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Rui Chen [*] Roxana Dumitrescu [†] Andreea Minca [‡] Agnès Sulem [§]

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Abstract

We study Mean-field BSDEs with jumps and a generalized mean-field operator that can capture higher order interactions such as those occurring on an inhomogeneous random graph. We provide comparison and strict comparison results. Based on these, we interpret the BSDE solution as a global dynamic risk measure that can account for the intensity of system interactions and therefore incorporate systemic risk. Using Fenchel-Legendre transforms, we establish a dual representation for the risk measure, and in particular we exhibit its dependence on the mean-field operator.

1 Introduction

We consider a Mean-field BSDE $(X, Z, \ell) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu$ satisfying the generalized dynamics

$$\begin{cases} -dX_t = f(t, \omega, F(t, X_t(\cdot)), X_t, Z_t, \ell_t(\cdot))dt - Z_t dW_t - \int_{\mathbf{E}} \ell_t(e)\tilde{N}(dt, de); \\ X_T = \xi. \end{cases} \tag{1.1}$$

where F is an $\mathcal{B}([0,T]) \times \mathcal{B}(L^2)$ measurable operator from $[0,T] \times L^2$ to $\mathbf{R}$. This dynamics is referred to Mean-field BSDE of first type.

An alternative specification for Mean-Field BSDEs, currently referred to as Mean-field BSDE of second type, was introduced in [4] [5] as a process $(X, Z, \ell) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu$ satisfying

$$\begin{cases} -dX_t = \mathbb{E}' \left[f(t, \omega, X'_t, Z'_t, \ell'_t(\cdot), X_t, Z_t, \ell_t(\cdot)) \right] dt - Z_t dW_t - \int_{\mathbf{E}} \ell_t(e)\tilde{N}(dt, de); \\ X_T = \xi, \end{cases} \tag{1.2}$$

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where

$$\mathbb{E}'f(t,\omega,X'_t,Z'_t,\ell'_t(\cdot),X_t,Z_t,\ell_t(\cdot)) = \int_\Omega f(t,\omega,X_t(\omega'),Z_t(\omega'),\ell_t(\cdot)(\omega'),X_t(\omega),Z_t(\omega),\ell_t(\cdot)(\omega))d\mathbb{P}(\omega').$$

The case with jumps for the Mean-field BSDE of second type is studied in [16]. The solution of the Mean-field BSDE of second type (1.2) is related to the solution (X^N, Z^N, ℓ^N) of the backward equation

$$\begin{cases} -dX_t^N = \frac{1}{N}\sum_{j=1}^N \left[f(t,\omega,X_t^{j,N},Z_t^{j,N},\ell_t^{j,N}(\cdot),X_t^N,Z_t^N,\ell_t^N(\cdot))\right]dt - Z_t^N dW_t - \int_{\mathbf{E}} \ell_t^N(e)\tilde{N}(dt,de); \\ X_T^N = \xi, \end{cases} \tag{1.3}$$

where the i.i.d. sequence $(X^{j,N}, Z^{j,N}, \ell^{j,N})$, $1 \le j \le N$, are following the same law as (X^N, Z^N, ℓ^N). The convergence is shown in [4] for the Brownian motion case. Namely, they show (X, Z, ℓ) to be the uniform limit of the solution (X^N, Z^N, ℓ^N) when $N \to \infty$.

In contrast, the first type of Mean-Field BSDE was introduced only recently by [1] with the specific form for the operator F as expectation: $F(t,X) = \mathbb{E}[\varphi(t,X)]$. In this simple case the first type Mean-Field BSDE can be seen as a limit of the coupled system

$$\begin{cases} -dX_t^{i,N} = f(t,\omega,\frac{1}{N}\sum_{j=1}^N \varphi(X_t^{j,N}),X_t^{i,N},Z_t^{i,N},\ell_t^{i,N}(\cdot))dt - Z_t^{i,N}dW_t - \int_{\mathbf{E}} \ell_t^{i,N}(e)\tilde{N}(dt,de); \\ X_T^{i,N} = \xi. \end{cases}$$

However, the operator F in the first type of mean field BSDEs (1.1) does not need to be limited to the linear or interactions of order one and can incorporate interactions of higher order.

For example, order two interactions can be captured by the following operator

$$F(t,X) = \mathbb{E}\left[\kappa(X,X')\right],$$

where κ is a kernel that captures the intensity of interactions, and X' is an independent variable with the same distribution as X, and this distribution can be parametrized by time t. In section 2.2, we provide an example where this operator represents the average intensity of interactions of processes living on the nodes of the inhomogeneous random graph introduced in [3]. For a finite graph of size N the inhomogeneous random graph model posits that interactions between node i and j occur according to a Poisson process of intensity $\kappa(X_t^{i,N}, X_t^{j,N})/N$, depending on the nodes' states $X_t^{i,N}$ and $X_t^{j,N}$. The average intensity of interactions in the system is given by

$$\frac{1}{N^2}\sum_{i,j}\kappa(X_t^{i,N}, X_t^{j,N}). \tag{1.4}$$

The convergence of the finite systems is beyond the scope of this paper. Here we directly introduce the first-type Mean-Field BSDE with generalized operator F as a model of dynamics for a typical node in a large system. The mean field operator is general enough to accommodate several models of interactions. The case of an inhomogeneous random graph serves as an example of interactions of interest in systemic risk. Existing mean field models of systemic risk, e.g., [6, 12, 13, 10], rely on a drift depending on the average state of the system. We suggest here that the states of the system define an intensity of interaction (or the existence of a link) among nodes. In addition to the average state of the system, the average intensity of interactions is critical in the driver.

Contributions. We provide results and properties for the (First-type) Mean-field BSDE with general Mean-field operators and the corresponding reflected Mean-field BSDE.

Under Lipschitz conditions, we show the existence/uniqueness and (strict) comparison results. While in [1], the authors considered Mean-field BSDEs of first type, they rely on the special case $F(t, X) = \mathbb{E}[\varphi(t, X)]$. This specification models first order interactions in a hypothetical large scale system. In contrast, here we allow for more general Mean-field operators, includes those modeling higher order interactions. We introduce *global dynamic risk measures* induced by the mean-field BSDE. When the mean field operator captures the average intensity of network interactions, the interpretation is that of a dynamic systemic risk measure.

A regulator can choose a level $-\eta$ for the acceptable capital or liquidity at a horizon T. Before time T and in order to become acceptable at the time horizon, the capital (liquidity) of the individual bank may have inflows/outflows from via the mean field term: the mean field operator can capture the intensity of such inflows or outflows. To be more precise : Let $T > 0$ be a time horizon and f be a Lipschitz driver. Then the risk measure is defined as

$$\rho_t(\eta, T) := -X_t(\eta), \quad 0 \leq t \leq T, \tag{1.5}$$

where $X_t(\eta)$ denotes the solution of mean-field the BSDE (2.11) with driver f, mean-field operator F and terminal condition $\eta \in L^2(\mathcal{F}_T)$. We provide properties for the global dynamic risk measures such as monotonicity, consistency, cash sub-additivity, convexity under appropriate hypotheses. The cash-subadditivity is an interesting property in the mean field case: under appropriate conditions, the mean field acts as a stabilizer and the liquidity needs are addressed to some extent at the system level. Our main result is a technical dual representation for the systemic risk measure. This is the expectation under a worst-case discount factor and the worst case probability measure of the final acceptable capital level $-\eta$ plus a penalty function. The dependence on the mean field operator (or its Fenchel-Legendre transform) in the worst-case elements and penalty is explicit, and distinguishes our results on past literature on classical dynamic risk measures. In this sense, our paper bridges the literature on dynamic risk measures and the literature on systemic risk measures. In the latter, [2, 11] for example, consider the case where capital is injected into the individual banks so that the loss aggregated at the level of the system becomes acceptable. In these works, the aggregation function allows for a wide variety of specifications, and in particular for a network (random or not) of interactions. These are powerful static approaches, which allow for different capital injections to the different banks. Complementary, we provide a dynamic approach that is applicable to a "representative" bank, and it is this bank's dynamics that is impacted by the system according to the mean field operator. Finally, we study the connection of Mean-field reflected BSDE to the optimal stopping problem for the dynamic global risk measure.

2 Mean-field BSDEs

2.1 Notation and definitions

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let W be a one-dimensional Brownian motion. Let $\mathbf{E} := \mathbf{R}^* := \mathbf{R} \setminus \{0\}$ and $\mathcal{B}(\mathbf{E})$ be its Borelian filtration. Suppose that it is equipped with a σ-finite positive measure ν and let $N(dt, de)$ be a Poisson random measure with compensator $\nu(de)dt$. Let $\tilde{N}(dt, de)$ be its compensated process. Let $I\!F = \{\mathcal{F}_t, t \geq 0\}$ be the completed natural filtration associated with W and N.

Notation. Let $\mathcal{P}$ be the predictable σ-algebra on $[0, T] \times \Omega$.

For each $T > 0$, we use the following notation: L^2 is the set of random variables ξ which are $\mathcal{F}_T$-measurable and square integrable; $I\!H^2$ is the set of real-valued predictable processes ϕ such that $\|\phi\|^2_{I\!H^2} := E\left[\int_0^T \phi_t^2 dt\right] < \infty$; $\mathcal{S}^2$ denotes the set of real-valued RCLL adapted processes ϕ such that $\|\phi\|^2_{\mathcal{S}^2} := E(\sup_{0 \leq t \leq T} |\phi_t|^2) < \infty$; $\mathcal{A}^2$ (resp. $\mathcal{A}^1$) is the set of real-valued non decreasing RCLL predictable processes A with $A_0 = 0$ and $E(A_T^2) < \infty$ (resp. $E(A_T) < \infty$). We also introduce the following spaces:

- L^2_ν is the set of Borelian functions $\ell : \mathbf{E} \to \mathbf{R}$ such that $\int_{\mathbf{E}} |\ell(e)|^2 \nu(de) < +\infty$.
  The set L^2_ν is a Hilbert space equipped with the scalar product
  $\langle \ell, \ell' \rangle_\nu := \int_{\mathbf{E}} l(e)\ell'(e)\nu(de)$ for all $\ell, \ell' \in L^2_\nu \times L^2_\nu$, and the norm $\|\ell\|^2_\nu := \int_{\mathbf{E}} |\ell(e)|^2 \nu(de)$.
- $I\!H^2_\nu$ is the set of all mappings $\ell : [0, T] \times \Omega \times \mathbf{E} \to \mathbf{R}$ that are $\mathcal{P} \otimes \mathcal{B}(\mathbf{E})/\mathcal{B}(\mathbf{R})$ measurable and satisfy $\|\ell\|^2_{I\!H^2_\nu} := E\left[\int_0^T \|\ell_t\|^2_\nu \, dt\right] < \infty$, where $\ell_t(\omega, e) = \ell(t, \omega, e)$ for all $(t, \omega, e) \in [0, T] \times \Omega \times \mathbf{E}$.

Moreover, $\mathcal{T}_0$ is the set of stopping times τ such that $\tau \in [0, T]$ a.s. and for each S in $\mathcal{T}_0$, we denote by $\mathcal{T}_S$ the set of stopping times τ such that $S \leq \tau \leq T$ a.s.

Definition 2.1 *([18] Definition 2.1) A progressive process (ϕ_t) is said to be left-upper semi-continuous (l.u.s.c.) along stopping times if for all $\tau \in \mathcal{T}_0$ and for each non decreasing sequence of stopping times (τ_n) such that $\tau^n \uparrow \tau$ a.s.,*

$$\phi_\tau \geq \limsup_{n \to \infty} \phi_{\tau_n} \quad \text{a.s.} \tag{2.6}$$

Definition 2.2 (Driver, Lipschitz driver) *A function f is said to be a* driver *if*

- $f : \Omega \times [0,T] \times \mathbf{R} \times \mathbf{R}^2 \times L^2_\nu \to \mathbf{R}$
  $(\omega, t, y', y, z, \ell(\cdot)) \mapsto f(\omega, t, y', y, z, \ell(\cdot))$ *is* $\mathcal{P} \otimes \mathcal{B}(\mathbf{R}^2) \otimes \mathcal{B}(L^2_\nu)-$ *measurable,*

- $f(.,0,0,0,0,0) \in I\!H^2$.

A driver f *is called a* Lipschitz driver *if moreover there exists a constant* $C \geq 0$ *such that* $dP \otimes dt$*-a.s. , for each* (y'_1, y_1, z_1, ℓ_1), (y'_2, y_2, z_2, ℓ_2),

$$|f(\omega, t, y'_1, y_1, z_1, \ell_1) - f(\omega, t, y'_2, y_2, z_2, \ell_2)| \tag{2.7}$$

$$\leq C(|y'_1 - y'_2| + |y_1 - y_2| + |z_1 - z_2| + \|\ell_1 - \ell_2\|_\nu). \tag{2.8}$$

We now introduce a mean-field operator F in the general form. As we will see below, the canonic example is the expectation or the expected intensity of state dependent interactions.

Definition 2.3 *An operator* F *on* L^2 *is said to be a mean-field operator if*

- $F : [0,T] \times L^2 \to \mathbf{R}$
  $(t, X) \mapsto F(t, X)$ *is* $\mathcal{B}([0,T]) \times \mathcal{B}(L^2)-$ *measurable,*

- *For each* $t \in [0,T]$, $F(t, 0) < +\infty$.

