A dynamic contagion risk model with recovery features

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Abstract

We introduce threshold growth in the classical threshold contagion model, or equivalently a network of Cramér-Lundberg processes in which nodes have downward jumps when there is a failure of a neighboring node. Choosing the configuration model as underlying graph, we prove fluid limits for the baseline model, as well as extensions to the directed case, state-dependent inter-arrival times and the case of growth driven by upward jumps. We obtain explicit ruin probabilities for the nodes according to their characteristics: initial threshold and in- (and out-) degree. We then allow nodes to choose their connectivity by trading off link benefits and contagion risk. We define a rational equilibrium concept in which nodes choose their connectivity according to an expected failure probability of any given link, and then impose condition that the expected failure probability coincides with the actual failure probability under the optimal connectivity. We show existence of an asymptotic equilibrium as well as convergence of the sequence of equilibria on the finite networks. In particular, our results show that systems with higher overall growth may have higher failure probability in equilibrium.

Keywords: Collective risk theory, systemic risk, default contagion, random graphs, interbank network, insurance-reinsurance networks, financial stability.

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1 Introduction

The random graph approach is a tool for systemic risk modeling when uncertainty stems from missing information on linkages. Such is the case for financial networks, see e.g. [7, 22, 24]. Instead of who is connected to whom, only aggregated information at the level of each node is available. One can think of these as node characteristics, and examples include capital, asset or liability size, degree of connectivity. The random graph approach allows one to compute the limit (when the size of the network is large) of the fraction of nodes that fail when a shock propagates. The assumption is that one can categorize nodes according to some of their characteristics, and within each category, nodes are exchangeable. Along this direction, [2] assume that connectivity of each node is known and that the underlying graph is the configuration model, chosen uniformly over all graphs with the prescribed degree sequence. Their exchangeability assumptions on the linkage weights ensure that a limit exists for the fraction of nodes with an initial threshold to contagion. The final fraction of affected nodes is given in closed form for all values of degrees and initial thresholds.

Of course, the configuration model is a graph where cycles become rare in the asymptotic limit, and therefore one has to be cautious about the scope its applicability to financial networks. First, they must be sufficiently large: networks of overlapping portfolios are large networks describing how firms impact each other through common asset holdings [21, 27]. Networks of interbank loans are of the order of thousands of nodes at the level of a large economic zone. Second, because of the presence of cycles in real-world networks, the asymptotic results on failures in the configuration model are interpreted as a lower bound on the real-world potential for contagion. They can serve as a mandate for regulators to collect data on that specific network and assess contagion via intensive computational methods. Without such mandate, data collection is only at aggregate level and existing methods for filling in the gaps are entropy methods, likely to understate contagion, or bayesian methods, computationally intensive even to generate one network compatible with the aggregated values. Threshold models of cascades in large random graphs originate in social network and have been used primarily to study the spread of influence [31, 35, 39]. Rigorous proofs of the asymptotic limits for various model flavors are newer [1, 2, 4, 12, 29, 34]. Existing models however assume that thresholds do not change during the contagion process.

Our main contribution in this paper is to extend the threshold contagion on the configuration model to the case when nodes' thresholds receive growth from the linkages. Because loss from the linkages and growth are intertwined, we call this *the recovery feature* of the threshold. We are motivated by the application to financial and insurance-reinsurance networks. Indeed, in financial networks thresholds represent –depending on the context – either capital or liquidity. An initial set of nodes fail exogenously and affect the nodes connected to them as they default on financial obligations. If those nodes' capital or liquidity is insufficient to absorb the losses, they will fail in turn. In other terms, if the number of failed neighbors reaches a node's threshold, then this node will fail as well, and so on. Since contagion takes time, there is the potential for the capital to recover before the next failure. It is therefore important to introduce a notion of growth.

Networks of insurers and reinsurers are the closest example of what we model here, for several reasons. First, as the recent literature [14, 30, 32] emphasizes, network effects are critical in insurance-reinsurance networks, where reinsurers insure the primary insurers and other reinsurers. Contagion proceeds as failed reinsurers cannot honor contracts to other institutions, and such failures can propagate via chains of reinsurance contracts. Such networks contain several thousand nodes with infrequent reporting, and consequently uncertainty about the reinsurance links, [37]. Therefore asymptotic results on the scope of contagion on a random graph model are appealing. Second, the growth model is particularly adequate, as both primary insurers and reinsurers receive premiums rather deterministically between instances of large losses.

The model we consider in this paper can be seen as a set of Cramér-Lundberg processes living on the nodes of a graph and which interact through the graph links. The capital grows linearly over time. In contrast to the Cramér-Lundberg process, losses do not arrive according to an exogenous Poisson process. Nodes have downward jumps when there is a failure of a neighboring node. When a node's capital or liquidity reaches zero, the node fails and it leads to downward jumps to its own neighbors. The notion of time is also important. Calendar time governs the growth of capital. On the other hand, jumps are governed by the interaction between nodes (specifically between a failed one and one of its neighbors, chosen according to a probability law dictated by the random graph model). There is a natural notion of interaction time and the link revealing filtration. Consequently, jump arrival times have to be translated from interaction time to calendar time. We assume that inter-arrival times are exponentials with mean inversly proportional to the size of the network.

