# Load Balancing Congestion Games and Their Asymptotic Behavior

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***Abstract*—A central question in routing games has been to establish conditions for the uniqueness of the equilibrium, either in terms of network topology or in terms of costs. This question is well understood in two classes of routing games. The first is the non-atomic routing introduced by Wardrop on 1952 in the context of road traffic in which each player (car) is infinitesimally small; a single car has a negligible impact on the congestion. Each car wishes to minimize its expected delay. Under arbitrary topology, such games are known to have a convex potential and thus a unique equilibrium. The second framework is splittable atomic games: there are finitely many players, each controlling the route of a population of individuals (let them be cars in road traffic or packets in the communication networks). In this paper, we study two other frameworks of routing games in which each of several players has an integer number of connections (which are population of packets) to route and where there is a constraint that a connection cannot be split. Through a particular game with a simple three link topology, we identify various novel and surprising properties of games within these frameworks. We show in particular that equilibria are not unique even in the potential game setting of Rosenthal with strictly convex link costs. We further show that non-symmetric equilibria arise in symmetric networks,**

**Kewords: Congestion Games, Load balancing. Several equilibria. Semi Splittable Games**

## I. Introduction

A central question in routing games has been to establish conditions for the uniqueness of the equilibria, either in terms of the network topology or in terms of the costs. A survey on these issues is given in [1].

The question of uniqueness of equilibria has been studied in two different frameworks. The first, which we call **F1**, is the *non-atomic routing* introduced by Wardrop on 1952 in the context of road traffic in which each player (car) is infinitesimally small; a single car has a negligible impact on the congestion. Each car wishes to minimize its expected delay. Under arbitrary topology, such games are known to have a convex potential and thus have a unique equilibrium [2]. The second framework, denoted by **F2**, is *splittable atomic games*. There are finitely many players, each controlling the route of a population of individuals. This type of games have already been studied in the context of road traffic by Haurie and Marcotte [3] but have become central in the telecom community to model routing decisions of Internet Service Providers that can decide how to split the traffic of their subscribers among various routes so as to minimize network congestion [4].

In this paper we study properties of equilibria in two other frameworks of routing games which exhibit surprising behavior. The first, which we call **F3**, known as *congestion games* [5], consists of atomic players with non splittable traffic: each player has to decide on the path to be followed by for its traffic and cannot split the traffic among various paths. This is a non-splittable framework. We further introduce a new semi-splittable framework, denoted by **F4**, in which each of several players has an integer number of connections to route. It can choose different routes for different connections but there is a constraint that the traffic of a connection cannot be split. In the case where each player controls the route of a single connection and all connections have the same size, this reduces to the congestion game of Rosenthal [5].

We consider in this paper routing games with additive costs (i.e. the cost of a path equals to the sum of costs of the links over the path) and the cost of a link is assumed to be convex increasing in the total flow in the link. The main goal of this paper is to study a particular symmetric game of this type in a simple topology consisting of three nodes and three links. We focus both on the uniqueness issue as well as on other properties of the equilibria.

This game has already been studied within the two frameworks **F1**-**F2** that we mentioned above. In both frameworks it was shown [6] to have a unique equilibrium. Our first finding is that in frameworks **F3** and **F4** there is a multitude of equilibria. The price of stability is thus different from the price of anarchy and we compute both. We show the uniqueness of the equilibrium in the limit as the number of players $N$ grows to infinity extending known results [3] from framework **F2** to the new frameworks. In framework **F2** uniqueness is in fact achieved not only for the limiting games but also for all $N$ large enough. We show that this *is not the case* for **F3**-**F4**: for any finite $N$ there may be several equilibria. We show however in **F3** that the whole set of equilibria corresponding to a givevn $N$ converge to the singleton corresponding to the equilibrium in **F1** as $N \to \infty$. We finally show a surprising property of **F4** that exhibits non symmetric equilibria in our symmetric network example while under **F1**, **F2** and **F3** there are no asymmetric equilibria.

The structure of the paper is as follows. We first introduce the model and the notations used in the study, we then move on to the properties of frameworks **F3** (Section III) and **F4** (Section IV) and their relation to frameworks **F1** and **F2**. We include in the Appendix the proofs of the theorems and propositions of the paper.

## II. Model and Notations

We shall use throughout the term *atomic game* to denote situations in which decisions of a player have an impact on other players' utility. It is *non-atomic* when players are infinitesimally small and are viewed like a fluid of players, such that a single player has a negligible impact on the utility of other players.

We consider a system of three nodes ($A$, $B$ and $C$) with two incoming traffic sources (respectively from node $A$ and $B$) and an exit node $C$. There are a total of $N$ connections originating from each one of the sources. Each connection can either be sent directly to node $C$ or rerouted via the remaining node. The system is illustrated in Figure 1.

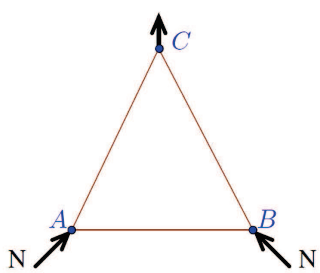


Fig. 1. Physical System

This model has been used to model load balancing issues in computer networks, see [6] and references therein. Jobs arrive to two computing centers represented by nodes A and B. A job can be processed locally at the node where it arrives or it may be forwarded to the other node incurring further communication delay. The costs of links $[AC]$ and $[BC]$ represent the processing delays of jobs processed at nodes A and B respectively. Once processed, the jobs leave the system. A connection is a collection of jobs with similar characteristics (e.g. belonging to the same application).

We introduce the following notations:

- A link between two nodes, say A and B, is denoted by $[AB]$. Our considered system has three links $[AB]$, $[BC]$ and $[AC]$.
- A route is simply referred by a sequence of nodes. Hence, the system has four types of connections (routes): two originating from node $A$ (route AC and ABC) and two originating from node $B$ (route BC and BAC).

Further, in the following, $n_{AC}$, $n_{BC}$, $n_{ABC}$ and $n_{BAC}$ will refer to the number of connections routed via the different routes while $n[AC]$, $n[BC]$ and $n[AB]$ will refer to the number of connections on each subsequent link. By conservation law, we have:

$$n_{AC} + n_{ABC} = n_{BC} + n_{BAC} = N$$

$$\text{and } \begin{cases} n[AC] = n_{AC} + n_{BAC}, \\ n[BC] = n_{ABC} + n_{BC}, \\ n[AB] = n_{BAC} + n_{ABC}. \end{cases}$$

For each route $r$, we also define the fraction (among $N$) of flow using it, i.e. $f_r = n_r/N$. The conservation law becomes $f_{AC} + f_{ABC} = f_{BC} + f_{BAC} = 1$.