A Mean-field operator F *is called a* Lipschitz Mean-field operator *if there exists a constant* $C \geq 0$, *such that for each* $(X_1, X_2) \in L^2 \times L^2$,

$$|F(t, X_1) - F(t, X_2)| \leq C\|X_1 - X_2\|_{L^2} \tag{2.9}$$

Remark 2.4 *An example is* $F(t, X) := \mathbb{E}[\phi(t, X)]$ *for* $X \in L^2$ *, where*

$$\phi : [0,T] \times \mathbf{R} \mapsto \mathbf{R}, (t, x) \mapsto \phi(t, x)$$

is a Lipschitz function such that $\phi(t, X) \in L^2$.

2.2 Mean field operator induced by inhomogeneous random graphs

We now consider the baseline example motivating the mean field operator modeling higher order interactions. This is based on the inhomogeneous graph model of [3].

Inhomogeneous random graph. We consider a sequence of N points $(X^{i,N})_{i=1,N}$ in a separable metric space that we here assume $\mathbf{R}$ equipped with a Borel probability measure μ. We assume that the empirical measure $\frac{1}{N}\sum_{i=1}^N \delta_{X^{i,N}}$ converges in probability to a measure μ as $N \to \infty$, with $\mu(\mathbf{R}) = 1$.[1] We define a kernel κ as a measurable function on $\mathbf{R} \times \mathbf{R}$. Given the sequence $(X^{i,N})_{i=1,N}$ we define $\mathcal{G}_N$ as the random graph where an edge between nodes i and j exists with probability

$$p_{ij} := \min(1, \kappa(X^{i,N}, X^{j,N})/N) \tag{2.10}$$

[1]This means that for any μ-continuity set A we have that $\#\{i,\ X^{i,N} \in A\}/N \xrightarrow{p} \mu(A)$.

independently of everything else. Then the expected number of edges in the graph is given by

$$e(\mathcal{G}_N) := \mathbb{E} \sum_{i,j \in 1,N} \min(1, \kappa(X^{i,N}, X^{j,N})/N)$$

and the average connectivity is

$$\bar{\lambda}_N := \frac{1}{N} \mathbb{E} \sum_{i,j \in 1,N} \min(1, \kappa(X^{i,N}, X^{j,N})/N).$$

Following [3], we say that a kernel κ is *graphical* if the following conditions hold:

(i) κ is continuous on $\mathbf{R} \times \mathbf{R}$;

(ii) $\kappa \in \mathcal{L}^1(\mathbf{R} \times \mathbf{R})$;

(iii) $\bar{\lambda}_N \longrightarrow \int_{\mathbf{R}\times\mathbf{R}} \kappa(s, s')\mu(ds)\mu(ds')$, as $N \to \infty$.

The two first conditions are natural technical conditions. The natural interpretation of κ is that it measures the density of edges, so the integral should be the average number of edges. Thus the third condition states that the graph has about the right number of edges.

Mean field operator. We now introduce a dynamic version of the inhomogeneous random graph model. We now let $(X_t^{i,N})_{i=1,N,t\in[0,T]} \in \mathcal{S}^2$, and as before we assume convergence of the unidimensional empirical distributions: for all $t \in [0, T]$, $\frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}$ converges in probability to a measure μ_t as $N \to \infty$. Over time, nodes i and j interact according to a Poisson process of intensity $\kappa(X_t^{i,N}, X_t^{j,N})/N$ (see [3][remark 2.4]). The last condition for a graphical kernel becomes

(iii') $\frac{1}{N}\mathbb{E}\sum_{i,j\in 1,N} \kappa(X_t^{i,N}, X_t^{j,N})/N \longrightarrow \int_{\mathbf{R}\times\mathbf{R}} \kappa(s, s')\mu_t(ds)\mu_t(ds')$, as $N \to \infty$, for all $t \in T$,

which can be interpreted as the average intensity of interactions converges to a limit as the size of the graph tends to infinity.

We can now interpret the second order mean field operator in the context of an inhomogeneous random graph as the average intensity of interactions in the network. Unlike classical mean-field models in which nodes' states depend on the average state of all other nodes, here what matters is the average intensity of interactions. If nodes do not interact with each other, there is little structural reason why the nodes' states depend on others. This suggests the following operator

$$F(t, X) = \int_{\mathbf{R}\times\mathbf{R}} \kappa(s, s')\mu_{t,X}(ds)\mu_{t,X}(ds'),$$

where $\mu_{t,X}$ is a distribution of X and is parametrized by t. We check that this operator satisfies Definition 2.3 and it is Lipschitz if the kernel κ is Lipschitz: there exists a constant $C > 0$ such that for each x, y, x', y', $|\kappa(x, y) - \kappa(x', y')| \leq C(|x - x'| + |y - y'|)$.

We now introduce the BSDEs with jumps, whose driver depends on the mean-field operator F defined above.

Definition 2.5 (Mean-field BSDEs) *A solution of a Mean-field BSDE with jumps with terminal time T, terminal condition ξ and driver f and operator F consists of a triple of processes $(X, Z, l) \in \mathcal{S}^2 \times I\!H^2 \times I\!H^2_\nu$ satisfying*

$$\begin{aligned} &-dX_t = f(t, \omega, F(t, X_t(\cdot)), X_t, Z_t, \ell_t(\cdot))dt - Z_t dW_t - \int_{\mathbf{E}} \ell_t(e)\tilde{N}(dt, de); \qquad (2.11)\\ &X_T = \xi. \end{aligned}$$

where X is a RCLL optional process, and Z (resp. k) is an $\mathbf{R}$-valued predictable process defined on $\Omega \times [0,T]$ (resp. $\Omega \times [0,T] \times \mathbf{R}^*$) such that the stochastic integral with respect to W (resp. $\tilde{N}$) is well defined. We denote by $(X(\xi,T), Z(\xi,T), l(\xi,T))$ the solution of the Mean-field BSDE associated with terminal time T and (ξ, f).

We now give existence and uniqueness results.

Theorem 2.6 *(Existence and Uniqueness Mean-field BSDE)*
Let f be a Lipschitz driver and F a Lipschitz Mean-field operator. The Mean-field BSDE (2.11) admits a unique solution $(X, Z, \ell(.)) \in \mathcal{S}^2 \times I\!H^2 \times I\!H^2_\nu$.

We note that an existence and uniqueness result statement was proven in [1] only for the case of a linear mean field operator, and in particular it does not allow for second order interactions. By using a priori estimates based on the Lipschitz property of both f and F, we establish the contraction property and the convergence of the Picard iterative sequence.
Proof. Let (X^n_s, Z^n_s, ℓ^n_s) be the solution of the following iterating BSDE with jumps

$$X^n_t = \xi + \int_t^T f(s, F(s, X^{n-1}_s(\cdot)), X^n_s, Z^n_s, \ell^n_s)ds - \int_t^T Z^n_s dB_s - \int_t^T \int_{\mathbf{E}} \ell^n_t(e)\tilde{N}(dt, de), \quad (2.12)$$

for $n \geq 1$ and $t \in [0,1]$ and where for $n = 0$ we set $(X^0_s, Z^0_s, \ell^0_s) = (0,0,0)$.
The existence and the uniqueness in each iteration is established by the classical results, see [20], and we denote by Φ the resulting map $(X^n, Z^n, \ell^n) = \Phi(X^{n-1}, Z^{n-1}, \ell^{n-1})$.[2]

We now show that Φ is a contraction on $I\!H^2 \times I\!H^2 \times I\!H^2_\nu$ (which implies that $(X^n, Z^n, \ell^n)_{n\geq 0}$ is a Cauchy sequence). Let $\bar{X}^n_t = X^n_t - X^{n-1}_t$; $\bar{Z}^n_t = Z^n_t - Z^{n-1}_t$; $\bar{\ell}^n_t = \ell^n_t - \ell^{n-1}_t$. For $\beta > 0$ we consider the norm $\|\cdot\|_\beta$, defined as $\|\phi\|_\beta := \mathbb{E}\left[\int_0^T e^{\beta s}\phi_s^2 ds\right]$. By applying Ito's formula to $e^{\beta s}|X^n_s - X^{n-1}_s|^2$, $n \geq 1$, we have analogously to the Proposition A.4 [17]

$$\begin{aligned} &e^{\beta t}(\bar{X}^n_t)^2 + \beta\int_t^T e^{\beta s}(\bar{X}^n_s)^2 ds + \int_t^T e^{\beta s}(\bar{Z}^n_s)^2 ds + \int_t^T e^{\beta s}\|\bar{\ell}^n_s\|^2_\nu ds \\ &= 2\int_t^T e^{\beta s}\bar{X}^n_s[f(s, F(s, X^{n-1}_s(\cdot)), X^n_s, Z^n_s, l^n_s) - f(s, F(s, X^{n-2}_s(\cdot)), X^{n-1}_s, Z^{n-1}_s, l^{n-1}_s)]ds \\ &\quad - 2\int_t^T e^{\beta s}\bar{X}^n_s\bar{Z}^n_s dW_s - \int_t^T e^{\beta s}\int_{\mathbb{R}^*}(2\bar{X}^n_{s^-}\bar{\ell}^n_s(u) + \bar{\ell}^n_s(u)^2)\tilde{N}(ds, de). \end{aligned}$$

[2] Note that in our current setting the map Φ does not depend on its last two arguments, but it would if we had a mean field term of the form $F(s, X^{n-1}_s(\cdot), Z^{n-1}(\cdot), \ell^{n-1}(\cdot))$. Existence and uniqueness results would go through in this case.

Taking the conditional expectation given $\mathcal{F}_t$ (local martingales are martingales since $X^n_\cdot, X^{n-1}_\cdot \in \mathcal{S}^2$) we get

$$e^{\beta t}(\bar{X}^n_t)^2 + \mathbb{E}_t\left[\beta\int_t^T e^{\beta s}(\bar{X}^n_s)^2 ds + \int_t^T e^{\beta s}[(\bar{Z}^n_s)^2 + \|\bar{\ell}^n_s\|^2_\nu]ds\right]$$
$$= 2\mathbb{E}_t\left[\int_t^T e^{\beta s}\bar{X}^n_s[f(s, F(s, X^{n-1}_s(\cdot)), X^n_s, Z^n_s, l^n_s) - f(s, F(s, X^{n-2}_s(\cdot)), X^{n-1}_s, Z^{n-1}_s, l^{n-1}_s)]ds\right].$$

Moreover,

$$|f(s, F(s, X^{n-1}_s(\cdot)), X^n_s, Z^n_s, l^n_s) - f(s, F(s, X^{n-2}_s(\cdot)), X^{n-1}_s, Z^{n-1}_s, \ell^{n-1}_s)|$$
$$\leq C_0[|F(s, X^{n-1}_s) - F(s, X^{n-2}_s)| + |\bar{X}^n_s| + |\bar{Z}^n_s| + \|\bar{\ell}^n_s\|_\nu]$$
$$\leq C[\|\bar{X}^{n-1}_s\|_2 + |\bar{X}^n_s| + |\bar{Z}^n_s| + \|\bar{\ell}^n_s\|_\nu] = C[(\mathbb{E}|\bar{X}^{n-1}_s|^2)^{\frac{1}{2}} + |\bar{X}^n_s| + |\bar{Z}^n_s| + \|\bar{\ell}^n_s\|_\nu].$$

Now, for all real numbers x, z, l and $\varepsilon > 0$
$2x(Cx + Cz + Cl) \leq \frac{x^2}{\varepsilon^2} + \varepsilon^2(Cx + Cz + Cl)^2 \leq \frac{x^2}{\varepsilon^2} + 3\varepsilon^2(C^2x^2 + C^2z^2 + C^2\ell^2)$ and
$\mathbb{E}[2\bar{X}^n_s(\mathbb{E}|\bar{X}^{n-1}_s|^2)^{\frac{1}{2}}] \leq \mathbb{E}[\frac{1}{\eta^2}|\bar{X}^n_s|^2 + \eta^2\mathbb{E}|\bar{X}^{n-1}_s|^2] = \frac{1}{\eta^2}\mathbb{E}|\bar{X}^n_s|^2 + \eta^2\mathbb{E}|\bar{X}^{n-1}_s|^2$
Thus we obtain that

$$e^{\beta t}(\bar{X}^n_t)^2 + \mathbb{E}_t\left[\beta\int_t^T e^{\beta s}(\bar{X}^n_s)^2 ds + \int_t^T e^{\beta s}[(\bar{Z}^n_s)^2 + \|\bar{\ell}^n_s\|^2_\nu]ds\right]$$
$$\leq \mathbb{E}_t\left[(\frac{C}{\eta^2} + \frac{1}{\epsilon^2})\int_t^T e^{\beta s}(\bar{X}^n_s)^2 ds + C\eta^2\int_t^T e^{\beta s}(\bar{X}^{n-1}_s)^2 ds + 3C^2\epsilon^2\int_t^T e^{\beta s}[(\bar{X}^n_s)^2 + (\bar{Z}^n_s)^2 + \|\bar{\ell}^n_s\|^2_\nu]ds\right]. \tag{2.13}$$

By choosing $\eta = \epsilon$, and β and ϵ such that $\beta - \frac{C+1}{\epsilon^2} - 3C\epsilon^2 \geq 2C\epsilon^2$ and $1 - 3C^2\epsilon^2 \geq 2C\epsilon^2$ we obtain the contraction inequality:

$$\|\bar{X}^n\|^2_\beta + \|\bar{Z}^n\|^2_\beta + \|\bar{\ell}^n\|^2_{\nu,\beta} \leq \frac{1}{2}(\|\bar{X}^{n-1}\|^2_\beta + \|\bar{Z}^{n-1}\|^2_\beta + \|\bar{\ell}^{n-1}\|^2_{\nu,\beta}) \tag{2.14}$$

Therefore the map ϕ is a contraction with respect to the norm $\|\cdot\|_\beta$, with $\beta > 0$. By the Banach fixed point theorem, the map Φ has a unique fixed point, (X, Z, ℓ). Now taking the limit in (2.12), we see that (X, Z, ℓ) is the unique solution of (2.11). □

2.3 Comparison Results

In this section, in order to compare the first components of the solutions of two mean-field BSDEs, we need additional monotonicity assumptions due to the presence of jumps and of the mean-field operator.