We assume that in each time unit, nodes' growth is proportional to nodes' number of linkages. The linear growth as in the Cramér-Lundberg is also consistent with models in the wider network literature that attribute a fixed reward (respectively cost in some models) to each link as a tradeoff to more contagion risk (respectively network rewards), see [15] and references therein.

We provide three extensions to the baseline model. First, we allow nodes' in- and out-degrees to differ. Second, we allow the inter-arrival times of interactions to depend on the state of the system, and in particular on the number of failed links (out-going links of failed nodes). This essentially provides a self-exciting process for the interactions on the network: the more failed linkages in the system, the higher the intensity with which counterparties are affected. Third, we consider the case when the intensity of the growth process can be prescribed and we provide fluid limits for the number of surviving agents.

Relation to prior literature. For the Cramér-Lundberg process, many extensions have been proposed and there is an extensive machinery for a variety of first passage problems. However the multi-dimensional case is considerably harder. In [19], authors provide Pollaczek-Khinchine type formula for the transform of ruin probabilities in the two-dimensional case. A risk network with a central branch has been proposed recently in [8]. A particular dependence that allows for tractability is when one of the processes models a central branch and another one a subsidiary, and the jumps form the central branch are driven by bailouts of the subsidiary. This particular two-dimensional case allows for approximations via a reduction to the one-dimensional case. Other explicit computations for the two dimensional case are provided in [9–11]. In the case of two-dimensional coupled Levy queues, two-dimensional Laplace transform for the "or" ruin probability is given in [17]. Our coupling of the individual risk processes is by the random graph given by the configuration model. This allows us to give explicit results which are asymptotic and thus quite different in nature.

Parallel to the development of network models for systemic risk, a recent series of works [18, 23, 25, 26, 26] introduced a reduced form approach to systemic risk analysis, based on mean field interaction models. Two works on interacting diffusions stand out as highly relevant to our work [33, 36]. The first one, [36] is closest in spirit. They study diffusions that live on the nodes of an infinite (directed, weighted) and complete graph, with weights that depend on the end nodes' types. There is an underlying stochastic kernel on the type set, and which encodes the networks structure of the model. Nodes impact each other via upward and downward jumps prescribed by the kernel when the processes hit certain barriers. Their network, while deterministic, is connected to the inhomogeneous random graph in [16]. Consistently with past results on contagion in inhomogeneous random graphs [3–5], the interacting particle system exhibits large macroscopic jumps (that we can interpret as cascades) that are linked to the largest eigenvalue of a matrix related to the type kernel.

The type kernel can be controlled by the agents via a mean-field game. While close in spirit, the interactions in our model live on a random graph that is a finite configuration model. We study the scaling limit, when the size of this random graph tends to infinity. The process that lives on the graph nodes is not a diffusion, but a Cramer-Lundberg process, with downward jumps at the failure times. Note that our model also features types, namely the initial thresholds, and can be extended to have more categorization, e.g. by having different in- and out-degrees. Agents control here the connectivity in a rational expectations equilibrium for which we prove convergence as the number of agents tends to infinity.

In [33], the authors study the scaling limit of interacting diffusions, where infractions do occur according to a random graph. While remarkably comprehensive, their model is not intended for cascades and there is no mechanism by which "small" jumps can add to macroscopic effects and our results on the size of a dynamic process on the network could not be obtained using their methodology. Indeed, we could not study the (scaled) limit of the stopping time of the contagion by analyzing the graph within a finite number of hops from a root. Indeed, as the size of the graph grows, so does the stopping time of the cascade.

Outline. The paper is organized as follows. In section 2 we give the main results on the scaled limit of failures when the model allows for recovery. We introduce the problem of choosing connectivity optimally in Section 3 and we give the equilibrium solution in the limit, the convergence of the equilibrium in the finite network as well as a numerical analysis of the equilibrium. Proofs of the main result are provided in Section 4. Finally, in Section 5, we give further results on different extensions.

Notations. We let $\mathbb{N}$ be the set of non-negative integers. For non-negative sequences x_n and y_n, we write $x_n = O(y_n)$ if there exist $N \in \mathbb{N}$ and $C > 0$ such that $x_n \leq C y_n$ for all $n \geq N$, and $x_n = o(y_n)$, if $x_n/y_n \to 0$, as $n \to \infty$.

Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued random variables on a probability space $(\Omega, \mathbb{P})$. If $c \in \mathbb{R}$ is a constant, we write $X_n \xrightarrow{p} c$ to denote that X_n *converges in probability to* c. That is, for any $\epsilon > 0$, we have $\mathbb{P}(|X_n - c| > \epsilon) \to 0$ as $n \to \infty$.

Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers that tends to infinity as $n \to \infty$. We write $X_n = o_p(a_n)$, if $|X_n|/a_n$ *converges to 0 in probability.* Additionally, we write $X_n = O_p(a_n)$, to denote that for any positive sequence $\omega(n) \to \infty$, we have $\mathbb{P}(|X_n|/a_n \geq \omega(n)) = o(1)$. If $\mathcal{E}_n$ is a measurable subset of Ω, for any $n \in \mathbb{N}$, we say that the sequence $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ occurs *with high probability (w.h.p.)* if $\mathbb{P}(\mathcal{E}_n) = 1 - o(1)$, as $n \to \infty$.