Finally, the performance measure considered in this work is the cost (delay) of connections experienced on their route. We consider a simple model in which the cost is additive (i.e. the cost of a connection on a route is simply taken as the sum of delays experienced by the connection over the links that constitute this route). The link costs are given by

$$\begin{cases} C_{[AB]} = a(f_{BAC} + f_{ABC}), \\ C_{[AC]} = b(f_{BAC} + f_{AC}), \\ C_{[BC]} = b(f_{BC} + f_{ABC}). \end{cases}$$

where $a(.)$ and $b(.)$ are some functions of the corresponding fractions of link flows. The path costs are given by:

$$\begin{array}{ll} C_{AB} = C_{[AB]}, & C_{ABC} = C_{[AB]} + C_{[BC]}, \\ C_{BC} = C_{[BC]}, & C_{BAC} = C_{[AB]} + C_{[AC]}. \end{array}$$

The cost for a user in frameworks F2-F4 is the average of path costs weighted by the fraction that the player sends over each of the paths. For framework F3 a single packet is sent by each player so the cost for the player is the cost for the path that it takes.

We shall frequently assume that the costs on each link are linear with coefficient $\alpha/N$ on link $[AB]$ and coefficient $\beta/N$ on link $[AC]$ and $[BC]$, i.e. for some positive constants $\alpha$ and $\beta$ we have

$$\begin{cases} C_{[AB]} = \dfrac{\alpha}{N}(f_{BAC} + f_{ABC}), \\ C_{[AC]} = \dfrac{\beta}{N}(f_{BAC} + f_{AC}), \\ C_{[BC]} = \dfrac{\beta}{N}(f_{BC} + f_{ABC}). \end{cases}$$

We restrict our study to the (pure) Nash equilibria which we express in terms of the corresponding flows marked by a star. By conservation law, the equilibria is uniquely determined by the specification of $f^*_{ABC}$ and $f^*_{BAC}$ (or equivalently $n^*_{ABC}$ and $n^*_{BAC}$).

The main contribution of the paper is the study of the above network within the following two types of decision models. In the first (**F3**), the decision is taken at the connection level (Section III), i.e. each connection has its own decision maker that seeks to minimize the connection's cost, and the connection cannot be split into different routes. In the second (**F4**), (Section IV) each one of the two source nodes decides on the routing of all the connections originating there. Each connection of a given source node (either A or B) can be routed independently but a connection cannot be split into different route. We hence refer to **F4** this semi-splittable framework. Note that the two-approaches (**F3** and **F4**) coincide when there is only $N = 1$ connection at each source, which we also detail later. We shall relate frameworks **F3** and **F4** to the frameworks **F1** and **F2** obtained as limits as the number of connections grows to infinity.

## III. Atomic Non-Splittable (F3 framework) Case and its non-atomic limit (F1 framework)

We consider here the case where each $2N$ players connection belongs to an individual user acting selfishly.

We first show that for fixed parameters, the game may have several equilibria, all of which are symmetric for any number of players. The number of distinct equilibria can be made arbitrary large by an appropriate choice of functions $a$ and $b$.

We then show properties of the limiting game obtained as the number of of players increases to infinity.

### A. Non-uniqueness of the equilibrium

**Theorem 1.** *Assume that $a$ is non-negative and non-decreasing and that $b$ is increasing. Then any equilibrium is symmetric i.e. $f^*_{BAC} = f^*_{ABC}$. Routing a fraction $2x$ players ($x$ from A and $x$ from B) to indirect links is an equilibrium if and only if*

$$a(2x) \le b(1 + 1/N) - b(1) \tag{1}$$

**Proof.** Consider an equilibrium $(f^*_{ABC}, f^*_{BAC})$. We first show that the equilibrium is symmetric. Assume on the contrary that $f^*_{ABC} > f^*_{BAC}$. Since the demands are the same this implies that $f^*_{BC} > f^*_{AC}$ and the total flow on link $BC$ is strictly larger than the flow on link $AC$. But then, any player that takes the route $ABC$ (note that by assumption there is at least one such player) would strictly decrease its cost if it deviates to the direct path $AC$. This contradicts the assumption of equilibrium. Hence $f^*_{ABC} = f^*_{BAC}$ and $f^*_{BC} = f^*_{AC}$.

At equilibrium a player that takes the direct path cannot gain by deviating. Thus a routing multistrategy is an equilibrium if and only if a player that takes the indirect path cannot strictly decrease its cost by deviation. Equivalently, routing a fraction $x$ of players via the indirect link is an equilibrium if and only if $b(1) + a(2x) \le b(1 + 1/N)$, which implies the Theorem. ∎

We shall call a multipolicy that routes $k$ connections to each of the indirect path a "k-policy".

**Corollary 1.** *Assume that $a(x)$ and $b(x)$ are increasing in $x$. Then, (i) if for some $k$, the $k$-policy is an equilibrium then for any $j < k$, the $j$-policy is also equilibrium.*
*(ii) If for some $N$, a $k$-policy is an equilibrium then it is also an equilibrium for smaller values of $N$.*

We calculate the number of equilibria for different cost functions. Let $k$ be the solution of eq. (1) obtained with equality. Hence the number of equilibira will is $\lfloor k \rfloor + 1$.

We have the following cases:

- When $b(x) = \beta x$ and $a(x) = \alpha x$, then Condition (1) reduces to
$$x \le \frac{\beta}{2N\alpha}.$$
So the number of equilibria is $\lfloor \frac{\beta}{2\alpha} \rfloor + 1$.
The plot of the number of equilibria with respect to $\beta$ for $\alpha = 1$ and $N = 10$ is given in Fig. 2.

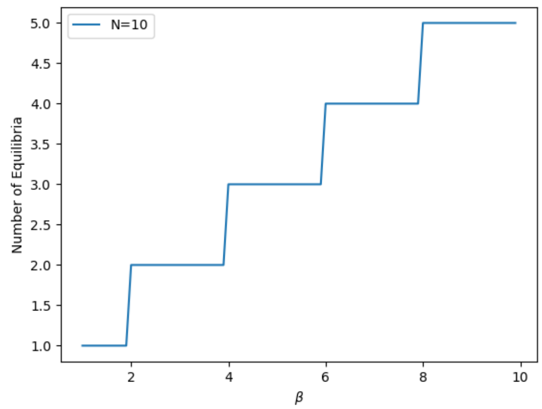


Fig. 2. Variation of number of equilibria with respect to $\beta$ (for $\alpha = 1$)

We have the following observations
1) The number of equilibria does not depend on the number $N$ of players.
2) The number of equilibria increases as the cost function $\beta$ increases for constant value of $\alpha$.

- When the cost function on the direct link is linear i.e., $b(x) = \beta x$ and indirect link is non-linear and is of the form $a(x) = x^\ell$ for $\ell \ge 0$, then Condition (1) reduces to
$$x \le \frac{1}{2}\left(\frac{\beta}{N}\right)^{\frac{1}{\ell}}.$$
So the number of equilibria is $\left\lfloor \frac{N}{2}\left(\frac{\beta}{N}\right)^{\frac{1}{\ell}} \right\rfloor + 1$.
The plot of the number of equilibria with respect to $\ell$ for $\beta = 1$ is shown in Fig. 3.