Assumption 2.1 *Assume that $dP\otimes dt$-a.s for each $(x', x, z, \ell_1, \ell_2) \in \mathbb{R}^3 \times (L^2_\nu)^2$, there exists a function $\theta^{x',x,z,\ell_1,\ell_2}_\cdot \in L^2_\nu$ such that*

$$f(t, x', x, z, \ell_1) - f(t, x', x, z, \ell_2) \geq \langle\theta^{x',x,z,\ell_1,\ell_2}_t, \ell_1 - \ell_2\rangle_\nu,$$

with

$$\theta_t^{x',x,z,\ell_1,\ell_2} : [0,T] \times \Omega \times \mathbb{R}^3 \times (L^2_\nu)^2 \to L^2_\nu\,;\ (t,\omega,x',x,z,\ell_1,\ell_2) \mapsto \theta_t^{x',x,z,\ell_1,\ell_2}(\omega,.)$$

$\mathcal{P} \otimes \mathcal{B}(\mathbf{R}^3) \otimes \mathcal{B}((L^2_\nu)^2)$-measurable, bounded, and satisfying $dP \otimes dt \otimes d\nu(u)$-a.s. , for each $(x',x,z,\ell_1,\ell_2) \in \mathbb{R}^3 \times (L^2_\nu)^2$,

$$\theta_t^{x',x,z,\ell_1,\ell_2}(u) \geq -1 \quad \textit{and} \quad |\theta_t^{x',x,z,\ell_1,\ell_2}(u)| \leq \psi(u), \tag{2.15}$$

where $\psi \in L^2_\nu$.

Theorem 2.7 (Comparison Theorem for Mean-field BSDEs) *Let $f_i = f_i(\omega,t,x',x,z,l)$, $i=1,2$, be two Lipschitz drivers, and one of them satisfy Assumption 2.1. Furthermore, we assume:*

- *One of the drivers is non-decreasing in x';*

F be a Lipschitz operator on L^2 and satisfying the following property:

- *F is nondecreasing in X in the following sense: let $X_1, X_2 \in L^2$, if $X_1 \leq X_2$ a.s., then for each $t \in [0,T]$, $F(t,X_1) \leq F(t,X_2)$.*

Let $\xi_1, \xi_2 \in L^2$ and denote by (X^1,Z^1,ℓ^1) and (X^2,Z^2,ℓ^2) the solution of the Mean-field BSDE with jumps (2.11) associated with (ξ_1,f_1) and (ξ_2,f_2). Then suppose that

- *$\xi_1 \geq \xi_2$, a.s.*
- *$f_1(\omega,t,x',x,z,\ell(\cdot)) \geq f_2(\omega,t,x',x,z,\ell(\cdot))$,a.s.*
  for all $(x',x,z,\ell(\cdot)) \in \mathbf{R}^2 \times L^2_\nu \times \mathbf{R}^2 \times L^2_\nu$

Then we have $X^1_t \geq X^2_t$, $\forall t \in [0,T]$ a.s.

Proof. For $i=1,2$, let $(X^{i,n}_s, Z^{i,n}_s, l^{i,n}_s)$ be the solution of the following iterating BSDE with jumps

$$X^{i,n}_s = \xi_i + \int_t^T f_i(s,F(s,X^{i,n-1}_s(\cdot)),X^{i,n}_s,Z^{i,n}_s,l^{i,n}_s)ds - \int_t^T Z^{i,n}_s dB_s - \int_t^T \int_{\mathbf{E}} \ell_t(e)\tilde{N}(dt,de) \tag{2.16}$$

For $n \geq 1$ and $t \in [0,1]$. For $n=0$, we set $(X^{i,0}_s, Z^{i,0}_s, l^{i,0}_s) = (0,0,0)$.
Without loss of generality, we assume that f_1 satisfies Assumption 2.1, while f_2 is nondecreasing in x'.

Now we define

$$\tilde{f}^n_1(s,x,z,l) = f_1(s,F(s,X^{1,n-1}_s(\cdot)),x,z,l), \tag{2.17}$$

$$\tilde{f}^n_2(s,x,z,l) = f_2(s,F(s,X^{2,n-1}_s(\cdot)),x,z,l) \tag{2.18}$$

Then obviously we have $\tilde{f}_1^1 \le \tilde{f}_2^1$ and $\tilde{f}_1^1$ satisfy the monotone assumption in Theorem 4.2 [17]. Thus by the classic comparison theorem for BSDE with jumps (Theorem 4.2 in [17]), we have

$$X_s^{1,1} \le X_s^{2,1} \text{ a.s.,} \quad s \in [0,T]. \tag{2.19}$$

Now since f_2 is nondecreasing in x', we have

$$\tilde{f}_1^2(s,x,z,l) = f_1(s, F(s, X_s^{1,1}(\cdot)), x, z, l) \tag{2.20}$$
$$\le f_2(s, F(s, X_s^{1,1}(\cdot)), x, z, l) \tag{2.21}$$
$$\le f_2(s, F(s, X_s^{2,1}(\cdot)), x, z, l) = \tilde{f}_2^2(s,x,z,l) \tag{2.22}$$

where the last inequality follows from (2.19) and f_2 and F are non decreasing. Using again the comparison results for classic BSDEs with jumps, we get

$$X_s^{1,2} \le X_s^{2,2} \text{ a.s.,} \quad s \in [0,T].$$

By the same argument above, we iteratively obtain that

$$X_s^{1,n} \le X_s^{2,n} \text{ a.s.,} \quad s \in [0,T], \quad n \ge 1. \tag{2.23}$$

Using the proof of the existence and uniqueness result, we have that for $i = 1,2$, $(X^{i,n}, Z^{i,n}, l^{i,n})_{n\ge 0}$ converges to the respective solution with drivers $(f_i)_{i=1,2}$, call these X^i, Z^i, l^i. We have that $X_t^1 \le X_t^2, t \in [0,T]$ a.s. follows directly from the fact that $X_t^{1,n} \le X_t^{2,n}, t \in [0,T]$ a.s. □

Remark 2.8 We can weaken the non-decreasing property of F, and consider it in the distribution sense. Let $D_1(x) = \mathbb{P}(X_1 \le x)$ and $D_2(x) = \mathbb{P}(X_2 \le x)$. Then we call F non decreasing in x, if $D_1(x) \ge D_2(x)$ implies $F(t, X_1) \ge F(t, X_2)$.

Take the example of Remark 2.4, when $F(X) = \mathbb{E}(\phi(t,X))$, for $X \in L^2$, F is non decreasing if ϕ is C^1 and non increasing. Assuming that D_1 and D_2 are sufficiently smooth, this can be verified by direct computation : $F(t,X_1) - F(t,X_2) = \mathbb{E}(\phi(t,X_1)) - \mathbb{E}(\phi(t,X_2)) = \int \phi(t,x) d(D_1 - D_2)(x) = \int \frac{\partial \phi}{\partial x}(t,x)(D_2 - D_1)(x)dx = \int \frac{\partial \phi}{\partial x}(t,x)[D_2(x) - D_1(x)]dx$.

In the example of an operator given by an inhomogeneous random graph of Section 2.2, we make the assumption that the kernel κ is ($D_1 \times D_1$ and $D_2 \times D_2$ - almost everywhere) differentiable and

$$\frac{\partial^2}{\partial x \partial y} \kappa(x,y) > 0. \tag{2.24}$$

In this case $F(t,X_1) = \int \kappa(x,y) dD_1(x) dD_1(y) = \int_{\mathbf{R}\times\mathbf{R}} \frac{\partial^2}{\partial x \partial y} \kappa(x,y) D_1(x) D_1(y) dx dy$ and the non-decreasing property of F follows from (2.24).

This condition, along the Lipschitz condition, are satisfied if one uses a truncated Gaussian kernel of coefficient σ $\kappa(x,y) = \begin{cases} e^{\frac{-(x-y)^2}{2\sigma^2}} & \textit{if } |x-y| < \sigma, \\ 0 & \textit{otherwise} \end{cases}$

We also notice that $X_1 \le X_2$ a.s. implies that $D_1(x) \ge D_2(x)$.

Remark 2.9 Symmetrically, if we assume one of the drivers is non-increasing in x' and F is an non-increasing operator, the arguments in Theorem 2.7 still hold.

We now provide a strict comparison theorem, which states that under *a strict inequality* on the map θ, two solutions of the BSDEs are equal at all times, if they are equal at the initial time.

Theorem 2.10 (Strict comparison for Mean-field BSDEs) *Suppose the assumptions of Theorem 2.7 hold. Moreover we assume one of the drivers satisfies Assumption 2.1 with strict inequality $\theta_t^{x',x,z,\ell^1,\ell^2}(u) > -1$ $dt \otimes dP$- a.s. If $X^1_{t_0} = X^2_{t_0}$ a.s. for some $t_0 \in [0,T]$, then $X^1_\cdot = X^2_\cdot$ a.s. on $[t_0, T]$.*

Proof. Let $\bar{X}_s = X^1_s - X^2_s$; $\bar{Z}_s = Z^1_s - Z^2_s$; $\bar{\ell}_s(u) = \ell^1_s(u) - \ell^2_s(u)$. Suppose that f_2 is nondecreasing in x'. Furthermore, we assume that Assumption 2.1 with strict inequality holds for f_1. Then

$$-dX_s = h_s ds - \bar{Z}_s dW_s - \int_{\mathbf{E}} \bar{\ell}_s(e)\tilde{N}(ds, de). \quad \bar{X}_T = \xi_1 - \xi_2.$$

where $h_s := f_1(s, F(s, X^1_s(\cdot)), X^1_s, Z^1_s, \ell^1_s(\cdot)) - f_2(s, F(s, X^2_s(\cdot)), X^2_s, Z^2_s, \ell^2_s(\cdot))$.
Let $\phi(s) := f_1(s, F(s, X^1_s(\cdot)), X^2_s, Z^2_s, \ell^2_s) - f_2(s, F(s, X^2_s(\cdot)), X^2_s, Z^2_s, \ell^2_s)$.

Note that we have $h_s = \phi_s + f_1(s, F(s, X^1_s(\cdot)), X^1_s, Z^1_s, l^1_s) - f_1(s, F(s, X^1_s(\cdot)), X^2_s, Z^2_s, \ell^2_s)$.

We can write $f_1(s, F(s, X^1_s(\cdot)), X^1_s, Z^1_s, l^1_s) - f_1(s, F(s, X^1_s(\cdot)), X^2_s, Z^2_s, \ell^2_s) =$
$f_1(s, F(s, X^1_s(\cdot)), X^1_s, Z^1_s, l^1_s) - f_1(s, F(s, X^1_s(\cdot)), X^2_s, Z^1_s, l^1_s) + f_1(s, F(s, X^1_s(\cdot)), X^2_s, Z^1_s, l^1_s)$
$- f_1(s, F(s, X^1_s(\cdot)), X^2_s, Z^2_s, l^1_s) + f_1(s, F(s, X^1_s(\cdot)), X^2_s, Z^2_s, l^1_s) - f_1(s, F(s, X^1_s(\cdot)), X^2_s, Z^2_s, \ell^2_s)$.