$\mathrm{Bin}(k, p)$ denotes a binomial distribution corresponding to the number of successes of a sequence of k independent Bernoulli trials each having probability of success p.

2 Model and results

2.1 A dynamic threshold model of cascades

We consider a system of n nodes (agents) $[n] = \{1, 2, \ldots, n\}$ endowed with a sequence of initial thresholds $(\theta_i)_{i\in[n]}$. To simplify the analysis, we assume that in-degree equals out-degree for each node $i \in [n]$, i.e., $\lambda_i^+ = \lambda_i^- = \lambda_i$. In Section 5.1, we state our main theorem for the case when in- and out-degree are allowed to differ. Given a connectivity λ_i, $i \in [n]$, nodes form links according to the random matching from the configuration model. In the (directed) configuration model with given degree sequence λ_i, $i \in [n]$, each node i is assigned λ_i in-coming half edges and λ_i

outgoing half edges. The (multi)graph results from uniform matching of the in-coming half edges and the outgoing half edges. Under some (usual) regularity conditions, as $n \to \infty$, self loops and parallel edges become rare, and the graph is simple with positive probability [28]. This means that any property that holds with high probability on the configuration model, also holds with high probability conditional on this random graph being simple. This graph is denoted by $\mathcal{G}^{(n)}$ and we write $(i,j) \in \mathcal{G}^{(n)}$ for the event that there's a link between i and j. We let $\mu^{(n)}_{\lambda,\theta}$ be the fraction of nodes with degree λ and threshold θ,

$$\mu^{(n)}_{\lambda,\theta} := \frac{\#\{i \in [n] \mid \theta_i = \theta,\ \lambda_i = \lambda\}}{n}. \tag{1}$$

We assume the following regularity conditions $\mu^{(n)}_{\lambda,\theta} \to \mu_{\lambda,\theta}$, as $n \to \infty$, for some distribution $\mu : \mathbb{N}^2 \to [0,1]$. We also assume that the average connectivity converges to a finite limit, as $n \to \infty$

$$\bar{\lambda}^{(n)} := \sum_{\lambda,\theta} \lambda \mu^{(n)}_{\lambda,\theta} \to \sum_{\lambda,\theta} \lambda \mu_{\lambda,\theta} =: \bar{\lambda} \in (0,\infty). \tag{2}$$

We let the fraction of nodes with threshold θ be defined as

$$\mu^{(n)}_{\theta} := \sum_{\lambda} \mu^{(n)}_{\lambda,\theta}.$$

This network is subject to contagion risk. After the network is formed, a shock occurs. For a set $\mathcal{D}_0$, representing a small fraction of the entire system, the threshold becomes zero meaning that they have failed. This initial set of failed nodes triggers a cascade of failed nodes, as we assume that failed nodes affect the nodes connected to them. Whenever a node's threshold is smaller than the number of failed (in-coming) links, i.e. linkages starting from failed nodes, then it fails due to contagion.

Recovery feature. During the cascade processes, there is a growth feature in the whole system. This feature is captured by introducing the global growth rate (per unit time) $\alpha \cdot n$ for the system with n nodes, where $0 \le \alpha < 1$. We think of $\alpha \cdot n$ is the rate of growth per unit time of the entire capital. We assume this global growth is distributed proportionally to the node's number of links. That is, the threshold of nodes with connectivity λ will grow with rate (per unit time)

$$\alpha \cdot n \cdot \frac{\lambda}{\sum_{\lambda,\theta} \lambda \mu^{(n)}_{\lambda,\theta} n} = \frac{\alpha \lambda}{\bar{\lambda}^{(n)}},$$

where $\sum_{\lambda,\theta} \lambda \mu^{(n)}_{\lambda,\theta} n = n\bar{\lambda}_n$ gives the total number of links in the system.

A very important feature is that only surviving nodes can grow.

We now introduce the dynamic model of contagion. At time 0 nodes in $\mathcal{D}_0$ fail. Each failed link, defined as a link between any node and a failed node, represents an interaction and the number of interactions is always lower than the total number of linkages in the network $\mathcal{G}^{(n)}$. In the dynamic model, we introduce the calendar time and relate it to interaction time. We will study the scaling limit of contagion size and we assume that the total (calendar) time for all interactions is independent of n. Since the number of links scales linearly with n (see (2)) then the average time between interactions must scale with $\frac{1}{n}$. For a system with n nodes, we define $T^{(n)}_k$ the calendar time of the k^{th} interaction and we refer to k to the interaction time.

We assume that the duration in calendar time between the two successive interactions is given by a random variable $\Delta_k^{(n)}$ follows an exponential distribution of parameter n, i.e.,

$$\Delta_k^{(n)} = T_k^{(n)} - T_{k-1}^{(n)} \sim \text{Exp}(n).$$

This reward mechanism in the dynamic case allows the threshold to grow $\frac{\alpha\lambda}{\bar{\lambda}^{(n)}}\,\Delta_k^{(n)}$ between the two interactions. Further extensions will be discussed in Section 5.