We have the following observations
1) The number of equilibria depends on the number of players, $N$. It increases with $N$ for $\ell > 1$ and decreases in $N$ for $\ell < 1$.
2) The number of equilibria increases in the power factor $\ell$.

**Remark 1.** *Consider the special case that $a = 0$. The problem is the equivalent to routing on parallel links. Now assume that $b$ is decreasing. Then the only equilibria are (i) Send no flow through AC and (ii) send no flow through BC.*

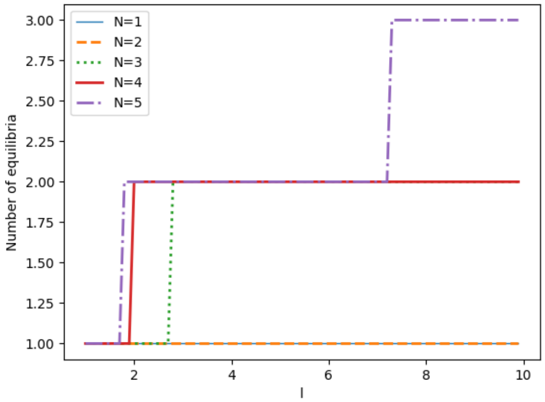


Fig. 3. Variation of number of equilibria with respect to $\ell$ (for $\beta = 1$)

**Corollary 2.** *Assume that the derivative $a'(0)$ of $a$ at zero and the derivative $b'(1)$ of $b$ at 1 exist. Then for large enough $N$, the $k$-policy is an equilibrium if*

$$2ka'(0) < b'(1)$$

*If moreover, $b$ is convex and $a$ concave (not necessarily strictly), then the above holds for every $n$. If the opposite inequality holds above then for all $n$ large enough the $k$ policy is not an equilibrium.*

**Proof.** The first part follows from (1). The second part follows from the fact that the slope $(f(x+y)-f(x))/y$ of a function increases in $y$ if the function is convex and decreases in $y$ if it is concave. ∎

*B. The potential and asymptotic uniqueness*

When the number of players $N$ grows to infinity, the limiting game becomes a non-atomic game with a potential [7] defined by

$$F_\infty(f_{ABC}, f_{BAC}) = \int_0^{r_1} a(s)ds + \int_0^{r_2} b(s)ds + \int_0^{r_3} b(s)ds$$

where $r_1 = f_{ABC} + f_{BAC}, r_2 = 1 - f_{ABC} + f_{BAC}$ and $r_3 = 1 + f_{ABC} - f_{BAC}$. In the special case of linear cost of the form $a(x) = \alpha x, b(x) = \beta x$, the above potential is equivalent to the following one (upto a constant)

$$F_\infty(f_{ABC}, f_{BAC}) \tag{2}$$
$$= \beta(f_{ABC} - f_{BAC})^2 + \frac{\alpha}{2}(f_{ABC} + f_{BAC})^2.$$

Hence we have the following:

**Proposition 1.** *If $a$ and $b$ are strictly increasing then the non-atomic game (framework **F1**) has a unique Nash equilibrium, which is $f^*_{ABC} = f^*_{BAC} = 0$.*

Uniqueness of the equilirium was shown to hold in [8], [9] under different conditions. More general topological setting are considered and more general definition of players. Yet it is assumed there that the costs are continuously differentiable which we do not assume here.

To show the uniqueness of the equilibrium in the limiting game, we make use of the fact that the limiting game has a potential which is convex. Yet, not only the limiting game has a convex potential, but also the original one, as we conclude from next theorem, whose proof is a direct application of [5].

**Theorem 2.** *For any finite number of players, the game is a potential game [10] with the potential function upto a constant:*

$$F(f_{ABC}, f_{BAC}) =$$
$$\sum_{i=0}^{Nr_1} a(i) + \sum_{i=0}^{Nr_2} b(i) + \sum_{i=0}^{Nr_3} b(i) \tag{3}$$

*For the case of linear costs this gives*

$$F(f_{ABC}, f_{BAC}) = \beta N^2(f_{ABC} - f_{BAC})^2$$
$$+\frac{\alpha N^2}{2}(f_{ABC} + f_{BAC})(f_{ABC} + f_{BAC} + 1/N). \tag{4}$$

Note that unlike the framework **F1** of non-atomic games, the fact that the game has a potential which is convex over the action set does not imply uniqueness. The reason for that is that in congestion games, the action space over which the potential is minimized is not a convex set (due to the non-splittable nature) so that it may have several local minima, each corresponding to another equilibrium, whereas a for a convex function over the Euclidean space, there is a unique local minimum which is also a global minimum of the function (and thus an equilibrium of the game).

*C. Efficiency*

Proposition 1 implies that

**Theorem 3.** *In the non-atomic setting, **F1**, the only Nash equilibrium is also the social optimum (i.e. the point minimizing the sum of costs of all players) of the system.*

**Proof.** The sum of costs of all players is

$$f_{ABC}C_{ABC} + f_{AC}C_{AC} + f_{BAC}C_{BAC} + f_{BC}C_{BC}$$
$$= (f_{ABC} + f_{BAC})a(f_{ABC} + f_{BAC}) + f_{ABC}b(f_{BC} + f_{ABC})$$
$$+f_{AC}b(f_{AC} + f_{BAC}) + f_{BAC}b(f_{AC} + f_{BAC})$$
$$+f_{BC}b(f_{ABC} + f_{BC}). \tag{5}$$

The minimum is hence obtained for $(f_{ABC}, f_{BAC}) = (0, 0)$. ∎

See [8], [9] for related results. Since the game possesses several equilibria, we can expect the PoA (Price of Anarchy - the largest ratio between the sum of costs at an equilibrium and the sum of costs at the social optimum) and PoS (Price of Stability - the smallest corresponding ratio) to be different.

Let $k^*$ be the largest integer such that $x^* = k^*/N$ satisfies eq (1). Then the equilibrium $(\hat{x}^*, \hat{x}^*)$ with largest cost corresponds to this $k$.

**Theorem 4.** *The price of stability of the game is* 1 *and the price of anarchy is*

$$PoA = \frac{x^* a(2x^*)}{b(1)} + 1$$

*with* $x^* = f^*_{ABC} = f^*_{BAC}$.

**Proof.** According to Theorem 1 we may restrict to symmetric equilibria i.e. $n^*_{ABC} = n^*_{BAC}$, then $f^*_{ABC} = f^*_{BAC} = x^*$. So the sum of costs of all the players becomes $2x^*a(2x^*) + 2b(1)$.

The sum of costs at social optimum is $2b(1)$ i.e., at $x^* = 0$.

The price of anarchy is equal to the largest ratio between the sum of costs at an equilibrium to the sum of costs at social optimum. So $PoA = \frac{2x^*a(2x^*)+2b(1)}{2b(1}$. Hence the result. ∎

**Note**: Substituting $x^* = k^*/N$, the price of anarchy becomes,

$$PoA = \frac{k^*}{Nb(1)} a\left(\frac{2k^*}{N}\right) + 1$$

We look into different cases of cost functions and calculate the price of anarchy using the value of $x$ and the theorem.