Then from the Assumption 2.1 on f_1, there exist bounded processes δ and β on $\bar{\Omega} \times [0,T]$, such that

$$f_1(s, F(s, X^1_s(\cdot)), X^1_s, Z^1_s, l^1_s) - f_1(s, F(s, X^1_s(\cdot)), X^2_s, Z^2_s, \ell^2_s) \geq \delta_s \bar{X}_s + \beta_s \bar{Z}_s + \langle \theta_s, \bar{\ell}_s \rangle_\nu$$

with

$$\delta_s := \frac{f_1(s, F(s, X^1_s(\cdot)), X^1_s, Z^1_s, l^1_s) - f_1(s, F(s, X^1_s(\cdot)), X^2_s, Z^1_s, l^1_s)}{\bar{X}_s}$$

$$\beta_s := \frac{f_1(s, F(s, X^1_s(\cdot)), X^2_s, Z^1_s, l^1_s) - f_1(s, F(s, X^1_s(\cdot)), X^2_s, Z^2_s, l^1_s)}{\bar{Z}_s}$$

and θ_s is as in Assumption 2.1.
Thus we have $h_s \geq \phi_s + \delta_s \bar{X}_s + \beta_s \bar{Z}_s + \langle \theta_s, \bar{\ell}_s \rangle_\nu$. For each $t \in [0,T]$, let $(\Gamma_{t,s})_{s \in [t,T]}$ be the unique solution of the forward SDE

$$d\Gamma_{t,s} = \Gamma_{t,s^-}\big[\delta_s ds + \beta_s dW_s + \int_{\mathbf{E}} \theta_s(e)\tilde{N}(dt, de)\big]; \quad \Gamma_{t,t} = 1.$$

By the comparison results with respect to a linear BSDE (see lemma 4.1 in [17]) we can derive that

$$\bar{X}_{t_0} \geq \mathbb{E}\big[\Gamma_{t_0,t}\bar{X}_t + \int_{t_0}^t \Gamma_{t_0,s}\phi(s)ds|\mathcal{F}_{t_0}\big], \quad t_0 \leq t \leq T.$$

Due to the Theorem 2.7, we have $\bar{X}_t = X_t^1 - X_t^2 \geq 0$ and the nondecreasing property of F, we can write

$$f_1(s, F(s, X_s^1(\cdot)), X_s^2, Z_s^2, \ell_s^2) \geq f_2(s, F(s, X_s^1(\cdot)), X_s^2, Z_s^2, \ell_s^2) \geq f_2(s, F(s, X_s^2(\cdot)), X_s^2, Z_s^2, \ell_s^2)$$

by the assumption on f_1 and f_2. Then we can conclude the proof by pointing out the fact that

$$\phi(s) = f_1(s, F(s, X_s^1(\cdot)), X_s^2, Z_s^2, \ell_s^2) - f_2(s, F(s, X_s^2(\cdot)), X_s^2, Z_s^2, \ell_s^2) \geq 0$$

and that if $\theta_s(u) > -1 dP \otimes ds \otimes d\nu(u)$-a.s., then $\Gamma_{t,s} > 0$ a.s. from Corollary 3.5 in [17]. □

Remark 2.11 In fact we can weaken the assumption by assuming the strict inequality $\theta_t^{x',x,z,\ell^1,\ell^2}(u) > -1$ *holds only through the solutions, that is* $\theta_t^{F(t,X_t^1),X_t^2,Z_t^2,\ell_t^1,\ell_t^2}(u) > -1$ $dt \otimes dP$*- a.s. In the symmetric case when* f_2 *is Lipshitz and nondecreasing in* x'*, we get the results by assuming* $\theta_t^{F(t,X_t^2),X_t^2,Z_t^2,\ell_t^1,\ell_t^2}(u) > -1$ $dt \otimes dP$*- a.s.*

3 Global dynamic risk measures

3.1 Definition and properties

Let $T > 0$ be a time horizon and f be a Lipschitz driver. Set

$$\rho_t(\eta, T) := -X_t(\eta), \quad 0 \leq t \leq T, \tag{3.25}$$

where $X_t(\eta)$ denotes the solution of mean-field the BSDE (2.11) with driver f, mean-field operator F and terminal condition $\eta \in L^2(\mathcal{F}_T)$.

If η represents a financial position at time T, then $\rho_t(\eta, T)$ is interpreted as the risk of η at time t. Note also that, in insurance, the functional $-\rho = X$ can represent a risk premium.

The functional $\rho : (\eta, T) \mapsto \rho_.(\eta, T)$ represents a *global dynamic risk measure* induced by the mean-field BSDE with driver f and mean-field operator F.

When the dependence on the time horizon T is clear from the context, we drop it from the notation and write $\rho_t(\eta)$ for $\rho_t(\eta, T)$.

We now provide properties of these global dynamic risk measures, which follow from the comparison results in the previous section. We work under the Assumption 2.1, which guarantees the monotonicity of the risk measure in the jumps. Contrary to the standard non mean- field case, the risk of a zero position may not be zero when the mean-field operator F is introduced. The first three properties are standard, and their proofs follow as in [17], by plugging in the comparison or uniqueness results specific for mean-field BSDEs.

1. *Continuity.* Let $\{\theta^\alpha, \alpha \in R\}$ be a family of stopping times converging a.s. to a stopping time θ as α tends to α_0. Let $\{\xi^\alpha, \alpha \in R\}$ be a family of random variables such that $\mathbb{E}[\operatorname{ess\,sup}_\alpha (\xi^\alpha)^2] < \infty$, and for each α, ξ^α is $\mathcal{F}_{\theta^\alpha}$ measurable. Suppose also that ξ^α converges a.s. to a $\mathcal{F}_{\theta^\alpha}$ measurable random variable ξ as α tends to α_0. Then for

each stopping time S, the random variable $\rho_S(\xi^\alpha) \to \rho_S(\xi)$ a.s. and the processes $\rho(\xi^\alpha) \to \rho_S(\xi)$ in $\mathcal{S}^2$ when $\alpha \to \alpha_0$. This property follows as in the technical proofs of [17, appendix], in which the a priori estimates for MFBSDEs are trivially extended to account for the mean field contribution.

2. *Monotonicity.* ρ is nonincreasing with respect to ξ. i.e. for each $\xi^1, \xi^2 \in L^2$. If $\xi^1 \geq \xi^2$ a.s., then $\rho_t(\xi^1) \leq \rho_t(\xi^2)$, $0 \leq t \leq T$ a.s.
Proof. This is the direct consequence of the comparison results of mean-field BSDEs (see Theorem 2.7) □

3. *Consistency.* By the flow property, ρ is consistent: more precisely, let S a stopping time, then for each time t smaller than S, the risk-measure associated with position ξ and maturity T coincides with the risk-measure associated with maturity S and position $-\rho_S(\xi, T) = X_S(\xi, T)$, that is

$$\forall t \leq S, \rho_t(\xi, T) = \rho_t(-\rho_S(\xi, T), S) \quad \text{a.s.}$$

The flow property is the consequence of the uniqueness result of the mean-field BSDEs

4. *Cash sub-additivity.* Suppose f is non-decreasing in both x' and x, while F is non-decreasing operator. Then for any $m > 0$, we have $\rho_t(\xi + m) \leq \rho_t(\xi) + m$.
Proof. It is straightforward that $X_t + m$ satisfying:

$$-d(X_t + m) = f(t, F(t, X_t), X_t, Z_t, \ell_t)dt - Z_t dW_t - \int_{\mathbf{E}} \ell_t(e)\tilde{N}(dt, de); X_T = \xi + m.$$

Thus, by the assumption, we have

$$-d(X_t+m) \leq f(t, F(t, X_t+m), X_t+m, Z_t, \ell_t)dt - Z_t dW_t - \int_{\mathbf{E}} \ell_t(e)\tilde{N}(dt, de); X_T = \xi+m.$$

let $(\bar{X}, \bar{Z}, \bar{l})$ be a solution of the Mean-field BSDE

$$-d\bar{X}_t = f(t, F(t, \bar{X}_t), \bar{X}_t, \bar{Z}_t, \bar{\ell}_t)dt - \bar{Z}_t dW_t - \int_{\mathbf{E}} \bar{\ell}_t(e)\tilde{N}(dt, de); \bar{X}_T = \xi + m.$$

Then by the (extended) comparison results, we have $X_t + m \leq \bar{X}_t$, which gives $\rho_t(\xi + m) \leq \rho_t(\xi) + m$. □

5. *Convexity.* If f is concave with respect to (x', x, z, l). We furthermore assume f nondecreasing in x' and F is nondreasing concave operator in x, then the dynamic risk-measure ρ is convex, that is for any $\lambda \in [0, 1]$, $\xi^1, \xi^2 \in L^2$

$$\rho(\lambda\xi^1 + (1-\lambda)\xi^2) \leq \lambda\rho(\xi^1) + (1-\lambda)\rho(\xi^2).$$

Proof. For $i = 1, 2$, let (X^i, Z^i, l^i) be a solution of the mean-field BSDE (2.11) associated to terminal time T, driver f, Mean-field operator F and terminal condition ξ^i. Set $\hat{\xi} := \lambda\xi^1 + (1-\lambda)\xi^2$, $\hat{X} := \lambda X^1 + (1-\lambda)X^2$, $\hat{Z} := \lambda Z^1 + (1-\lambda)Z^2$, $\hat{l} := \lambda\ell^1 + (1-\lambda)\ell^2$. We have

$$\begin{aligned}-d\hat{X}_t = [\lambda f(t, F(t, X_t(\cdot)), X_t, Z_t, \ell_t(\cdot)) + (1-\lambda)f(t, F(t, X_t(\cdot)), X_t, Z_t, \ell_t(\cdot))]dt- \\ \hat{Z}_t dW_t - \int_{\mathbf{E}} \hat{l}_t(e)\tilde{N}(dt, de); \\ \hat{X}_T = \hat{\xi}.\end{aligned}$$

By the assumption on f, and F, we have

$$\begin{aligned}&\lambda f(t, F(t, X_t^1), X_t^1, Z_t^1, \ell_t^1) + (1-\lambda)f(t, F(t, X_t^2), X_t^2, Z_t^2, \ell_t^2) \\ &\le f(t, \lambda F(t, X_t^1) + (1-\lambda)F(t, X_t^2), \lambda X_t^1 + (1-\lambda)X_t^2, \lambda Z_t^1 + (1-\lambda)Z_t^2, \lambda\ell_t^1 + (1-\lambda)\ell_t^2) \\ &\le f(t, F(t, \lambda X_t^1 + (1-\lambda)X_t^2), \lambda X_t^1 + (1-\lambda)X_t^2, \lambda Z_t^1 + (1-\lambda)Z_t^2, \lambda\ell_t^1 + (1-\lambda)\ell_t^2) \\ &= f(t, F(t, \hat{X}_t), \hat{X}_t, \hat{Z}_t, \hat{l}_t)\end{aligned}$$

Thus

$$-d\hat{X}_t \le f(t, F(t, \hat{X}_t), \hat{X}_t, \hat{Z}_t, \hat{l}_t)dt - \hat{Z}_t dW_t - \int_{\mathbf{E}} \hat{l}_t(e)\tilde{N}(dt, de); \hat{X}_T = \hat{\xi}.$$

let $(\bar{X}, \bar{Z}, \bar{l})$ be a solution of the mean-field BSDE (2.11) associated to terminal time T, driver f, mean-field operator F and terminal condition $\hat{\xi}$. i.e.

$$-d\bar{X}_t = f(t, F(t, \bar{X}_t), \bar{X}_t, \bar{Z}_t, \bar{\ell}_t)dt - \bar{Z}_t dW_t - \int_{\mathbf{E}} \bar{\ell}_t(e)\tilde{N}(dt, de); \bar{X}_T = \hat{\xi}.$$

by (extended) comparison results Th.2.7, we obtain $\hat{X}_t \le \bar{X}_t$ which gives the results. □

Suppose furthermore in Assumption 2.1 we have $\theta_t^{x', x, z, \ell^1, \ell^2}(u) > -1$.

6. *No Arbitrage.* For each $\xi^1, \xi^2 \in L^2$, if $\xi^1 \ge \xi^2$ a.s. and if $\rho_t(\xi^1) = \rho_t(\xi^2)$ a.s. on an event $A \in \mathcal{F}_t$. then $\xi^1 = \xi^2$ a.s. on A.
   This is the directly consequence of strict comparison results (Theorem 2.10).

In the classical case without the mean field term and when the drift term is independent of the current position, the risk measure is cash additive. We have the Translation invariance (cash additivity) property: If f does not depend on x' and x, then the associated risk-measure satisfies the following *translation invariance* property:

$$\rho_t(\xi + \xi', T) = \rho_t(\xi, T) - \xi', \quad \text{for any } \xi \in L^2(\mathcal{F}_T) \text{ and } \xi' \in L^2(\mathcal{F}_T)$$

With the mean field, it is understood that cash is transferred via the mean field term. When f is non-decreasing in the mean field term, then the mean field is acting as a stabilizer.

This means that individually, a bank needs less capital to reach the same final position as without the stabilizing effect.

In systemic risk models where the mean field has a stabilizing effect [6, 12, 13, 10], the function f captures interbank lending: banks with high liquidity position lend to banks with low liquidity position, where high and low is given with respect to the average liquidity in the system. Here, we do not consider formally a model of interbank lending, but our risk processes behave similarly. In addition, when f is convex we give a dual representation of the risk measures.

Interpretation as systemic risk measure .

We may regard (2.11) as a limit as $n \mapsto \infty$ of the following system:

$$- dX_t^{i,N} = f(t,\omega, \frac{1}{N}\sum_{j=1}^{N} X_t^{j,N}(\cdot), X_t^{i,N}, Z_t^{i,N}, \ell_t^{i,N}(\cdot))dt - Z_t^{i,N} dW_t - \int_{\mathbf{E}} \ell_t^{i,N}(e)\tilde{N}(dt,de); \tag{3.26}$$

$$X_T^{i,N} = \xi.$$

As discussed before, the more interesting case is when the mean field term represents the average intensity of interactions in the system, and the interactions of any two nodes are state-dependent and captured by a kernel κ. The mean field term then is

$$\frac{1}{N^2}\mathbb{E}\sum_{i,j\in 1,N} \kappa(X_t^{i,N}, X_t^{j,N}).$$

Here $-X_t^{i,N}$ represents the liquidity or capital reserve of bank i. In a system framework, its evolution depends on the capital of the other banks. The first case where the mean field term is the average capital in the system captures a situation where a joint liquidity fund is used to stabilize banks, for example a central clearinghouse.

