not show that MELL is undecidable. But it would show that the proof idea of [3] cannot work.

## 5. Conclusion

Whenever a paper is published whose main proof is faulty, it is always easy to blame the author or the editor or the referee or some other victim. However, in this case this would be too short-sighted. I think that the publication of [3] is the consequence of a systemic problem in the field of structural proof theory. Namely, that it is studied by two different communities: one coming from a computer science background and the other coming from a philosophical background. These two communities use different notation and terminology and do not talk to each other. Furthermore, they also consider each other to be less skilled and take this as a justification for not taking each other's papers seriously.

On the one hand, the philosophers rightfully accuse the computer scientists of ignoring the vast amount of literature on substructural logic that existed *before linear logic*, and the computer scientists rightfully accuse the philosophers of considering linear logic as just another substructural logic, ignoring the semantic consideration that gave rise to it and the vast amount of literature that came *after linear logic*, exhibiting its enormous influence in many areas of theoretical computer science that makes linear logic very special among the zoo of substructural logics.

The decision problem for MELL is open for three decades now, and many people in the computer science community worked on it, but nobody has observed its containment in RMELL, even though decision problems for relevant logics have been studied since the 1960's, and the relation between MELL and RMELL is obvious to anyone with a background in philosophical logic. This observation leads naturally to Bimbó's proof idea: Can we bound the proof search for MELL by using information that we can extract from the RMELL proofs?

On the one hand, it is quite embarrassing for the computer science community that nobody has explored this idea before Bimbó.[11] On the other hand, the gap in Bimbó's reasoning could be spotted immediately by anybody familiar with the peculiarities of MELL. However, Bimbó's error is very easy to overlook for someone not familiar with linear logic, and this led to the unfortunate publication of [3].

We should take this as a lesson to take each other more seriously in the future. It seems that with the decidability of MELL we have a problem that needs for its solution both communities, philosophy and computer science.

*Acknowledgments*

Many thanks to Sylvain Schmitz, Dale Miller, Willem Heijltjes, and Revantha Ramanayake for helpful comments and discussions improving the readability and comprehensiveness of this short paper.

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[11] At this point it is worth mentioning that the Karp-Miller trees [19], which are finite structures that are crucial for the decidability proof for VASS, play in that proof a similar role as the (|RMELL|) derivations in Bimbó's proof attempt for MELL. Given that it is still a big leap from Karp-Miller trees to the full decidability of the reachability problem for VASS, a proof of Conjecture 4.6 will very likely be much harder than it seems on first sight, if it exists at all.

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