The dynamics of interactions is as follows: links that belong to failed nodes are revealed one by one (initially all such links are unrevealed). At each interaction time, a link belonging to a failed node is revealed[1] and the survival condition of the counterparty node is checked according to its current threshold. If the number of failed links of the counterparty exceeds its current threshold, the node fails and its links become unrevealed failed links. The cascade progresses until there are no more unrevealed failed links. Therefore it stops at most after $n\bar{\lambda}^{(n)}$ interactions.

Formally, we let $S_{\lambda,\theta,\ell}^{(n)}(k)$ be the number of surviving nodes with initial threshold θ, λ outgoing links and ℓ failed (incoming) links at time $T_k^{(n)}$. We have that $\mathbf{S}^{(n)}(k) = \left(S_{\lambda,\theta,\ell}^{(n)}(k)\right)_{\lambda,0\le\ell\le\lambda,0\le\theta\le\lambda}$ is a Markov chain and its transitions are given in Section 4.1.

Remark 1 (Threshold at k-th interaction)**.** *It is easy to see that any (surviving) node with λ outgoing links and initial threshold θ will have a threshold*

$$\theta + \alpha\frac{\lambda}{\bar{\lambda}^{(n)}}T_k^{(n)}$$

at the k-th interaction.

The number of failed nodes among those with connectivity λ and initial threshold θ is then

$$D_{\lambda,\theta}^{(n)}(k) = n\mu_{\lambda,\theta}^{(n)} - \sum_{0\le\ell<\lceil\theta+\alpha\frac{\lambda}{\bar{\lambda}^{(n)}}T_k^{(n)}\rceil} S_{\lambda,\theta,\ell}^{(n)}(k).$$

Contagion stops at a time when all failed links have been revealed

$$k_{\text{stop}}^{(n)} = \inf\left\{k = 0, 1, \ldots, n\bar{\lambda}^{(n)} : \sum_{\lambda,\theta}\lambda D_{\lambda,\theta}^{(n)}(k) = k\right\}.$$

We let $\mathcal{D}_f^{(n)}$ the set of failed nodes at the end of the contagion process. The number of failed nodes at the end of the contagion process is thus given by

$$|\,\mathcal{D}_f^{(n)}| = \sum_{\lambda,\theta} D_{\lambda,\theta}^{(n)}(k_{\text{stop}}^{(n)}).$$

2.2 Limit theorem in the case without growth

In the case without recovery feature, i.e. $\alpha = 0$, the asymptotic fraction of failed nodes is characterized by the following theorem:

[1]The choice is uniform among all unrevealed links belonging to failed nodes.

Theorem 2 ([2]). *Let π^* be the smallest fixed point of the map J in $[0,1]$, where*

$$J(\pi) := \sum_{\lambda,\theta} \frac{\lambda\mu_{\lambda,\theta}}{\bar{\lambda}} \cdot B_{\lambda,\theta}(\pi),$$

and,

$$B_{\lambda,\theta}(\pi) := \mathbb{P}(\mathrm{Bin}(\lambda,\pi) \geq \theta) = \sum_{\ell=\theta}^{\lambda} \binom{\lambda}{\ell} \pi^{\ell}(1-\pi)^{\lambda-\ell}.$$

(i) *If $\pi^* = 1$, i.e., if $J(\pi) > \pi$ for all $\pi \in [0,1)$, then asymptotically (as $n \to \infty$) almost all nodes fail during the cascade.*

(ii) *If $\pi^* < 1$ and π^* is a stable fixed point of J, i.e., $J'(\pi^*) < 1$, then the final fraction of failed nodes converges in probability to*

$$\frac{|\mathcal{D}_f^{(n)}|}{n} \xrightarrow{p} \sum_{\lambda,\theta} \mu_{\lambda,\theta} \sum_{\theta} B_{\lambda,\theta}(\pi^*). \tag{3}$$

Furthermore, the asymptotic fraction of surviving nodes with degree λ and initial threshold θ converges in probability (as $n \to \infty$)

$$\frac{S_{\lambda,\theta}^{(n)}(k_{\text{stop}}^{(n)})}{n} \xrightarrow{p} \mu_{\lambda,\theta}\left(1 - B_{\lambda,\theta}(\pi^*)\right).$$

Theorem 2 states that in the asymptotic limit the fraction of failed nodes can be described by means of the binomial distribution (which governs the solution on the infinite Galton-Watson tree): given the probability π^* that a link is failed, then the probability of failure of a node with connectivity λ and threshold θ is approximated by $B_{\lambda,\theta}(\pi^*)$, the probability of failure *as if* the links failure events were independent. For this reason we will refer to π^* as the global failure probability, which is the probability that a link chosen at random leads to a failed node at the end of the cascade process, in the limit when $n \to \infty$. The fraction of failed nodes is given by (42). Note that a fixed point as in theorem always exists as we check that $J(0) \geq 0$ and $J(1) \leq 1$ and the function J is continuous.

2.3 Scaled limit of contagion with recovery features

We can now extend Theorem 2 to the case where the global growth rate is $\alpha > 0$. We will prove that a similar convergence result also holds in this case. The proof is more involved because the threshold at any point in time is no longer constant and equal to θ, but grows at a rate α. Nodes that fail during the contagion process will not benefit from the recovery feature, only surviving nodes can grow at any given time. This presents challenges in the description and the analysis of the system.