For the second case, where banks have state dependent interactions, we can set for example $\kappa(x,y) = \phi(x-y)$, where ϕ is the gaussian kernel. This means that we have a tiered structure where banks with similar capital interact, while there is little interaction for banks of dissimilar size. This yields a core-periphery structure, shown to be quite realistic [7].

Finally, for a system with these stylized dependence structures, a regulator is choosing a level $-\xi$ for the acceptable capital or liquidity at a horizon T (ξ is then the financial position at time T). Before time T and in order to become acceptable at the time horizon, the capital (liquidity) of the individual bank may have inflows/outflows from via the mean field term.

3.2 Dual representation of convex global risk measures

We now provide a representation for global dynamic risk measures induced by concave mean-field BSDEs in terms of the value of a stochastic control problem. In this case, the risk measure is convex. This dual representation is given via a control problem over a set of probability measures which are absolutely continuous with respect to P.

Let f be a Lipschitz driver and F be an Lipschitz operator. For each (ω, t), we denote by f^* the Fenchel-Legendre transform of f, defined for each $(\beta, q, \alpha_1, \alpha_2) \in \mathbf{R}^3 \times L^2_\nu$ and F^* the Fenchel-Legendre transform of F, defined for each $\delta \in L^2$, that is,

$$f^*(\omega, t, q, \beta, \alpha_1, \alpha_2) = \sup_{(x', x, z, l) \in \mathbf{R}^3 \times L^2_\nu} [f(\omega, t, x', x, z, l) - qx' - \beta x - \alpha_1 z - \langle \alpha_2, l \rangle_\nu]$$

$$F^*(t, \delta) = \sup_{X \in L^2} [F(t, X) - \langle X, \delta \rangle_{L^2}]$$

For each predictable processes $\alpha_t = (\alpha^1_t, \alpha^2_t(\cdot))$, let $\mathcal{Q}^\alpha$ be the probability absolutely continuous with respect to $\mathcal{P}$ which admits Γ^α_T as density with respect to P on $\mathcal{F}_T$, where Z^α is the solution of

$$d\Gamma^\alpha_t = \Gamma^\alpha_{t-}\big(\alpha^1_t dW_t + \int_{\mathbf{R}^*} \alpha^2_t(u) d\tilde{N}(dt, du)\big); \quad \Gamma^\alpha_0 = 1. \tag{3.27}$$

Now we define the control set consisting of probability measures in terms of their densities. Let $\mathcal{A}_T$ be the set of predictable processes $\alpha_s = (\alpha^1_s, \alpha^2_s)$ such that

- $\int_0^T (\alpha^1_s)^2 ds + \int_0^T \|\alpha^2_s\|^2_\nu ds$ is bounded
- $\alpha^2_s(u) > -1 \quad \nu(du) - a.s.$

We have from Proposition 3.1 and 3.2 [17] that for all $\alpha. \in \mathcal{A}_T$, $\Gamma^\alpha_t > 0, 0 \leq t \leq T$ a.s. and $(\Gamma^\alpha_t)_{0 \leq t \leq T} \in \mathcal{S}^2$.
And let $\bar{\mathcal{A}}_T$ be the set of processes $(\gamma_t, \beta_t, q_t, \alpha^1_t, \alpha^2_t)$ where $(\beta_t, q_t, \alpha^1_t, \alpha^2_t)$ are predictable and γ_t is progressively measurable, such that

- $(f^*(\omega, t, q_t, \beta_t, \alpha^1_t, \alpha^2_t))_{t \in [0,T]}$ belongs to $I\!H^2$
- $\alpha_t = (\alpha^1_t, \alpha^2_t(\cdot))_{t \in [0,T]}$ belongs to $\mathcal{A}_T$.
- $0 \leq q_t \leq C$, $\forall t \in [0, T]$, $\quad dP$ a.s.
- The processes $(\Gamma^\alpha_t e^{-\int_0^t \gamma_s ds})_{t \in [0,T]}$ belongs to $I\!H^2$.

In the sequel, we assume that f is nondecreasing with respect to x' and satisfies Assumption 2.1 with strict inequality $\theta_t(u) > -1$ $dt \otimes dP$- a.s.

3.2.1 Technical lemmas

We begin by the following technical lemma on part of the bounds of the control set which appear in the definition of the Fenchel transform of f.

Lemma 3.1 *For each (s,ω), $D_s(\omega)$ is the non empty set of $(\bar{q},\bar{\beta},\bar{\alpha}_1,\bar{\alpha}_2) \in \mathbf{R}^3 \times L^2_\nu$ such that $f^*(\omega,s,\bar{q},\bar{\beta},\bar{\alpha}_1,\bar{\alpha}_2) < +\infty$. Then $D_s(\omega)$ is included in the set Usatisfying the following properties:*

- *$q \geq 0$ and is bounded by C.*
- *β and α_1 are bounded by C.*
- *$\alpha_2(u) > -1 \quad$ and $\quad |\alpha_2(u)| \leq C \quad \nu(du) - a.s.$*

where C is the Lipschitz constant of f.

Proof. Let us suppose that $q < 0$, we will show that this assumption leads to contradiction. By the definition of f^* we have

$$f^*(t,q,\beta,\alpha_1,\alpha_2) \geq f(t,x',0,0,0) - x'q,$$

which holds for each x'. This holds in particular for $x_n := n$, $n \in \mathbb{N}$. We thus get

$$f^*(t,q,\beta,\alpha_1,\alpha_2) \geq f(t,n,0,0,0) - nq \geq f(t,0,0,0,0) - nq$$

where the last inequality follows by the non-decreasingness of the map f with respect to x'. By letting $n \to +\infty$ in the above inequality, we get $\lim_{n\to+\infty} f(t,0,0,0,0) - qn = +\infty$, since $q < 0$. This implies that $f^*(t,q,\beta,\alpha_1,\alpha_2) < +\infty$ which provides the expected contradiction, We thus have proved that $q \geq 0$. The fact that q, β and α are included in the bounded domain $[-C,C]$ is due to the uniform Lipschitz property of f. Finally, for the properties of (α_1,α_2), we apply the similar proof as in Lemma 5.4 [17]. Suppose by contradiction that

$$\nu(\{u \in \mathbb{R}^*, \alpha_2(u) \leq -1\}) > 0.$$

Since f satisfies Assumption 2.1, the following inequality holds for each $l \in L^2_\nu$.

$$f(\omega,t,0,l) \geq f(\omega,t,0,0) + \langle \theta_t^{0,l,0}(\omega), l\rangle_\nu.$$

Again by the definition of f^* we have

$$f^*(\omega,t,\alpha_1,\alpha_2) \geq f(\omega,t,0,l) - \langle \alpha_2, l\rangle_\nu \geq f(\omega,t,0,0) + \langle \theta_t^{0,l,0}(\omega) - \alpha_2, l\rangle_\nu.$$

This holds in particular for $l := n\mathbf{1}_{\{\alpha_2 \leq -1\}}$ where $n \in \mathbb{N}$. We thus derive

$$f^*(\omega,t,\alpha_1,\alpha_2) \geq f(\omega,t,0,0) + n\int_{\{\alpha_2 \leq -1\}} \big(\theta_t^{0,l,0}(\omega,u) - \alpha_2(u)\big)\nu(du)$$

Since by assumption, $\theta_t^{0,l,0}(\omega,u) > -1$, thus $\theta_t^{0,l,0}(\omega,u) - \alpha_2(u) > 0$ on $\{\alpha_2 \leq -1\}$. By letting n tend to $+\infty$ in this inequality, we get $f^*(\omega,t,\alpha_1,\alpha_2) = +\infty$, which provides the expected contradiction. We thus have proven that $\alpha_2 > -1$ ν-a.s. By similar arguments, one can show that α_1 is bounded by C and $|\alpha_2(u)| \leq C \quad \nu(du) - a.s.$, which ends the proof. □

The following technical lemma is used in the proof of the dual representation Theorem 3.1. It provides part of the bounds of the control set which appear in the definition of the Fenchel transform of F.

Lemma 3.2 *Assume that operator F is nondecreasing in the sense of Theorem 2.7. Then for each $t \in [0,T]$, the non empty set of $\{\delta \in L^2 | F^*(t,\delta) < +\infty\}$ is included in the set satisfying the following properties:*

- *$\delta \geq 0$ dP a.s. and $\|\delta\|_{L^2} \leq C$, where C is the Lipschitz constant of F.*

Proof. Suppose $\delta \geq 0$ dP a.s. is not true. We denote $A = \{\omega \in \Omega | \delta(\omega) < 0\}$ Then $P(A) > 0$. By the definition of F^*, we have for each $X \in L^2$

$$F^*(t,\delta) \geq F(t,X) - \langle X, \delta \rangle_{L^2} = F(t,X) - \mathbb{E}^P[X\delta].$$

This holds in particular for $X_n(\omega) := -n\delta \mathbf{1}_A(\omega)$ where $n \in \mathbb{N}$. This gives $X_n \geq 0$ dP a.s. and thus by the nondecreasing properties of F, we obtain

$$F^*(t,\delta) \geq F(t,X^n) - \mathbb{E}^P[X^n\delta] \geq F(t,0) - \mathbb{E}^P[X^n\delta] = F(t,0) + n\int_A |\delta(\omega)|^2 dP(\omega)$$

By letting $n \to +\infty$ in the above inequality, we get $F^*(\delta) = +\infty$, which gives the contradiction to the assumption. Thus $P(A) = 0$ which implies $\delta \geq 0$ dP a.s. The boundedness of δ is a direct results of the Lipschitz property of F. □

The following technical lemma establishes the existence of the solution of a particular mean-field SDE.

Lemma 3.3 *Suppose $(\alpha_s^1, \alpha_s^2(\cdot))_{s \geq t}$ belongs to $\mathcal{A}_T$, $(U_s)_{s\geq t}$ are bounded in L^2 and $(h_s)_{s\geq t}$ are bounded almost surely. Then the following SDE admits a solution $(V_s)_{s\geq t} \in \mathcal{S}$*

$$dV_s = V_s[\alpha_s^1 dW_s + \int_{\mathbf{R}^*} \alpha_s^2(u) d\tilde{N}(ds,de)] + U_s \mathbb{E}[V_s h_s] ds, \ t \leq s \leq T$$
$$V_t = 1. \tag{3.28}$$

Proof. We define inductively the sequence (V^n) of the processes by setting $V^0 \equiv V_0$ and for $n \geq 1$

$$V_u^n = V_t + \int_t^u V_s^{n-1} dM_s + \int_t^u U_s \mathbb{E}[V_s^{n-1} h_s] ds,$$

where $dM_s = \alpha_s^1 dW_s + \int_{\mathbf{R}^*} \alpha_s^2(e) d\tilde{N}(ds,de)$. Because any two real numbers h and k satisfy $(h+k)^2 \leq 2(h^2+k^2)$, we have

$$\mathbb{E}\left[\sup_{t\leq s\leq u} |V_s^{n+1} - V_s^n|^2\right] \leq 2\mathbb{E}\left[\sup_{t\leq s\leq u}\left(\int_t^s (V_r^n - V_r^{n-1}) dM_r\right)^2 + \sup_{t\leq s\leq u}\left(\int_t^s U_r \mathbb{E}[V_r^n - V_r^{n-1}] dr\right)^2\right]. \tag{3.29}$$

By Doob and Cauchy-Schwarz inequality, it follows that

$$\mathbb{E}\left[\sup_{t\le s\le u}|V_s^{n+1}-V_s^n|^2\right]\le 8\mathbb{E}\left[\left(\int_t^u (V_s^n-V_s^{n-1})dM_s\right)^2\right]+2T\mathbb{E}\left[\int_t^u |U_s\mathbb{E}[V_s^n-V_s^{n-1}]|^2ds\right]$$
$$\le 8\mathbb{E}\left[\int_t^u (V_s^n-V_s^{n-1})^2d[M,M]_s\right]+2T\left[\int_t^u \mathbb{E}|U_s|^2|\mathbb{E}[V_s^n-V_s^{n-1}]|^2ds\right]. \quad (3.30)$$

Since $d[M,M]_s=(\alpha_s^1)^2ds+\int_{\mathbf{R}^*}(\alpha_s^2(e))^2d\tilde{N}(ds,de)+\int_{\mathbf{R}^*}(\alpha_s^2(e))^2\nu(de)ds$ and by assumption, $(\alpha_s^1,\alpha_s^2(\cdot))_{s\ge t}$ belongs to $\mathcal{A}_T$, $(U_s)_{s\ge 0}$ is bounded in L^2, we obtain there exists a constant K such that

$$\mathbb{E}\left[\sup_{t\le s\le u}|V_s^{n+1}-V_s^n|^2\right]\le 2K(4+T)\mathbb{E}\left[\int_t^u |V_s^n-V_s^{n-1}|^2ds\right]$$
$$\le 2K(4+T)\mathbb{E}\left[\int_t^u \sup_{t\le s\le u}|V_s^n-V_s^{n-1}|^2ds\right]. \quad (3.31)$$

We set $C=2K(4+T)$ and let $D:=\mathbb{E}\left[\sup_{t\le s\le T}|V_s^1-V_s^0|^2\right]$. It then follows from the above computation that for each $t\le u\le T$ and n,

$$\mathbb{E}\left[\sup_{t\le s\le u}|V_s^n-V_s^{n-1}|^2\right]\le\frac{DC^nT^n}{n!}$$

consequently

$$\sum_{n=1}^{\infty}\left\|\sup_{t\le s\le u}|V_s^n-V_s^{n-1}|\right\|_2<\infty$$

Thus it gives the series $\sum_{n=1}^{\infty}\sup_{t\le s\le u}|V_s^n-V_s^{n-1}|$ converges a.s. and as a result, V^n converges a.s. uniformly on every bounded interval to a right-continuous process V which is a solution to (3.28) □

3.2.2 Dual representation theorem

We now give the main result of this section, the dual representation theorem of the mean risk measure. Note that at time zero, this is the risk measure itself.