We have the following cases:

- When the cost function on both the direct and indirect link is linear and is of the form $b(x) = \beta x$ and $a(x) = \alpha x$, then $PoA \leq \frac{\beta}{2\alpha N^2} + 1$. The exact value of price of anarchy can be obtained by substituting the exact value of $k$. So,

$$PoA = \frac{2\alpha}{N^2 \beta} \left\lfloor \frac{\beta}{2\alpha} \right\rfloor^2 + 1.$$

The plot of the $PoA$ with respect to varying $\beta$ for a constant $\alpha = 1$ is shown in Fig.4.

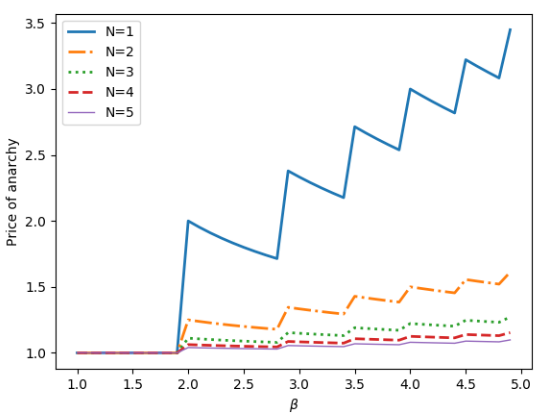


Fig. 4. Variation of price of anarchy with respect to $\beta$ (for $\alpha = 1$)

We have the following observations

1) As the number of player increases, the $PoA$ decreases.
2) For large $N$, the price of anarchy may asymptotically reach to 1.
3) The $PoA$ increases as the cost function $\beta$ increases.
4) If $\beta > 2\alpha$, the $PoA$ never becomes 1 for any value of $N$.

- When the cost function on the direct is linear i.e., $b(x) = \beta x$ and indirect link is non- linear and is of the form $a(x) = x^\ell$ for $l \geq 0$, then the exact value of price of anarchy can be obtained by substituting the exact value of $k$. So,

$$PoA = \frac{2^\ell}{\beta N^{\ell+1}} \left\lfloor \frac{N}{2}\left(\frac{\beta}{N}\right)^{\frac{1}{\ell}} \right\rfloor^{\ell+1} + 1.$$

The plot of the $PoA$ with respect to varying $\ell$ for a constant $\beta = 1$ is given in Fig. 5.

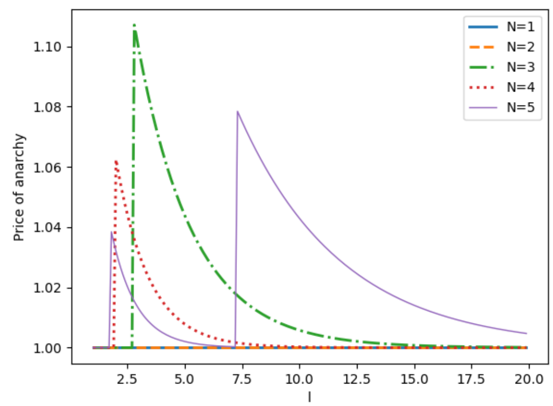


Fig. 5. Variation of number of equilibria with respect to $\ell$ (for $\beta = 1$)

We have the following observations from the graph:

1) There is no monotonicity in the graph with either respect to the number of players or with the power factor $l$.
2) If $\beta < 2^\ell N^{1-\ell}$, the $PoA$ becomes 1.
3) For large $\ell$, the $PoA$ again becomes 1.
4) For $N = 1, 2$, the $PoA$ is 1 for all values of $\ell$.

We also make the following observations:

(i) In the splittable atomic games studied in [6] the PoA was shown to be greater than one for sufficiently small number of players (smaller than some threshold), and was 1 for all large enough number of players (larger than the same threshold). Here for any number of players, the PoS is 1 and the PoA is greater than 1.

(ii) The PoA decreases in $N$ and tends to 1 as $N$ tends to infinity, the case of splittable games.

(iii) We have shown that the PoA is unbounded: for any real value $K$ and any number of players one can choose the cost parameters $a$ and $b$ so that the PoA exceeds $K$. This corresponds to what was observed in splittable games [8] and contrast with the non-atomic setting of single commodity flows (i.e. when there is only one source node instead of two) see [11].

## IV. Atomic Semi-Splittable Case and Its Splittable Limit (F4 framework)

We restrict in the rest of the paper to the linear cost. The game can be expressed as a 2-player matrix game where each player (i.e. each source node $A$ and $B$) has $N+1$ possible actions, for each of the $N+1$ possible values of $f_{ABC}$ and $f_{BAC}$ respectively.

The utility for player $A$ is

$$\begin{aligned} U_A(f_{ABC}, f_{BAC}) &= f_{AC}C_{AC} + f_{ABC}C_{ABC} \\ &= b - bf_{ABC} + bf_{BAC} \\ &\quad +(a-2b)f_{ABC}f_{BAC} + (a+2b)f_{ABC}^2 \end{aligned} \tag{6}$$

Similarly, for player $B$:

$$\begin{aligned} U_B(f_{ABC}, f_{BAC}) &= f_{BC}C_{BC} + f_{BAC}C_{BAC} \\ &= b - bf_{BAC} + bf_{ABC} \\ &\quad +(a-2b)f_{BAC}f_{ABC} + (a+2b)f_{BAC}^2 \end{aligned} \tag{7}$$

Note that

$$\frac{\partial U_A}{\partial f_{ABC}} = -b + (a-2b)f_{BAC} + 2(a+2b)f_{ABC}$$

and $\dfrac{\partial U_B}{\partial f_{BAC}} = -b + (a-2b)f_{ABC} + 2(a+2b)f_{BAC}$.

Hence $\dfrac{\partial^2 U_A}{\partial f_{ABC}^2} = 2(a+2b) = \dfrac{\partial^2 U_B}{\partial f_{BAC}^2}$. Therefore, both $u_A : f_{ABC} \mapsto U_A(f_{ABC}, f_{BAC})$ and $u_B : f_{BAC} \mapsto U_B(f_{ABC}, f_{BAC})$ are (strictly) convex functions. This means that for each action of one player, there would be a unique best response to the second player if its action space was the interval $(0,1)$. Hence, for the limit case (when $N \to \infty$), the best response is unique. In contrast, for any finite value of $N$, there are either 1 or 2 possible best responses which are the discrete optima of functions $u_A : f_{ABC} \mapsto U_A(f_{ABC}, f_{BAC})$ and $u_B : f_{BAC} \mapsto U_B(f_{ABC}, f_{BAC})$. We will however show that in the finite case, there may be up to $2 \times 2 = 4$ Nash equilibria while in the limit case the equilibrium is always unique.