In the case without growth, it was sufficient at any time to keep track of the number of failed linkages since failure happens when the number of failed linkages reaches the initial threshold. In contrast, here nodes fail at the first time when the number of failed links reaches the initial threshold plus the growth up to that time, so it is insufficient to keep track only of the current number of failed links. We need to keep track of cumulative failed links process (which is an increasing jump process with jump size one). If this process has ever crossed the threshold (with linear growth), then the node has failed, so the failure of a node is a first passage problem.

Remarkably, one can give a heuristic to compute the probability that a node fails based on a notion of average growth and the notion of global failure probability. The rigorous proof is given below as proof to Theorem 4.

Heuristic of node failure probability computation. We introduce $B^{\alpha}_{\lambda,\theta}(\pi^*)$ as the failure probability of a node with degree λ, threshold θ when the growth rate is α and the global failure probability is π^*. We also introduce $\beta^{\alpha}_{\lambda,\theta,\ell}(\pi^*)$ as the probability that the nodes with connectivity λ, initial threshold θ and under the growth threshold rate α survive when the global failure probability is π^* and have ℓ failed links at the end of cascade. The quantity $B^{\alpha}_{\lambda,\theta}(\pi^*)$ can be calculated by:

$$B^{\alpha}_{\lambda,\theta}(\pi^*) := 1 - \sum_{\ell=0}^{\lambda} \beta^{\alpha}_{\lambda,\theta,\ell}(\pi^*).$$

By Remark 1 the threshold at the end of the cascade is (approximately)

$$\theta + \alpha \frac{\lambda}{\bar{\lambda}^{(n)}} \frac{k^{(n)}_{\text{stop}}}{n}.$$

As we will see, side result of the proof to Theorem 4 is that

$$\frac{k^{(n)}_{\text{stop}}}{\bar{\lambda}^{(n)} n} \xrightarrow{p} \pi^*.$$

The global failure probability π^* can also be thought of as the duration of the contagion in calendar time: the higher the global failure probability, the longer the contagion lasts. In turn, if the cascade lasts for longer then nodes that survive have also recovered for longer. Heuristically, the final threshold $\theta + \alpha\lambda\pi^*$ is the initial threshold plus the growth. A (necessary) condition for the node to survive is that the number of failed links ℓ does not exceed the final threshold. Hence,

$$\beta^{\alpha}_{\lambda,\theta,\ell}(\pi^*) = 0 \quad \text{for} \quad \ell \geq \lceil \theta + \alpha\lambda\pi^* \rceil.$$

Note that $B^{\alpha}_{\lambda,0}(\pi^*) = 1$. This follows by definition, since $\beta_{\lambda,0,\ell}(\pi^*) = 0$. Moreover, we can check that $B^{\alpha}_{\lambda,\theta}(0) = 0$ for $\theta > 0$ (all not initially failed nodes survive). This is due to the following:

(i) If $\ell = 0$, then $\beta^{\alpha}_{\lambda,\theta,\ell}(0) = \binom{\lambda}{0} 0^0 (1-0)^{\lambda} = 1$;

(ii) If $0 < \ell \leq \theta$, then $\beta^{\alpha}_{\lambda,\theta,\ell}(0) = \binom{\lambda}{\ell} 0^{\ell} (1-0)^{\lambda-\ell} = 0$;

(iii) If $\ell > \theta$, then $\beta^{\alpha}_{\lambda,\theta,\ell}(0) = 0$.

In general, when $0 \leq \pi \leq \frac{\ell-\theta}{\alpha\lambda}$, we have $\beta^{\alpha}_{\lambda,\theta,\ell}(\pi) = 0$ (see Theorem 4 (ii)).

We now proceed to give the heuristic for the computation of β. For the case when the number of failed links ℓ is smaller or equal than the initial threshold θ, then the survival probability $\beta^{\alpha}_{\lambda,\theta,\ell}(\pi^*)$ is simply the binomial distribution (the probability to have ℓ failed links)

$$\beta^{\alpha}_{\lambda,\theta,\ell}(\pi^*) = \binom{\lambda}{\ell} (\pi^*)^{\ell} (1-\pi^*)^{\lambda-\ell} \quad \text{for} \quad \ell \leq \theta,$$

since each link is exposed to failure probability π^*. For the case when ℓ is larger than the initial threshold θ, the calculation of $\beta^{\alpha}_{\lambda,\theta,\ell}(\pi^*)$ is more involved.