For $(\gamma,q,\beta,\alpha^1,\alpha^2)\in\bar{\mathcal{A}}_T$, we denote

$$D_{t,s}^{\beta,\gamma}:=\exp(-\int_t^s(\beta_u+\gamma_u\mathbf{1}_{\mathbb{E}[q_u]>0})du), 0\le t\le s\le T,$$

which can be interpreted as a discount factor. We recall that the process Γ^α follows the dynamics defined in (3.27).

The following lemma will be used for the existence of the optimal control that appears in the dual representation theorem. In the classical setting without the mean field the two components $\bar{\beta}_s,\bar{\alpha}_s$ of the optimal control are shown to exist in previous literature. The challenge we now solve is the construction of the other components, which are associated to the mean field operator.

Lemma 3.4 *Given the predictable processes processes $(X_s, \bar{q}_s, \bar{\beta}_s, \bar{\alpha}_s)_{s\geq t}$, there exist a progressively measurable processes $(\bar{\gamma}_s)_{s\geq t}$ such that $(\Gamma^{\bar{\alpha}_s} e^{-\int_t^s \bar{\gamma}_u du})_{s\geq t}$ belongs to $I\!H^2$ and satisfying the following equation:*

$$F(s, X_s) - \frac{\mathbb{E}^{\mathcal{Q}^{\bar{\alpha}}}[X_s D_{t,s}^{\bar{\beta},\bar{\gamma}} \bar{\gamma}_s]}{\mathbb{E}^{\mathcal{Q}^{\bar{\alpha}}}[D_{t,s}^{\bar{\beta},\bar{\gamma}} \bar{q}_s]} = F^*(s, \frac{\Gamma_s^{\bar{\alpha}} D_{t,s}^{\bar{\beta},\bar{\gamma}} \bar{\gamma}_s}{\mathbb{E}^{\mathcal{Q}^{\bar{\alpha}}}[D_{t,s}^{\bar{\beta},\bar{\gamma}} \bar{q}_s]}) \tag{3.32}$$

Proof. Since F is concave and Lipschitz on L^2, the conjugacy relation of (F, F^*) gives that for each s, there exists $Y_s \in L^2$ such that

$$F(s, X_s) - \mathbb{E}^P[X_s Y_s] = F^*(s, Y_s) \tag{3.33}$$

and since F is nondecreasing, by lemma 3.2 we have $Y_s \geq 0\ dP$ a.s. and $\|Y_s\|_{L_P} \leq C$. Now let $(V_s)_{s\geq t} \in \mathcal{S}$ be the solution of (3.28) with $U_s = e^{\int_t^s \bar{\beta}_u du} Y_s$, $h_s = e^{-\int_t^s \bar{\beta}_u du} \bar{q}_s$ and $V_t = 1$. We apply Ito's formula to $V_s(\Gamma_s^{\bar{\alpha}})^{-1}$ and we obtain

$$d(V(\Gamma^{\bar{\alpha}})^{-1})_s = (\Gamma_s^{\bar{\alpha}})^{-1} e^{\int_0^s \bar{\beta}_u du} Y_s \mathbb{E}[V_s e^{-\int_0^s \bar{\beta}_u du} \bar{q}_s] ds \tag{3.34}$$

Thus $V^1 := V(\Gamma^{\bar{\alpha}})^{-1}$ satisfies the random differential equation

$$d(V^1)_s = (\Gamma_s^{\bar{\alpha}})^{-1} e^{\int_0^s \bar{\beta}_u du} Y_s \mathbb{E}[\Gamma_s^{\bar{\alpha}} V_s^1 e^{-\int_0^s \bar{\beta}_u du} \bar{q}_s] ds \tag{3.35}$$

Since $\Gamma_s^{\bar{\alpha}}$, $\bar{q}_s > 0$, $Y_s \geq 0\ dP$ a.s., and $V_t^1 := V_t = 1 > 0$ a.s. we have $V_s^1 > 0$ a.s. Thus for each $(s, \omega) \in [t, T] \times \Omega$, we can choose $\bar{\gamma}_s(\omega) = -\frac{d}{ds}(\log V_s^1)(\omega)$ which is well defined due to (3.35). This gives the process $\bar{\gamma}$ satisfying that $e^{-\int_t^s \bar{\gamma}_u du} = V_s^1$ and from (3.35) we obtain that

$$\bar{\gamma}_s e^{-\int_t^s \bar{\gamma}_u du} ds = d(e^{-\int_t^s \bar{\gamma}_u du})_s = (\Gamma_s^{\bar{\alpha}})^{-1} e^{\int_t^s \bar{\beta}_u du} Y_s \mathbb{E}[\Gamma_s^{\bar{\alpha}} e^{-\int_t^s \bar{\gamma}_u du} e^{-\int_t^s \bar{\beta}_u du} \bar{q}_s] ds \tag{3.36}$$

which implies that $(\bar{\gamma}_s)_{s\geq t}$ satisfies

$$\frac{\Gamma_s^{\bar{\alpha}} D_{t,s}^{\bar{\beta},\bar{\gamma}} \bar{\gamma}_s}{\mathbb{E}[\Gamma_s^{\bar{\alpha}} D_{t,s}^{\bar{\beta},\bar{\gamma}} \bar{q}_s]} = Y_s\ a.s.$$

and $\Gamma_s^{\bar{\alpha}} e^{-\int_t^s \bar{\gamma}_u du} = V_s$ belongs to $I\!H^2$.

Now we show that $(\bar{\gamma}_s)_{s\geq t}$ is progressively measurable. Since the Hilbert space L^2 is separable, hence by the measurable selection theorem (see e.g. [21]), the map $Y : [t, T] \mapsto L^2, s \mapsto Y_s(\cdot)$ is measurable with respect to $\mathcal{B}([t, T])$ and $\mathcal{B}(L^2)$. Fix now $s \in [t, T]$. For each Y in $L^2(\mathcal{F}_s)$ and feasible, by Jensen's inequality applied to F, we have for each $X \in L^2$.

$$F(s, X) - \mathbb{E}^P[XY] \leq F(s, \mathbb{E}[X|\mathcal{F}_s]) - \mathbb{E}^P[\mathbb{E}[X|\mathcal{F}_s]Y]$$

This gives

$$F^*(s, Y) = \sup_{X \in L^2(\mathcal{F}_T)} [F(s, X) - \langle X, Y \rangle_{L^2}] = \sup_{X \in L^2(\mathcal{F}_s)} [F(s, X) - \langle X, Y \rangle_{L^2}]$$

Thus we could restrict the operator F on the subspace $L^2(\mathcal{F}_s)$. This implies we could choose $Y_s \in L^2(\mathcal{F}_s)$ in (3.33) and for each $u \in [0,T]$, $Y : [t,u] \mapsto L^2(\mathcal{F}_u), s \mapsto Y_s(\cdot)$ are measurable with respect to $\mathcal{B}([t,u])$ and $\mathcal{B}(L^2(\mathcal{F}_u))$. Now for each $u \in [t,T]$, since $L^2(\mathcal{F}_u)$ is a separable Hilbert space, there exists a countable orthonormal basis $(e^i_u)_{i\in\mathbb{N}}$ of $L^2(\mathcal{F}_u)$. For each $i \in \mathbb{N}$, define $\lambda^i_u = \langle Y_u, e^i\rangle_P$. Since the map $\langle ., e^i_u\rangle_P$ is continuous on $L^2(\mathcal{F}_u)$, $\lambda^i : [t,u] \mapsto \mathbf{R}, s \mapsto \lambda^i_s$ is $\mathcal{B}([t,u])$ measurable. As $Y_s(\omega) = \sum_i \lambda^i_s e_i(\omega)$, it follows that $Y : [t,u] \times \Omega \mapsto \mathbf{R}, (s,\omega) \mapsto Y_s(\omega)$ is $\mathcal{B}([t,u]) \otimes \mathcal{F}_u$ measurable. This holds for each $u \in [t,T]$, thus $(Y_s)_{s\geq t}$ is progressive measurable. And this gives the progressive measurability of $(\bar{\gamma}_s)_{s\geq t}$.

□

We are now ready to give the main result. The risk measure can be interpreted as the expectation (under a worst-case discount factor and the worst case probability measure) of the final position ξ plus a penalty function. The lemmas in the previous section ensure that the supremum is finite as the relevant domain is bounded. We define the discount factor as follows. For $(\gamma, q, \beta, \alpha^1, \alpha^2) \in \bar{\mathcal{A}}_T$, we denote

$$D^{\beta,\gamma}_{t,s} := exp(-\int_t^s (\beta_u + \gamma_u \mathbf{1}_{\mathbb{E}[q_u]>0})du), 0 \leq t \leq s \leq T. \tag{3.37}$$

Theorem 3.1 *Suppose that the Hilbert space L^2_ν is separable. Let f be a Lipschitz driver with Lipschitz constant C satisfying Assumption 2.1. Suppose moreover that f is concave with respect to (x', x, z, l) and non-decreasing in x', and that F is a Lipschitz concave operator in X and satisfies the following property: for each $s, t \in [0,T]$ and $X \in L^2$, $F(t, \mathbb{E}[X|\mathcal{F}_s]) \geq F(t, X)$.*

Then, for each $t \in [0,T]$, the mean of the convex risk-measure ρ_t, that is $\mathbb{E}\rho(., T)$ has the following representation: for each $\xi \in L^2$,

$$\mathbb{E}\rho_t(\xi, T) = \sup_{(\gamma,\beta,q,\alpha)\in\bar{\mathcal{A}}_T} \Big[\mathbb{E}^{\mathcal{Q}^\alpha} D^{\beta,\gamma}_{t,T}(-\xi) - \zeta(\gamma, \beta, q, \alpha, T)\Big] \tag{3.38}$$

where the function ζ, called penalty function, is defined, for each T and $(\gamma, q, \beta, \alpha^1, \alpha^2) \in \bar{\mathcal{A}}_T$ by

$$\zeta(\gamma, \beta, q, \alpha, T) := \int_t^T \left(\mathbb{E}^{\mathcal{Q}^\alpha}[D^{\beta,\gamma}_{t,s} f^*(s, q_s, \beta_s, \alpha_s)] + \mathbb{E}^{\mathcal{Q}^\alpha}[D^{\beta,\gamma}_{t,s} q_s] F^*(t, \frac{\Gamma^\alpha_s D^{\beta,\gamma}_{t,s}\gamma_s}{\mathbb{E}^{\mathcal{Q}^\alpha}[D^{\beta,\gamma}_{t,s} q_s]})\mathbf{1}_{\mathbb{E}^{\mathcal{Q}^\alpha}[q_s]>0}\right) ds,$$

with Γ^α_s following the dynamics defined in (3.27). Moreover, for each $\xi \in L^2$, there exists $(\bar{\gamma}_t, \bar{q}_t, \bar{\beta}_t, \bar{\alpha}^1_t, \bar{\alpha}^2_t) \in \bar{\mathcal{A}}_T$ achieving the supremum in (3.38).