### A. Efficiency

Note that the total cost of the players is

$$\begin{aligned} &\Sigma(f_{ABC}, f_{BAC}) = U_A(f_{ABC}, f_{BAC}) + U_B(f_{ABC}, f_{BAC}) \\ &= 2b + 2(a-2b)f_{ABC}f_{BAC} + (a+2b)(f_{ABC}^2 + f_{BAC}^2) \\ &= 2b + a(f_{ABC} + f_{BAC})^2 + 2b(f_{ABC} - f_{BAC})^2 \\ &\geq 2b. \end{aligned}$$

Further, note that $\Sigma = 2(F_\infty + b)$. Hence $\Sigma$ is strictly convex. Also $\Sigma(0,0) = 2b$. Therefore $(0,0)$ is the (unique) social optimum of the system. Yet, for sufficiently large $N$ (that is, as soon as we add enough flexibility in the players' strategies), this is not a Nash equilibrium, as stated in the following theorem:

**Theorem 5.** *The point $(f_{ABC}, f_{BAC}) = (0,0)$ is a Nash equilibrium if and only if $N \leq (a/b) + 2$.*

**Proof.** By symmetry and as $u_A : f_{ABC} \mapsto U_A(f_{ABC}, f_{BAC})$ is convex, then $(0,0)$ is a Nash equilibrium iff $U_A(0,0) \leq U_A(1/N, 0) = b - b/N + (a+2b)/N^2$ which leads to the conclusion. ∎

Also, we can bound the total cost by:

$$\begin{aligned} &\Sigma(f_{ABC}, f_{BAC}) = \\ &= 2b + 2(a-2b)f_{ABC}f_{BAC} + (a+2b)(f_{ABC}^2 + f_{BAC}^2) \\ &\leq 2b + (a-2b)(f_{ABC}^2 + f_{BAC}^2) + (a+2b)(f_{ABC}^2 + f_{BAC}^2) \\ &\leq 2b + 2a(f_{ABC}^2 + f_{BAC}^2) \\ &\leq 2b + 4a \end{aligned}$$

This bound is attained at $\Sigma(1,1) = 2b + 2(a-2b) + 2(a+2b) = 4a + 2b$. Yet, it is not obtained at the Nash equilibrium for sufficiently large values of $N$:

**Theorem 6.** *$(1,1)$ is a Nash equilibrium if and only if $N \leq \dfrac{2b+a}{3a+b}$.*

**Proof.** We have $U_A(1,1) = b + 2a$ and

$$U_A(1-1/N, 1) = 2a + b - 3a/N - b/N + 2b/N^2 + a/N^2.$$

Therefore $U_A(1-1/N, 1) \geq U_A(1,1) \Leftrightarrow 2b + a \geq (3a+b)N$. The conclusion follows by convexity. ∎

Therefore, for $N \geq \max(\frac{a}{b}+2, \frac{2b+a}{3a+b})$ the Nash equilibria are neither optimal nor worse-case strategies of the game.

### B. Case of N=1

In case of $N = 1$ (one flow arrives at each source node and there are thus two players) the two approach coincides: the atomic non-splittable case (**F3**) is also a semi-splittable atomic game (**F4**). $f_{ABC}$ and $f_{BAC}$ take values in $\{\{0\},\{1\}\}$. From Eq. 6 and Eq. 7, the matrix game can be written

$$\begin{pmatrix} (b\,, b) & (2b\,, a+2b) \\ (a+2b\,, 2b) & (2a+b\,, 2a+b) \end{pmatrix}$$

and the potential of Eq. 4 becomes

$$\begin{pmatrix} 0 & a+b \\ a+b & 3a \end{pmatrix}.$$

Then, assuming that either $a$ or $b$ is non null, we get that $(0,0)$ is always a Nash equilibrium and that $(1,1)$ is a Nash equilibrium if and only if $3a \leq a+b$, i.e. $2a < b$.

We next consider any integer $N$ and identify another surprising feature of the equilibrium. We show that depending on the sign of $a-2b$, non-symmetric equilibria arise in our symmetric game. In all frameworks other than the semi-splittable games there are only symmetric equilibria in this game. We shall show however that in the limit (as $N$ grows to infinity), the limiting game has a single equilibrium.

### C. Case $a - 2b < 0$

In this case, there may be multiple equilibria, as shown in the following example.

**Example 1.** *Consider $a = 1$, $b = 3$ and $N = 4$, then the cost matrices are given below, with the two Nash equilibria of the game represented in bold letters:*

$$U_A = \frac{1}{16}\begin{pmatrix} 48 & 60 & 72 & 84 & 96 \\ 43 & \mathbf{50} & 57 & 64 & 71 \\ 52 & 54 & \mathbf{56} & 58 & 60 \\ 75 & 72 & 69 & 66 & 63 \\ 112 & 104 & 96 & 88 & 80 \end{pmatrix}, \textit{ and}$$

$$U_B = \frac{1}{16}\begin{pmatrix} 48 & 43 & 52 & 75 & 112 \\ 60 & \mathbf{50} & 54 & 72 & 104 \\ 72 & 57 & \mathbf{56} & 69 & 96 \\ 84 & 64 & 58 & 66 & 88 \\ 96 & 71 & 60 & 63 & 80 \end{pmatrix}.$$

Note that due to the shape of $U_A$ and $U_B$ the cost matrices of the game are transpose of each other. Therefore in the following, we shall only give matrix $U_A$.

We have the following theorem:

**Theorem 7.** *All Nash equilibria are symmetric, i.e.*

$$f^*_{ABC} = f^*_{BAC}.$$

The proof is given in Appendix A.

### *D. Case $a = 2b$ (with $a > 0$)*

When $a = 2b$, we shall show that some non-symmetric equilibria exists.

**Theorem 8.** *If $a = 2b$, there are exactly either 1 or 4 Nash equilibria. For any $N$, let $\overline{N} = \lfloor N/8 \rfloor$.*

- *If $N$ mod $8 = 4$, there are 4 equilibria $(n^*_{ABC}, n^*_{BAC})$, which are $(\overline{N}, \overline{N})$, $(\overline{N}+1, \overline{N})$, $(\overline{N}, \overline{N}+1)$ and $(\overline{N}+1, \overline{N}+1)$.*
- *Otherwise, there is a unique equilibrium, which is $(\overline{N}, \overline{N})$ if $N$ mod $8 < 4$ or $(\overline{N}+1, \overline{N}+1)$ if $N$ mod $8 > 4$.*

**Proof.** The Nash equilibria are the optimal points for both $u_A$ and $u_B$. They are therefore either interior or boundary points (i.e. either $f_{ABC}$ or $f_{BAC}$ are in $0, 1$). We detail the interior point cases in Appendix B. The rest of the proof derives directly from the definition of $\frac{\partial U_A}{\partial f_{ABC}}$ and $\frac{\partial U_B}{\partial f_{BAC}}$. Indeed:

$$\frac{\partial U_A}{\partial f_{ABC}} = (a-2b)f_{BAC} + 2(2b+a)f_{ABC} - b = 8bf_{ABC} - b$$

$$\frac{\partial U_B}{\partial f_{BAC}} = (a-2b)f_{ABC} + 2(a+2b)f_{BAC} - b = 8bf_{BAC} - b.$$