If the number of failed links is $\ell > \theta$, then the node is definitely failed if the growth cannot cover $\ell - \theta$, i.e., if $\alpha\pi^*\lambda \le \ell - \theta$, which gives

$$\beta^{\alpha}_{\lambda,\theta,\ell}(\pi) = 0 \quad \text{for} \quad 0 \le \pi \le \frac{\ell - \theta}{\alpha\lambda}.$$

In contrast, if $\pi > \frac{\ell-\theta}{\alpha\lambda}$ then we need to make sure that the node survives. Namely, it needs to satisfy the survival condition at the end of contagion and also at each time before. This makes the computation more involved. Remarkably, the solution has a combinatorial representation, that we show in Figure 1. The key point is to identify the critical times when the threshold process could be crossed by the cumulative failed links process. Let t be the current time in the spread of contagion. As we will show, the cascade will end at a time (step) $k^{(n)}_{\text{stop}}$ when all failed linkages have been explored and is related to the global link failure probability π^* by a scaling constant: $t_{\text{stop}} \approx \bar{\lambda}\pi^*$. The longer the contagion lasts the larger the global link failure probability. We note that the threshold process for a node with initial threshold θ and connectivity λ is $\theta + \alpha\frac{\lambda}{\bar{\lambda}}t$, $t \in [0, t_{\text{stop}}]$. Recall that we are computing the survival probability of such a node, *given* that its final number of failed linkages is ℓ. For every $u \in [\theta + 1, \ell]$ we let

$$t_u = t_{\lambda,\theta,u} := \frac{u - \theta}{\alpha\lambda},$$

the scaled time (real time divided by average degree $\bar{\lambda}$) when the node's threshold process is equal to u and thus the node can withstand up to u failed links at this time. In order to ensure that the node survives, we need to check that the number of failed links at time t_ℓ is lower than ℓ. The first case is when this number is $r \le \theta$. In this case, we are sure that the cumulative failed links process has not crossed the threshold process at any time $s \in [0, t_{\lambda,\theta,\ell}]$. Therefore the survival probability in this case is simply the probability that there are r failed linkages between 0 and t_ℓ times the probability that there are $\ell - r$ failed linkages between the time t_ℓ and t_{stop}. The proof of Theorem 4 suggests that the failure probability of a linkage in any time interval is proportional to the length of that interval (with the scaling $\frac{1}{\bar{\lambda}}$).

The second case is when the number of failed links at time t_ℓ is $\theta + u_m$, for some $u_m \in [1, \ell - \theta - 1]$. Then we have a backward recursion by which we determine previous times when we need to check that the crossing has not happened. The previous time when such crossing could have happened is $t_{\theta+u_m}$, since between $t_{\theta+u_m}$ and t_ℓ, the threshold processes is definitely above $\theta + u_m$. Thus we only need to check that the number of failed links at time $t_{\theta+u_m}$ is given by $\theta + u_{m-1}$ for $u_{m-1} < u_m$. By the same reasoning, we need to check at time $t_{\theta+u_{m-1}}$ the number of failed links is given by $\theta + u_{m-2}$ for a $u_{m-2} < u_{m-1}$ and so on, until at time $t_{\theta+u_1}$ we need to check that the number of failed links is $r \le \theta$. Then the survival probability is the product of the probabilities that there are r, $\theta + u_1 - r$ $\ldots$, $u_{m-1} - u_{m-2}$, $u_m - u_{m-1}$ and $\ell - \theta - u_m$ in the respective time intervals $[0, t_{\theta+u_1}]$, $[t_{\theta+u_1}, t_{\theta+u_2}]$ $\ldots$, $[t_{\theta+u_{m-1}}, t_{\theta+u_m}]$, $[t_{\theta+u_m}, t_\ell]$ and finally $[t_\ell, t_{\text{stop}}]$. It is understood that m is a discrete variable which takes values in $[1, \ell - 1 - \theta]$ (this is the number of times it would be possible to cross the threshold process).

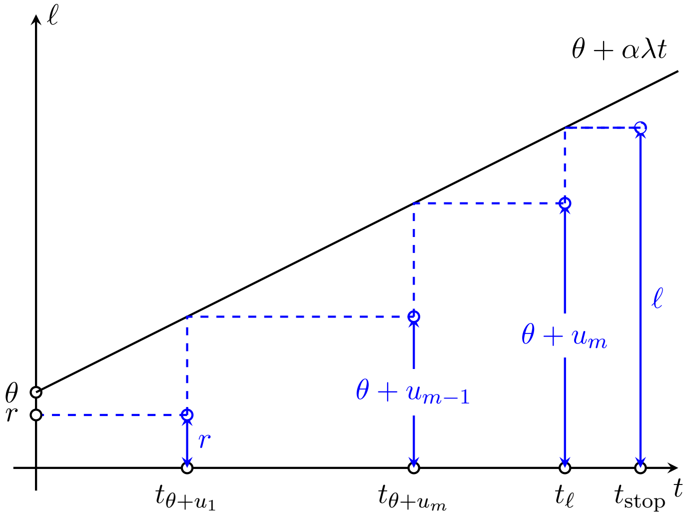

Figure 1: Heuristic of node failure probability computation

It turns out that

$$\frac{\beta^{\alpha}_{\lambda,\theta,\ell}(\pi)}{\binom{\lambda}{\ell}\pi^{\ell}(1-\pi)^{\lambda-\ell}} = \sum_{r=0}^{\theta}\sum_{m=1}^{\ell-\theta-1}\sum_{0<u_1<\ldots<u_m<\ell-\theta}\frac{\ell!}{r!(\theta+u_1-r)!(u_2-u_1)!\ldots(u_m-u_{m-1})!(\ell-\theta-u_m)!}$$
$$\left(\frac{u_1}{\alpha\lambda\pi}\right)^{r}\left(\frac{u_2-u_1}{\alpha\lambda\pi}\right)^{\theta+u_1-r}\left(\frac{u_3-u_2}{\alpha\lambda\pi}\right)^{u_2-u_1}\cdots\left(\frac{u_m-u_{m-1}}{\alpha\lambda\pi}\right)^{u_{m-1}-u_{m-2}}$$
$$\left(\frac{\ell-\theta-u_m}{\alpha\lambda\pi}\right)^{u_m-u_{m-1}}\left(1-\frac{\ell-\theta}{\alpha\lambda\pi}\right)^{\ell-\theta-u_m}.$$