Proof. For each predictable processes $(\gamma_s, q_s, \beta_s, \alpha^1_s, \alpha^2_s) \in \bar{\mathcal{A}}_T$, we apply Itô's formula to $D^{\beta,\gamma}_{t,s} X_s$ between t and T, where (X, Z, l) is the solution of Mean-field BSDE (2.11). We obtain

$$X_t = D^{\beta,\gamma}_{t,T}\xi + \int_t^T D^{\beta,\gamma}_{t,s}[-\beta_s X_s - \gamma_s \mathbf{1}_{\mathbb{E}[q_s]>0} X_s - \alpha^1_s Z_s - \langle \alpha^2_s, l_s\rangle_\nu + f(s, F(s, X_s(\cdot)), X_s, Z_s, l_s)]ds \\ - \int_t^T dM^{\mathcal{Q}^\alpha}_s \tag{3.39}$$

where $dM^{\mathcal{Q}^\alpha}_s = D^{\beta,\gamma}_{t,s} Z_s dW^{\mathcal{Q}^\alpha}_s + \int_{\mathbf{E}} D^{\beta,\gamma}_{t,s} l_s(e) d\tilde{N}^{\mathcal{Q}^\alpha}(dt, de)$. For each $s \in [t, T]$, we have

$$-\beta_s X_s - \gamma_s \mathbf{1}_{\mathbb{E}[q_s]>0} X_s - \alpha^1_s Z_s - \langle \alpha^2_s, l_s\rangle_\nu + f(s, F(s, X_s(\cdot)), X_s, Z_s, l_s) \\ = -\beta_s X_s - q_s F(s, X_s) - \alpha^1_s Z_s - \langle \alpha^2_s, l_s\rangle_\nu + f(s, F(s, X_s(\cdot)), X_s, Z_s, l_s) + (q_s F(s, X_s) - \gamma_s \mathbf{1}_{\mathbb{E}[q_s]>0} X_s) \tag{3.40}$$

Since $q_s \geq 0$ dP a.s., we notice that

$$q_s F(s, X_s) - \gamma_s \mathbf{1}_{\mathbb{E}[q_s]>0} X_s = (q_s F(s, X_s) - \gamma_s X_s)\mathbf{1}_{\mathbb{E}[q_s]>0}$$

By taking expectation at time $t = 0$ on both sides, we can obtain that

$$\mathbb{E}[X_t] = \mathbb{E}^{\mathcal{Q}^\alpha}\left[D^{\beta,\gamma}_{t,T}\xi + \int_t^T D^{\beta,\gamma}_{t,s}[-\beta_s X_s - q_s F(X_s) - \alpha^1_s Z_s - \langle \alpha^2_s, l_s\rangle_\nu + f(s, F(s, X_s(\cdot)), X_s, Z_s, l_s)]\right] ds \\ + \int_t^T \mathbb{E}^{\mathcal{Q}^\alpha}[D^{\beta,\gamma}_{t,s} q_s]\left[F(s, X_s) - \frac{\mathbb{E}^{\mathcal{Q}^\alpha}[X_s D^{\beta,\gamma}_{t,s}\gamma_s]}{\mathbb{E}^{\mathcal{Q}^\alpha}[D^{\beta,\gamma}_{t,s} q_s]}\right]\mathbf{1}_{\mathbb{E}[q_s]>0} ds \tag{3.41}$$

holds for all processes $(\gamma_s, \beta_s, q_s, \alpha^1_s, \alpha^2_s) \in \bar{\mathcal{A}}_T$. Since $\mathcal{Q}^{\bar{\alpha}}$ and P are equivalent measures and $q_s \geq 0$ dP a.s., we have $\mathbf{1}_{\mathbb{E}[q_s]>0} = \mathbf{1}_{\mathbb{E}^{\mathcal{Q}^\alpha}[q_s]>0}$

By the definition of Fenchel-Legendre transform, we have $f(s, F(s, X_s), X_s, Z_s, l_s) - q_s F(s, X_s) - \beta_s X_s - \alpha^1_s Z_s - \langle \alpha^2_s, l_s\rangle_\nu \leq f^*(s, q_s, \beta_s, \alpha^1_s, \alpha^2_s)$ a.s. and $F(s, X_s) - \frac{\mathbb{E}^{\mathcal{Q}^\alpha}[X_s D^{\beta,\gamma}_{t,s}\gamma_s]}{\mathbb{E}^{\mathcal{Q}^\alpha}[D^{\beta,\gamma}_{t,s} q_s]} = F(s, X_s) - \frac{\mathbb{E}[\Gamma^\alpha_s X_s D^{\beta,\gamma}_{t,s}\gamma_s]}{\mathbb{E}[\Gamma^\alpha_s D^{\beta,\gamma}_{t,s} q_s]} \leq F^*(s, \frac{\Gamma^\alpha_s D^{\beta,\gamma}_{t,s}\gamma_s}{\mathbb{E}^{\mathcal{Q}^\alpha}[D^{\beta,\gamma}_{t,s} q_s]})$

Since by assumption $D^{\beta,\gamma}_{t,s} \geq 0$ and $q_s \geq 0$ $d\mathcal{Q}^\alpha$ a.s., we obtain

$$\mathbb{E}X_t \leq \inf_{(\beta,\gamma,q,\alpha)\in\mathcal{A}_T} \mathbb{E}^{\mathcal{Q}^\alpha}\left[D^{\beta,\gamma}_{t,T}\xi + \int_t^T D^{\beta,\gamma}_{t,s} f^*(s, q_s, \beta_s, \alpha^1_s, \alpha^2_s)\right] ds \\ + \int_t^T \mathbb{E}^{\mathcal{Q}^\alpha}[D^{\beta,\gamma}_{t,s} q_s]\left[F^*(s, \frac{\Gamma^\alpha_s D^{\beta,\gamma}_{t,s}\gamma_s}{\mathbb{E}^{\mathcal{Q}^\alpha}[D^{\beta,\gamma}_{t,s} q_s]})\right]\mathbf{1}_{\mathbb{E}^{\mathcal{Q}^\alpha}[q_s]>0} ds \tag{3.42}$$

Recall that for any $(\omega, s) \in \Omega \times [0, T]$, f is Lipschitz, concave in (x', x, z, l), the following conjugacy relation of (f, f^*) holds. Let U be the set introduced in Lemma 3.1, we have $D_s(\omega) \subset U$. Using the same arguments as in Lemma 5.5 in [17], we have

$$f(\omega, s, x', x, z, l) = \inf_{(q,\beta,\alpha^1,\alpha^2)\in\bar{U}} \{f^*(\omega, s, q, \beta, \alpha^1, \alpha^2) + qx' + \beta x + \alpha^1 z + \langle \alpha^2, l\rangle_\nu\} \\ = f^*(\omega, s, \bar{q}, \bar{\beta}, \bar{\alpha}^1, \bar{\alpha}^2) + \bar{q}x' + \bar{\beta}x + \bar{\alpha}^1 z + \langle \bar{\alpha}^2, l\rangle_\nu \tag{3.43}$$

where $\bar{U}$ is the closure of set U, that is the set in which α_2 satisfies $\alpha_2(u) \geq -1$ instead of the strict inequality.

Now since $\bar{U}$ is strongly closed and convex, we obtain there exists $(\bar{q}, \bar{\beta}, \bar{\alpha}^1, \bar{\alpha}^2) \in \bar{U}$ that satisfy (3.43). However Lemma 3.1 gives in fact that $(\bar{q}, \bar{\beta}, \bar{\alpha}^1, \bar{\alpha}^2) \in D_s(\omega)$.

Since, by assumption, $\mathbf{R}^3 \times L^2_\nu$ is separable, we can apply the measurable selection theorem (Appendix of Ch.III [8]) as in the lemma 5.5 of [17] to assert the existence of a predictable processes $(\bar{q}_s, \bar{\beta}_s, \bar{\alpha}^1_s, \bar{\alpha}^2_s)_{s\geq t}$

$$f(s, F(s, X_s(\cdot)), X_s, Z_s, l_s) = \bar{\beta}_s X_s + \bar{q}_s F(X_s) + \bar{\alpha}^1_s Z_s + \langle \bar{\alpha}^2_s, l_s \rangle_\nu + f^*(s, \bar{\beta}_s, \bar{q}_s, \bar{\alpha}^1_s, \bar{\alpha}^2_s) \quad a.s. \tag{3.44}$$

Similarly, since F is Lipschitz and concave, the conjugacy relation also holds for (F, F^*). Given the predictable processes $(X_s, \bar{q}_s, \bar{\beta}_s, \bar{\alpha}_s)_{s\geq t} \in \mathcal{S} \times \bar{\mathcal{A}}_T$, we now introduce $\tilde{q}_s = \bar{q}_s \mathbf{1}_{\mathbb{E}^{\mathcal{Q}^{\bar{\alpha}}}[\bar{q}_s]>0} + C\mathbf{1}_{\mathbb{E}^{\mathcal{Q}^{\bar{\alpha}}}[\bar{q}_s]=0}$ with C the Lipschitz constant of f. We can show (lemma 3.4) there exist a predictable processes $(\bar{\gamma}_s)_{s\geq t} \in \bar{\mathcal{A}}_T$ such that

$$F(s, X_s) - \frac{\mathbb{E}^{\mathcal{Q}^{\bar{\alpha}}}[X_s D^{\bar{\beta},\bar{\gamma}}_{t,s} \bar{\gamma}_s]}{\mathbb{E}^{\mathcal{Q}^{\bar{\alpha}}}[D^{\bar{\beta},\bar{\gamma}}_{t,s} \tilde{q}_s]} = F^*(s, \frac{\Gamma^{\bar{\alpha}}_s D^{\bar{\beta},\bar{\gamma}}_{t,s} \bar{\gamma}_s}{\mathbb{E}^{\mathcal{Q}^{\bar{\alpha}}}[D^{\bar{\beta},\bar{\gamma}}_{t,s} \tilde{q}_s]}) \tag{3.45}$$

And since $\tilde{q}_s = \bar{q}_s$ for any s such that $\mathbb{E}[\bar{q}_s] > 0$, thus we obtain

$$\mathbb{E}X_t = \mathbb{E}^{\mathcal{Q}^{\bar{\alpha}}}\left[D^{\bar{\beta},\bar{\gamma}}_{t,T}\xi + \int_t^T D^{\bar{\beta},\bar{\gamma}}_{t,s} f^*(s, \bar{\beta}_s, q_s, \bar{\alpha}^1_s, \bar{\alpha}^2_s)\right] ds + \int_t^T \mathbb{E}^{\mathcal{Q}^{\bar{\alpha}}}[D^{\bar{\beta},\bar{\gamma}}_{t,s} \bar{q}_s]\left[F^*(s, \frac{\Gamma^{\bar{\alpha}}_s D^{\bar{\beta},\bar{\gamma}}_{t,s} \bar{\gamma}_s}{\mathbb{E}^{\mathcal{Q}^{\bar{\alpha}}}[D^{\bar{\beta},\bar{\gamma}}_{t,s} \bar{q}_s]})\right] \mathbf{1}_{\mathbb{E}^{\mathcal{Q}^{\bar{\alpha}}}[\bar{q}_s]>0} ds \tag{3.46}$$

Together with (3.42), we obtain (3.38).
Finally, (3.44) implies that the processes $f^*(\omega, t, \bar{q}_t, \bar{\beta}_t, \bar{\alpha}^1_t, \bar{\alpha}^2_t)$ belongs to $I\!H^2_T$ since by assumption $(X, Z, l(.)) \in \mathcal{S}^2 \times I\!H^2 \times I\!H^2_\nu$ and $(\bar{q}_t, \bar{\beta}_t, \bar{\alpha}^1_t, \bar{\alpha}^2_t)$ are bounded. $\square$

Remark 3.5 We give an example of the F such that the assumptions in Theorem 3.1 hold. Define $F(t, X) := \mathbb{E}[\phi(t, X)]$ for $X \in L^2(\Omega, P, \mathcal{F}_T)$, where $\phi : [0, T] \times \mathbf{R} \mapsto \mathbf{R}, (t, x) \mapsto \phi(t, x)$ is a concave Lipschitz function such that $\phi(t, X) \in L^2$. Then by conditional Jensen's inequality

$$\mathbb{E}\left[\phi(t, \mathbb{E}[X|\mathcal{F}_s])\right] \geq \mathbb{E}\left[(\mathbb{E}[\phi(t, X)|\mathcal{F}_s])\right] = \mathbb{E}[\phi(t, X)]$$

An example is take $\phi(t, x) = -(x - k_s)^-$ where $(k_s)_{s\geq 0}$ is a deterministic function.

Similarly, for the example in Section 2.2, if we assume that the kernel κ is Lipschitz and convex, then F satisfies the assumptions in Theorem 3.1.

4 Optimal stopping for global dynamic risk measures

We start this section by describing the optimal stopping problem for global dynamic risk measures. For each stopping time $S \in \mathcal{T}_0$, we introduce the value function of the following optimization problem

$$Y_S = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_S} \rho_S(\xi_\tau, \tau) \quad \text{a.s.} \tag{4.47}$$

Our aim is to provide a representation for Y_S in terms of reflected mean-field BSDEs. In order to do this, we give below the definition of a reflected mean-field BSDE.

Definition 4.1 *A process $(Y, Z, k(.), A)$ is said to be a solution of the Mean-field reflected BSDE associated with driver f, operator F and obstacle $\xi_.$ if*

$$(Y, Z, k(.), A) \in \mathcal{S}^2 \times I\!H^2 \times I\!H^2_\nu \times \mathcal{S}^2$$
$$-\,dY_t = f(t, \omega, F(t, X_t(\cdot)), X_t, Z_t, k_t(\cdot))dt + dA_t - Z_t dW_t - \int_{\boldsymbol{E}} k_t(u)\tilde{N}(dt, du); \quad Y_T = \xi_T, \tag{4.48}$$

$Y_t \geq \xi_t,\ 0 \leq t \leq T$ *a.s.,*

A is a nondecreasing RCLL continous process with $A_0 = 0$ and such that

$$\int_0^T (Y_t - \xi_t) dA^c_t = 0 \text{ a.s. and } \Delta A^d_t = -\Delta Y_t\, \mathbf{1}_{\{Y_{t-} = \xi_{t-}\}} \quad \text{a.s.}$$

For $\beta > 0$ and $\phi \in I\!H^{2,T}$, we introduce the norm $\|\phi\|^2_{\beta,T} := E[\int_0^T e^{\beta s}\phi_s^2 ds]$ and for $l \in I\!H^{2,T}_\nu$, we set $\|l\|^2_{\nu,\beta,T} := E[\int_0^T e^{\beta s}\|l_s\|^2_\nu\, ds]$.