Both are minimum for $1/8$. Therefore, it is attained if $N$ is a multiple of 8. Otherwise, the best response of each player is either $\lfloor N/8 \rfloor / N$ if $N \mod 8 \leq 3$ or $\lceil N/8 \rceil / N$ if $N \mod 8 \geq 5$. If $N \mod 8 = 4$, then each player has 2 best responses which are $\frac{1}{N}\frac{N-4}{8}$ and $\frac{1}{N}\frac{N+4}{8}$. Then, one can check that the boundary points follow the law of Theorem 11 when $\overline{N} = \lfloor N/8 \rfloor = 0$. ∎

### *E. Case $a - 2b > 0$*

**Theorem 9.** *If $a - 2b > 0$, there are exactly either 1, 2 or 3 Nash equilibria.*
*Let $\alpha = \frac{a+2b}{3a+2b}$, $\beta = \frac{2a}{3a+2b}$ and $\gamma = \frac{b}{3a+2b}$.*

*Define further $\widetilde{N} = \lfloor N\gamma \rfloor$ and $z(N) = N\gamma - \widetilde{N}$. The equilibria are of the form*

- *Either $(\widetilde{N}, \widetilde{N})$, $(\widetilde{N}+1, \widetilde{N})$, $(\widetilde{N}, \widetilde{N}+1)$ if $N$ is such that $z(N) = \alpha$ (mode 3-A in Figure 6)*
- *Or $(\widetilde{N}+1, \widetilde{N}+1)$, $(\widetilde{N}+1, \widetilde{N})$, $(\widetilde{N}, \widetilde{N}+1)$ if $N$ is such that $z(N) = \beta$ (mode 3-B)*
- *Or $(\widetilde{N}, \widetilde{N}+1)$, $(\widetilde{N}+1, \widetilde{N})$ if $N$ is such that $\alpha < z(N) < \beta$ (mode 2)*
- *Or $(\widetilde{N}, \widetilde{N})$ if $N$ is such that $\beta < z(N) < \alpha + 1$ (mode 1).*

Fig. 6. Different modes according to different values of $N$.

We illustrate the different modes in the following example.

**Example 2.** *Suppose that $a = 10$ and $b = 3$ (we represent only the part of the matrices corresponding to $1/N \leq f_{ABC}, f_{BAC} \leq 4/N$).*

*If $N = 24$, there are 3 Nash equilibria:*

$$\begin{matrix} 1152 & 1200 & 1248 & 1296 \\ 1118 & \mathbf{1172} & \mathbf{1226} & 1280 \\ 1112 & \mathbf{1172} & 1232 & 1292 \\ 1134 & 1200 & 1266 & 1332 \end{matrix}$$

*If $N = 26$, there are 2 Nash equilibria:*

$$\begin{matrix} 1352 & 1404 & 1456 & 1508 \\ 1314 & 1372 & \mathbf{1430} & 1488 \\ 1304 & \mathbf{1368} & 1432 & 1496 \\ 1322 & 1392 & 1462 & 1532 \end{matrix}$$

*If $N = 27$, there are 3 Nash equilibria:*

$$\begin{matrix} 1458 & 1512 & 1566 & 1620 \\ 1418 & 1478 & \mathbf{1538} & 1598 \\ 1406 & \mathbf{1472} & \mathbf{1538} & 1604 \\ 1422 & 1494 & 1566 & 1638 \end{matrix}$$

*If $N = 28$, there is a single Nash equilibrium:*

$$\begin{matrix} 1568 & 1624 & 1680 & 1736 \\ 1526 & 1588 & 1650 & 1712 \\ 1512 & 1580 & \mathbf{1648} & 1716 \\ 1526 & 1600 & 1674 & 1748 \end{matrix}$$

### *F. Limit Case: Perfectly Splittable Sessions*

We focus here in the limit case where $N \to +\infty$.

**Theorem 10.** *There exists a unique Nash equilibrium and it is such that*

$$f^*_{BAC} = f^*_{ABC} = \frac{b}{3a+2b}.$$

**Proof.** Note that $\frac{\partial U_A}{\partial f_{ABC}}(1) > 0$ and $\frac{\partial U_B}{\partial f_{BAC}}(1) > 0$. If $f_{ABC} = 0$ then $f_{BAC} = \frac{b}{2a+4b}$ which implies that $-b + \frac{b(a-2b)}{2a+4b} \geq 0$, which further implies that $-a-6b > 0$ which is impossible. Hence $f_{ABC} > 0$. Similarly $f_{BAC} > 0$ which concludes the proof. ∎

Recall that the optimum sum (social optimum) is given by $(0,0)$ and that the worse case is given by $(1,1)$. Hence, regardless of the values of $a$ and $b$, at the limit case, we observe that there is a unique Nash equilibrium, that is symmetric, and is neither optimal (as opposed to **F3**), nor the worst case scenario. The price of anarchy is then:

$$PoA = PoS = \frac{2b + 2f^{*^2}_{ABC}a}{2b} = 1 + \frac{ab}{(3a+2b)^2}.$$

## V. Conclusions

We revisited in this paper a load balancing problem within a non-cooperative routing game framework. This model had already received much attention in the past within some classical frameworks (the Wardrop equilibrium analysis and the atomic splittable routing game framework). We studied this game under other frameworks - the non splittable atomic game (known as congestion game) as well as a the semi-splittable framework. We have identified many surprising features of equilibria in both frameworks. We showed that unlike the previously studied frameworks, there is no uniqueness of equilibrium, and non-symmetric equilibria may appear (depending on the parameters). For each of the frameworks we identified the different equilibria and provided some of their properties. We also provided an efficiency analysis in terms of price of anarchy and price of stability. In the future we plan to investigate more general cost structures and topologies.

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## Appendix

### A. Proof of Theorem 7.

Suppose that $(f^*_{ABC}, f^*_{BAC})$ is a Nash equilibrium with $f^*_{ABC} \neq f^*_{BAC}$. Then, by definition:

$$U_A(f^*_{ABC}, f^*_{BAC}) \leq U_A(f^*_{BAC}, f^*_{BAC}) \text{ and}$$
$$U_B(f^*_{ABC}, f^*_{BAC}) \leq U_B(f^*_{ABC}, f^*_{ABC}),$$

which gives, after some manipulations,

$$\begin{cases} (a-2b)f^*_{ABC}f^*_{BAC} \leq \\ \qquad 2af^{*2}_{BAC} + bf^*_{ABC} - bf_{BAC} - (a+2b)f^{*2}_{ABC} \\ (a-2b)f^*_{ABC}f^*_{BAC} \leq \\ \qquad 2af^{*2}_{ABC} + bf^*_{BAC} - bf^*_{ABC} - (a+2b)f^{*2}_{BAC}. \end{cases}$$

Therefore $2(a-2b)f^*_{ABC}f^*_{BAC} \leq (a-2b)(f^{*2}_{ABC} + f^{*2}_{BAC})$ and hence $0 \leq (a-2b)(f^*_{ABC} - f^*_{BAC})^2$ which is impossible.