Let $U_1^\pi, U_2^\pi, \ldots, U_\ell^\pi$ be i.i.d. uniform distribution on $[0,\pi]$ and the order statistics be

$$U^\pi_{(1)} \le U^\pi_{(2)} \le \cdots \le U^\pi_{(\ell)}.$$

As the above heuristic arguments suggest, we expect to have

- for $\ell = 0, 1, \ldots, \theta$:
$$\beta^{\alpha}_{\lambda,\theta,\ell}(\pi) = \binom{\lambda}{\ell}\pi^{\ell}(1-\pi)^{\lambda-\ell}.$$
- for $\ell = \theta+1, \ldots, \lambda$:
$$\beta^{\alpha}_{\lambda,\theta,\ell}(\pi) = \binom{\lambda}{\ell}\pi^{\ell}(1-\pi)^{\lambda-\ell}\mathbb{P}\left(U^\pi_{(\theta+1)} > t_{\theta+1}, U^\pi_{(\theta+2)} > t_{\theta+2}, \ldots, U^\pi_{(\ell)} > t_\ell\right),$$
  where $t_u = t_{\lambda,\theta,u} = \frac{u-\theta}{\alpha\lambda}$ for all $u = \theta+1, \ldots, \lambda$.

Let us denote by

$$P_{\lambda,\theta,\ell}(\pi) := \mathbb{P}\left(U^\pi_{(\theta+1)} > t_{\theta+1}, U^\pi_{(\theta+2)} > t_{\theta+2}, \ldots, U^\pi_{(\ell)} > t_\ell\right), \tag{4}$$

for $\ell = \theta+1, \ldots, \lambda$ and $P_{\lambda,\theta,\ell}(\pi) = 1$ for $\ell = 0, 1, \ldots, \theta$.

Remark 3. *The density of* $U^{\pi}_{(\theta+1)}, U^{\pi}_{(\theta+2)}, \ldots, U^{\pi}_{(\ell)}$ *is given by*

$$f(x_{\theta+1}, x_{\theta+2}, \ldots, x_{\ell}) = \frac{\ell!\,(x_{\theta+1})^{\theta}}{\theta!\pi^{\ell}} \mathbb{1}_{\{0<x_{\theta+1}<x_{\theta+2}<\cdots<x_{\ell}<\pi\}}.$$

With the intuition behind the survival probability computation, we can now turn to finding the global link failure probability π^*. Our theorem below shows that this quantity is the solution to the fixed point equation $\pi = J^{\alpha}(\pi)$, where the function J^{α} makes use of the survival probability B^{α}. To see that this fixed point represents the global link failure probability, let us multiply both sides of the fixed point equation with $n\bar{\lambda}$, which represents the total number of links in the network. Then on the lefthand side we have $\pi^* n\bar{\lambda}$ which represents the expected total number of failed links present in the system at the end of the cascade. The number of failed links at the end of the cascade can be also accounted as the sum of expected failed links across different nodes. Indeed, $\lambda n \mu_{\lambda,\theta} B^{\alpha}_{\lambda,\theta}(\pi^*)$ gives the expected number of failed links from nodes with threshold θ and connectivities λ, since $B^{\alpha}_{\lambda,\theta}(\pi^*)$ represents the failure probability of node with threshold θ and connectivities λ while $\lambda n \mu_{\lambda,\theta}$ counts the total number of links belongs to such nodes. Summing up over θ and λ we have that $\sum_{\lambda,\theta} \lambda n \mu_{\lambda,\theta} B^{\alpha}_{\lambda,\theta}(\pi^*)$ also gives the total number of failed links. The fixed point equation $\pi^* = J^{\alpha}(\pi^*)$ states that the second way to account for failed links reaches the same value as the first one.

Theorem 4. *Let* π^* *be the relaxed fixed point of the map* J^{α} *defined as*

$$\pi^* := \min\{\pi \in [0,1] \mid J^{\alpha}(\pi) \leq \pi\},$$

where

$$J^{\alpha}(\pi) := \sum_{\lambda,\theta} \frac{\lambda \mu_{\lambda,\theta}}{\bar{\lambda}} \cdot B^{\alpha}_{\lambda,\theta}(\pi),$$

and for fixed $\alpha, \lambda, \theta, \pi$,

$$B^{\alpha}_{\lambda,\theta}(\pi) := 1 - \sum_{\ell=0}^{\min\{\lceil \theta+\alpha\lambda\pi \rceil -1, \lambda\}} \binom{\lambda}{\ell} \pi^{\ell}(1-\pi)^{\lambda-\ell} P_{\lambda,\theta,\ell}(\pi).$$