We now show an existence and uniqueness result for mean-field reflected BSDEs with jumps, in the general case of RCLL obstacle.

Theorem 4.2 *(Existence and Uniqueness for Mean-field reflected BSDEs)*
Let $\xi_.$ be a RCLL adapted process in $\mathcal{S}^2$ and let f be a Lipschitz driver and F a Lipschitz Mean-field operator. The Mean-field RBSDE (4.48) admits a unique solution $(Y, Z, k(.), A) \in \mathcal{S}^2 \times I\!H^2 \times I\!H^2_\nu \times \mathcal{S}^2$. If (ξ_t) is left-upper semicontinuous (l.u.s.c.) over stopping times, then (A_t) is continuous.

Proof. Denote by $I\!H^2_\beta$ the space $I\!H^2 \times I\!H^2 \times I\!H^2_\nu$ equipped with the norm $\|Y, Z, k(\cdot)\|^2_\beta := \|Y\|^2_\beta + \|Z\|^2_\beta + \|k\|^2_{\nu,\beta}$. We define a mapping Φ from $I\!H^2_\beta$ into itself as follows. Given $(U, V, l) \in I\!H^2_\beta$, let $(Y, Z, k) = \Phi(U, V, l)$ be the the solution of the standard RBSDE associated with driver $f^1(s) := f(s, F(s, U_s(\cdot)), U_s, V_s, l_s)$. Let A be the associated nondecreasing process. The mapping Φ is well defined by Lemma 2.5 [18].

Now we prove that the mapping Φ is a contraction from $I\!H^2_\beta$ into $I\!H^2_\beta$. Let $(\hat{U}, \hat{V}, \hat{l})$ be another element of $I\!H^2_\beta$ and let $(\hat{Y}, \hat{Z}, \hat{k}) := \Phi(\hat{U}, \hat{V}, \hat{l})$, that is, the solution of the RBSDE associated with driver process $f(s, F(s, \hat{U}_s(\cdot)), \hat{U}_s, \hat{V}_s, \hat{l}_s)$.

Set $\bar{U} = U - \hat{U}$, $\bar{V} = V - \hat{V}$, $\bar{\ell} = l - \hat{l}$, $\bar{Y} = Y - \hat{Y}$, $\bar{Z} = Z - \hat{Z}$, $\bar{k} = k - \hat{k}$.

Let $\Delta f_. := f(\cdot, F(s, U_s(\cdot)), U_s, V_s, l_s) - f(\cdot, F(s, \hat{U}_s(\cdot)), \hat{U}_s, \hat{V}_s, \hat{l}_s)$. From Lipschitz continuity of f, we have $\mathbb{E}|\Delta f|^2 = \mathbb{E}|f(\cdot, F(s, U_s(\cdot)), U_s, V_s, l_s) - f(\cdot, F(s, \hat{U}_s(\cdot)), \hat{U}_s, \hat{V}_s, \hat{l}_s)|^2 \leq 4C^2\mathbb{E}[|\overline{U}|^2 + |\overline{V}|^2 + \|\bar{l}\|^2_\nu]$. Here we have used the fact that $|F(s, U_s(\cdot)) - F(s, \hat{U}'_s(\cdot))|^2 \leq C\|U - \hat{U}\|^2_2 = \mathbb{E}|\overline{U}|^2$ Now recall that $\|\Delta f\|^2_\nu = [\int_0^T e^{\beta s}\mathbb{E}|\Delta f|^2\, ds]$. Using estimates (A.58) and (A.59) in [18] with $\eta \leq \frac{1}{2C^2}$ and Lipschitz constant equal to 0 (since the driver f^1 does not depend on the solution), we get

$$\|\bar{Y}\|_\beta^2 + \|\bar{Z}\|_\beta^2 + \|\bar{k}\|_{\nu,\beta}^2 \leq \eta(T+2)\|\Delta f\|_\beta^2 \leq \eta(T+2)4C^2(\|\bar{U}\|_\beta^2 + \|\bar{V}\|_\beta^2 + \|\bar{\ell}\|_{\nu,\beta}^2),$$

Choosing $\eta = \frac{1}{(T+2)8C^2}$, we deduce $\|(\overline{Y}, \overline{Z}, \overline{k})\|_\beta^2 \leq \frac{1}{2}\|(\overline{U}, \overline{V}, \overline{l})\|_\beta^2$. Hence, Φ is a contraction and thus admits a unique fixed point (Y, Z, k) in $I\!H_\beta^2$, which is the solution of RBSDE (4.48). For the second assertion, assuming that ξ is l.u.s.c. over stopping times, we now show that for any predictable stopping time $\tau \in \mathcal{T}_0$, $\Delta A_\tau = 0$ a.s. Since $\Delta A_t^d = -\Delta Y_t \, \mathbf{1}_{\{Y_{t^-} = \xi_{t^-}\}}$, we obtain

$$\Delta A_\tau^d = \mathbf{1}_{\{Y_{\tau^-} = \xi_{\tau^-}\}}(Y_{\tau^-} - Y_\tau)^+ = \mathbf{1}_{\{Y_{\tau^-} = \xi_{\tau^-}\}}(\xi_{\tau^-} - Y_\tau)^+ \leq \mathbf{1}_{\{Y_{\tau^-} = \xi_{\tau^-}\}}(\xi_\tau - Y_\tau)^+$$

The last inequality follows from the inequality $\xi_{\tau^-} \leq \xi_\tau$ a.s. Since $\xi \leq Y$, we derive that $\Delta A_\tau^d \leq 0$ a.s. Thus due to the nondecreasing property of A, we have $\Delta A_\tau^d = 0$ for all predictable stopping time and the continuity of A follows directly.

□

We now give the characterization result of the value function as the solution of a reflected mean-field BSDE.

Theorem 4.3 (Characterization of the value function) *Let $T > 0$ be the terminal time. Let $(\xi_t, 0 \leq t \leq T)$ be an RCLL process in $\mathcal{S}^2$ and let f be a Lipschitz driver satisfying Assumption 2.1 and F a nondecreasing operator. Furthermore, we assume that f is non-decreasing in X. Suppose now that $(Y, Z, k(\cdot), A)$ is the solution of the mean-field reflected BSDE (4.48). Then*

- *For each stopping time $S \in \mathcal{T}_0$, we have*

$$Y_S = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_S} X_S(\xi_\tau, \tau) \quad \text{a.s.} \tag{4.49}$$

  where for $\tau \in \mathcal{T}_S$, $X_\cdot(\xi_\tau, \tau)$ is the solution of the Mean-field BSDE (2.11) associated with terminal time τ, terminal condition ξ_τ, and driver f.

- *For each $S \in \mathcal{T}_0$ and each $\varepsilon > 0$, let τ_S^ε be the stopping time defined by*

$$\tau_S^\varepsilon = \inf\{t \geq S,\ Y_t \leq \xi_t + \varepsilon\}. \tag{4.50}$$

  We have

$$Y_S \;\leq\; X_S(\xi_{\tau_S^\varepsilon}, \tau_S^\varepsilon) + K\varepsilon \quad \text{a.s.}\,, \tag{4.51}$$

  *where $K = K(T, C)$ is a constant which only depends on T and the Lipschitz constant C of f. In other words, τ_S^ε is a $(K\varepsilon)$-*optimal stopping time *for* (4.49).

Proof. The proof is quite standard. For the reader's convenience, we give it below. Let $\tau \in \mathcal{T}_S$. Firstly we note that the process $(Y_s, Z_s, k_s; 0 \leq s \leq \tau)$ is the solution of the mean-field reflected BSDE (4.48) associated with terminal time τ, terminal condition Y_τ, and (generalized) driver

$$f(s, y', y, z, k)ds + dA_s.$$

We have $f(s, y', y, z, k)ds + dA_s \geq f(s, y', y, z, k)ds$ a.s. and $Y_\tau \geq \xi_\tau$ a.s.

Since f satisfies Assumption 2.1 and is nondecreasing in y', the comparison theorem 2.7 for mean-field BSDEs can be applied and gives $Y_s \geq X_s(\xi_\tau, \tau)$, $0 \leq s \leq \tau$ a.s. for each $\tau \in \mathcal{T}_S$. In particular $Y_S \geq X_S(\xi_\tau, \tau)$. By taking the supremum over $\tau \in \mathcal{T}_S$, we derive that

$$Y_S \geq \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_S} X_S(\xi_\tau, \tau) \quad \text{a.s.} \tag{4.52}$$

It remains to show the converse inequality. For each $S \in \mathcal{T}_0$ and for each $\varepsilon > 0$, let τ_S^ε be the stopping time defined by

$$\tau_S^\varepsilon := \inf\{t \geq S,\ Y_t \leq \xi_t + \varepsilon\}. \tag{4.53}$$

Firstly, by the definition of τ_S^ε and the right-continuity of (ξ_t) and (Y_t), we have

$$Y_{\tau_S^\varepsilon} \leq \xi_{\tau_S^\varepsilon} + \varepsilon \quad \text{a.s.}$$

Next we show that the process $(Y_t, S \leq t \leq \tau_S^\varepsilon)$ is the solution of the mean-field BSDE associated with terminal time τ_S^ε, terminal condition $Y_{\tau_S^\varepsilon}$, driver f and Mean-field operator F, that is

$$Y_t = X_t(Y_{\tau_S^\varepsilon}, \tau_S^\varepsilon) \quad S \leq t \leq \tau_S^\varepsilon \ \text{ a.s.} \tag{4.54}$$

We first note that $\tau_S^\varepsilon \in \mathcal{T}_S$. Fix $\varepsilon > 0$. By definition of τ_S^ε, for a.e. ω, if $t \in [S(\omega), \tau_S^\varepsilon(\omega)[$, then $Y_t(\omega) > \xi_t(\omega) + \varepsilon$ and hence $Y_t(\omega) > \xi_t(\omega)$. It follows that for a.e. ω, the function $t \mapsto A_t^c(\omega)$ is constant on $[S(\omega), \tau_S^\varepsilon(\omega)]$ and $t \mapsto A_t^d(\omega)$ is constant on $[S(\omega), \tau_S^\varepsilon(\omega)[$. Also, $Y_{(\tau_S^\varepsilon)^-} \geq \xi_{(\tau_S^\varepsilon)^-} + \varepsilon$ a.s. Since $\varepsilon > 0$, it follows that $Y_{(\tau_S^\varepsilon)^-} > \xi_{(\tau_S^\varepsilon)^-}$ a.s., which implies that $\Delta A^d_{\tau_S^\varepsilon} = 0$ a.s. Hence, the process $(Y_t, S \leq t \leq \tau_S^\varepsilon)$ is a solution of the mean-field BSDE associated with terminal time τ_S^ε, terminal condition $Y_{\tau_S^\varepsilon}$ and driver f. By uniqueness of the solution of Lipschitz mean-field BSDEs, we get (4.54).

Finally we can prove inequality (4.51).

By (4.52) and by the comparison theorem for mean-field BSDEs, we derive that for each $\varepsilon > 0$,

$$Y_S = X_S(Y_{\tau_S^\varepsilon}, \tau_S^\varepsilon) \leq X_S(\xi_{\tau_S^\varepsilon} + \varepsilon, \tau_S^\varepsilon) \quad \text{a.s.} \tag{4.55}$$

Now, by classical estimates on BSDEs in Proposition A4 [17], we have

$$|X_S(\xi_{\tau_S^\varepsilon} + \varepsilon, \tau_S^\varepsilon) - X_S(\xi_{\tau_S^\varepsilon}, \tau_S^\varepsilon)|^2 \leq e^{\beta(T-S)}\varepsilon^2 \quad \text{a.s.}$$

where $\beta := 3C^2 + 4C$. This with (4.55) leads to inequality (4.51). Hence, for each $\varepsilon > 0$,

$$Y_S \ \leq\ X_S(\xi_{\tau_S^\varepsilon}, \tau_S^\varepsilon) + K\varepsilon \ \leq\ \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_S} X_S(\xi_\tau, \tau) + K\varepsilon \quad \text{a.s.} \tag{4.56}$$

where $K := e^{\frac{\beta T}{2}}$. It follows that $Y_S \le \operatorname{ess\,sup}_{\tau \in \mathcal{T}_S} X_S(\xi_\tau, \tau)$ a.s. By (4.52), this inequality is an equality. Moreover, the $K\varepsilon$-optimality property of τ_S^ε follows from (4.56). □

The following stream of results in [18] generalize to the mean field case.

1. The comparison results for Reflected Mean-field BSDE.
2. Given the comparison results, the approximation results for the optimal stopping time.

We note that in a wide variety of papers starting with the seminal paper [9], the key steps for these results are very similar and refer the reader to [18].

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