### B. Boundary equilibria when a=2b.

**Theorem 11.** *If $a = 2b$, there exists a single Nash equilibrium of the form $(0, f^*_{BAC})$ and $(f^*_{BAC}, 0)$ with $f^*_{BAC}$ non null. It is obtained for $N = 4$ and $f^*_{BAC} = 1/4$. The points $(0,0)$ are Nash equilibria if and only if $N \leq 4$. Further, there are no equilibrium of the form $(f_{ABC}, 1)$ or $(1, f_{BAC})$.*

**Proof.** We first study the equilibria of the form $(0, f_{ABC})$. $(0, \gamma)$ is a Nash equilibrium iff

$$\begin{cases} U_A(0,\gamma) \leq U_A\left(\frac{1}{N}, \gamma\right) \\ U_B(0,\gamma) \leq U_B\left(0, \gamma + \frac{1}{N}\right) \\ U_B(0,\gamma) \leq U_B\left(0, \gamma - \frac{1}{N}\right) \end{cases} \Leftrightarrow \begin{cases} b \leq \frac{2b+a}{N} \\ b \leq (a+2b)(2\gamma + \frac{1}{N}) \\ b \geq (a+2b)(2\gamma - \frac{1}{N}) \end{cases}$$

$$\Leftrightarrow \begin{cases} 1 \leq \frac{4}{N} \\ 1 \leq 4(2\gamma + \frac{1}{N}) \\ 1 \geq 4(2\gamma - 1/N) \end{cases} \Leftrightarrow \begin{cases} N \leq 4 \\ \frac{N/8 - 1/2}{N} \leq \gamma \\ \qquad \leq \frac{N/8 + 1/2}{N} \end{cases}$$

If $N \leq 3$ then $N/8 + 1/2 \leq 7/8 < 1$ which cannot be obtained by the player otherwise than in 0. For $N = 4$, the second inequality becomes $0 \leq \gamma \leq \frac{1}{4}$ which hence leads to the only non null Nash equilibrium.

We next study the potential equilibria of the form $(f_{ABC}, 1)$. Let $(\gamma, 1)$ be a Nash equilibrium. Then $U_B(\gamma, 1) \leq U_B(\gamma, 1 - 1/N)$. Then
$b\gamma + a + 2b \leq b - b(1 - 1/N) + b\gamma + (a+2b)(1-1/N)^2$
$\Rightarrow a + 2b \leq b/N + (a+2b)(1 + 1/N^2 - 2/N)$
$\Rightarrow 0 \leq b + (a+2b)(1/N - 2)$
$\Rightarrow 2a + 3b \leq (a+2b)/N \Rightarrow N \leq 1/4.$
∎

*C. Boundary equilibria when $a-2b>0$.*

**Theorem 12.** *$(0,\alpha)$ and $(\alpha,0)$ are Nash equilibria iff:*

$$\frac{b}{a-2b}-\frac{1}{N}\frac{a+2b}{a-2b}\leq\alpha\leq\frac{b}{2(a+2b)}+\frac{1}{2N}.$$

*Further, there are no Nash equilibrium of the form $(A,1)$.*

**Proof.** We first focus on the Nash equilibria of the form $(0,A)$. Since $U_A(.,f_{BAC})$ and $U_B(f_{ABC},.)$ are convex, $(0,\gamma)$ is a Nash equilibrium iff

$$\begin{cases} U_A(0,\gamma)\leq U_A\left(\frac{1}{N},\gamma\right) \\ U_B(0,\gamma)\leq U_B\left(0,\gamma+\frac{1}{N}\right) \\ U_B(0,\gamma)\leq U_B\left(0,\gamma-\frac{1}{N}\right) \end{cases}$$

$$\Leftrightarrow\begin{cases} b\leq(a-2b)\gamma+\frac{2b+a}{N} \\ b\leq(a+2b)(2\gamma+\frac{1}{N}) \\ b\geq(a+2b)(2\gamma-\frac{1}{N}) \end{cases}\Leftrightarrow\begin{cases} \gamma\geq\frac{bN-2b-a}{N(a-2b)} \\ \gamma\geq\frac{bN-a-2b}{2N(a+2b)} \\ \gamma\leq\frac{bN+a+2b}{2N(a+2b)} \end{cases}$$

But $\frac{bN-2b-a}{N(a-2b)}\geq\frac{bN-a-2b}{2N(a+2b)}$ which concludes the proof. and hence $\frac{bN-a-2b}{2N(a+2b)}\leq\gamma\leq\frac{bN+a+2b}{2N(a+2b)}$

We now study the potential equilibria of the form $(A,1)$. Let $(A,1)$ be a Nash equilibrium. Then $U_B(A,1)\leq U_B(A,1-1/N)$. Then

$$\begin{aligned} -b+(a-2b)A+(a+2b)&\leq-b(1-1/N) \\ &+(a-2b)A(1-1/N)+(a+2b)(1-1/N)^2 \end{aligned}$$

$$\Rightarrow 0\leq b-(a-2b)A+(a+2b)(-2+1/N)$$

$$\Rightarrow(a-2b)A\leq-2a-3b+(a+2b)/N\Rightarrow$$

$$\Rightarrow 2a+3b\leq(a-2b)A+2a+3b\leq(a+2b)/N$$

But $2a+3b\leq(a+2b)/N\Rightarrow N\leq\frac{a+2b}{2a+3b}<1$. ∎

*D. Proof of Theorem 9.*

We first start by showing that there are at most 4 interior Nash equilibria and that they are of the form: $(A,A)$,$(A+1,A)$,$(A,A+1)$,$(A+1,A+1)$.

**Proof.** Let $f_{ABC},f_{BAC}$ be a Nash equilibrium in the interior (i.e. $0<f_{ABC}<1$ and $0<f_{BAC}<1$). Then $f_{ABC}$ and $f_{BAC}$ are the (discrete) minimizers of $x\mapsto U_A(x,f_{BAC})$ and $x\mapsto U_B(f_{ABC},x)$ respectively. Further:

$$\begin{cases} \frac{\partial U_A}{\partial f_{ABC}}=-b+(a-2b)f_{BAC}+2(2b+a)f_{ABC} \\ \frac{\partial U_B}{\partial f_{BAC}}=-b+(a-2b)f_{ABC}+2(a+2b)f_{BAC} \end{cases}$$

The optimum values are therefore respectively:

$$x_A=\frac{b-\theta f_{BAC}}{\lambda}\text{ and }x_B=\frac{b-\theta f_{ABC}}{\lambda}$$

with $\lambda=2(2b+a)$ and $\theta=a-2b$. Therefore:

$$\begin{cases} x_A-1/(2N)\leq f_{ABC}\leq x_A+1/(2N), \\ x_B-1/(2N)\leq f_{BAC}\leq x_B+1/(2N). \end{cases}$$

Hence

$$\begin{aligned} \frac{b}{\lambda}-\frac{\theta}{\lambda}\left(\frac{b}{\lambda}-\frac{\theta}{\lambda}f_{ABC}+\frac{1}{2N}\right)-\frac{1}{2N}\leq f_{ABC}\leq\frac{1}{2N} \\ +\frac{b}{\lambda}-\frac{\theta}{\lambda}\left(\frac{b}{\lambda}-\frac{\theta}{\lambda}f_{ABC}-\frac{1}{2N}\right) \end{aligned}$$

Then

$$\frac{b}{\lambda+\theta}-\frac{\lambda}{2N(\lambda-\theta)}\leq f_{ABC}\leq\frac{\lambda}{2N(\lambda-\theta)}+\frac{b}{\lambda+\theta}$$

Then $\frac{b}{\lambda+\theta}=\frac{b}{2b+3a}$, $\frac{\lambda}{2N(\lambda-\theta)}=\frac{4b+2a}{2N(6b+a)}$ and $\frac{\lambda}{2N(\lambda-\theta)}=\frac{2(a+2b)}{2N(6b+a)}$, which gives

$$\frac{b}{2b+3a}-\frac{a+2b}{N(6b+a)}\leq f_{ABC}\leq\frac{2b+a}{N(6b+a)}+\frac{b}{2b+3a}.$$

Similarly, we have

$$\frac{b}{2b+3a}-\frac{(2b+a)}{N(6b+a)}\leq f_{BAC}\leq\frac{b}{2b+3a}+\frac{2b+a}{N(6b+a)}.$$

Note that $\frac{1}{2}<\frac{2b+a}{6b+a}<1$. Therefore there are either 1 or 2 possible values, which are identical for $f_{ABC}$ and $f_{BAC}$. There are therefore 4 possible equilibria. ∎

Now, the potential equilibria are of the form $(A,A)$, $(A,A+1)$, $(A+1,A)$ and $(A+1,A+1)$. By symmetry, note that if $(A,A+1)$ is a Nash equilibrium, then $(A+1,A)$ also is. The following lemma reduces the number of combinations of equilibria:

**Lemma 1.** *If $(A,A)$ is a Nash equilibrium then $(A+1,A+1)$ is not a Nash equilibrium.*

**Proof.** Suppose that $(A,A)$ and $(A+1,A+1)$ are two Nash equilibria. Then $U_A(A,A)\leq U_A(A+1,A)$ and $U_A(A+1,A+1)\leq U_A(A,A+1)$, which implies

$$\begin{cases} -bAN+(a-2b)A^2+(2b+a)A^2\leq \\ \quad -b(A+1)N+(a-2b)A(A+1)+(2b+a)(A+1)^2 \\ -b(A+1)N+(a-2b)(A+1)^2+(2b+a)(A+1)^2\leq \\ \quad -bAN+(a-2b)A(A+1)+(2b+a)A^2 \end{cases}$$

$$\Rightarrow\begin{cases} bN\leq(a-2b)A+(2b+a)(2A+1) \\ (a-2b)(A+1)+(2b+a)(2A+1)\leq bN \end{cases}$$

$$\Rightarrow(a-2b)(A+1)\leq bN-(2b+a)(2A+1)\leq(a-2b)A$$

Hence $(a-2b)(A+1)\leq(a-2b)A$ and therefore $a-2b\leq 0$ which is impossible. ∎

Therefore the different possible combinations are mode 1, mode 2, mode 3-A and mode 3-B in Figure 6).

We first start by the occurrence of mode 3-A:

**Lemma 2.** *Suppose that* $a-2b>0$. *Suppose that* $(A,A)$ *and* $(A+1,A)$ *are two Nash equilibria. Then*

$$A=\frac{bN-2b-a}{3a+2b}.$$

**Proof.** Suppose that $(A,A)$ and $(A+1,A)$ are two Nash equilibria. Then necessarily $U_A(A,A)=U_A(A+1,A)$. Hence

$$\begin{aligned}&-bAN+(a-2b)A^2+(2b+a)A^2\\&=-b(A+1)N+(a-2b)A(A+1)+(2b+a)(A+1)^2\end{aligned}$$

i.e.

$$bN=(a-2b)A+(2b+a)(2A+1)\Rightarrow bN-2b-a=(3a+2b)A$$

which leads to the conclusion. ∎

Hence, the system is in mode 3-A iff $bN-2b-a$ is divisible by $3a+2b$ or in other words, if $N$ is of the form $[(3a+2b)K+2a]/b$ for some integer $K$.

We then move on to Mode 3-B:

**Lemma 3.** *Suppose that* $a-2b>0$. *Suppose that* $(A+1,A+1)$ *and* $(A+1,A)$ *are two Nash equilibria. Then*

$$A=\frac{bN-2a}{3a+2b}.$$

**Proof.** Suppose that $(A+1,A+1)$ and $(A,A+1)$ are two Nash equilibria, then $U_1(A+1,A+1)=U_1(A,A+1)$. This implies

$$\begin{aligned}-Nb(A+1)+(a-2b)(A+1)^2+(2b+a)(A+1)^2=\\-NbA+(a-2b)A(A+1)+(2b+a)A^2\end{aligned}$$

$$\Rightarrow(a-2b)(A+1)+(2b+a)(2A+1)=Nb$$

$$\Rightarrow(3a+2b)A=Nb-2a$$

which concludes the proof. ∎

Hence, the system is in mode 3-B iff $bN-2a$ is divisible by $3a+2b$ or in other words, if $N$ is of the form $[(3a+2b)K+2b+a]/b$ for some integer $K$.

Finally, for Mode 2:

**Lemma 4.** *Suppose that* $a-2b>0$. *Suppose that* $(A,A+1)$ *and* $(A+1,A)$ *are only two Nash equilibria. Then*

$$(3a+2b)A+2b+a<bN<(3a+2b)A+2a.$$

**Proof.** Suppose that $(A,A+1)$ and $(A+1,A)$ are two Nash equilibria, then:

$$\begin{aligned}&U_A(A,A+1)\le U_A(A+1,A+1)\text{ and}\\&U_A(A+1,A)\le U_A(A,A)\end{aligned}$$

ie

$$\begin{cases}bN\le(3a+2b)A+2a\\(3a+2b)A+2b+a\le bN\end{cases}$$

The conclusion comes from Lemma 2 and 3, since neither $(A,A)$ nor $(A+1,A+1)$ are Nash equilibria. ∎

Finally the system is in mode 1 if it is not in any over modes. One can then check that the boundary cases found in Theorem 12 corresponds to the case where $A=0$ which concludes the proof.