We have:

(i) If $\pi^* = 1$*, i.e., if* $J^{\alpha}(\pi) > \pi$ *for all* $\pi \in [0,1)$*, then asymptotically (as* $n \to \infty$*) almost all nodes fail during the cascade.*

(ii) If $\pi^* < 1$ *and* π^* *is a stable fixed point of* J^{α}*, then the final fraction of failed nodes converges in probability to*

$$\frac{|\,\mathcal{D}^{(n)}_f|}{n} \xrightarrow{p} \sum_{\lambda,\theta} \mu_{\lambda,\theta} B^{\alpha}_{\lambda,\theta}(\pi^*). \tag{5}$$

Furthermore, the asymptotic fraction of surviving nodes with degree λ *and initial threshold* θ *converges in probability (as* $n \to \infty$*)*

$$\frac{S^{(n)}_{\lambda,\theta}(k^{(n)}_{\text{stop}})}{n} \xrightarrow{p} \mu_{\lambda,\theta}\left(1 - B^{\alpha}_{\lambda,\theta}(\pi^*)\right).$$

Note that when $\alpha = 0$, we have $B^{0}_{\lambda,\theta}(\pi) = B_{\lambda,\theta}(\pi)$ and we recover the result of Theorem 2.

Remark 5 (Existence of a fixed point)**.** *The relaxed fixed point always exists since $J^\alpha(0) \geq 0$ and $J^\alpha(1) \leq 1$, even if J^α is discontinuous. When the function J^α is continuous on $[0,1)$, the relaxed fixed point is the same as the standard fixed point.*

The main theorem is established by describing the contagion process using a Markov chain of lower dimension than the initial system, in which we aggregate nodes according to their connectivity, threshold, and number of failed counterparties. From the point of view of the evolution of the cascade, the nodes in the same class are exchangeable. We then show that, as the network size increases, the Markov chain rescaled by network size converges in probability to a limit described by a system of ordinary differential equations, which can be solved in closed form. This readily gives us the asymptotic fraction of surviving nodes (in each class of connectivity and threshold) at each time of the cascade spread. The stopping time of the cascade $k_{\text{stop}}^{(n)}$ is the first time when there are no more unexplored failed linkages. We can relate the stopping time of the cascade to the global failure risk captured by π^* and in the sequel we will use this quantity to define the nodes' performance criteria and define their connectivity optimization problem.

3 Agents' optimal connectivity choice in equilibrium

In this section we introduce the problem of choosing connectivity optimally. In choosing their connectivity, agents (nodes) face the following tradeoff: as they add more connectivity, they increase the risk of contagion. At the same time they derive more rewards from their linkages. We capture these rewards in a simple way, by assuming that there exists a numéraire and that surviving nodes with connectivity λ receive λ (units of the numéraire) at the end of the cascade. They receive no reward from their linkages if they fail.

We are now ready to define the nodes' reward. In the case without growth, the analysis of the optimal connectivity is treated in [20]. For its tractability, we keep the same reward definition here and we analyze the effect of growth on the network in equilibrium.

Definition 6 (Nodes' asymptotic reward)**.** *We define the asymptotic reward for a node with degree λ and threshold θ as the expected benefit of linkages, and it is given in the asymptotic limit by*

$$\lambda\big(1 - B^\alpha_{\lambda,\theta}(\pi^*)\big),$$

with π^* given in Theorem 4.

Note that the global failure probability of a link π^* depends on the connectivity choice of all nodes. Therefore, the optimal connectivity is an outcome of an equilibrium.

Of course in the network of size n, nodes with degree λ and threshold θ compute a reward

$$\lambda\left(1 - \frac{D^{(n)}_{\lambda,\theta}(k^{(n)}_{\text{stop}})}{n}\right) = \lambda\big(1 - B^\alpha_{\lambda,\theta}(\pi^*) + o_p(1)\big). \tag{6}$$

We proceed in two steps to determine this equilibrium with the asymptotic criterion. In the first step, we let nodes choose their connectivity according to an expected failure probability π of a link, i.e., a node with threshold θ chooses a connectivity $\lambda_\theta(\pi)$

$$\lambda^*_\theta(\pi) \in \arg\max_\lambda \lambda\big(1 - B^\alpha_{\lambda,\theta}(\pi)\big). \tag{7}$$

In the second step, we will impose an equilibrium condition that the expected failure probability of a link coincides with the actual failure probability of a link, under the optimal connectivity.

It is understood that all other nodes' connectivities and nodes optimize the same asymptotic criterion in all networks of size n. A finite optimizer exists, and as such it can be used to obtain an equilibrium.

Proposition 7 (Existence)**.** *The optimization problem (7) admits a finite optimizer $\lambda_\theta^*(\pi)$.*

Proof. We define $V(\lambda) := \lambda\big(1 - B^\alpha_{\lambda,\theta}(\pi)\big)$ which is (upper) bounded by the following quantity:

$$U(\lambda) := \lambda\Bigg(\sum_{\ell\le\theta+\alpha\lambda\pi}\binom{\lambda}{\ell}(1-\pi)^{\lambda-\ell}\pi^\ell\Bigg).$$

We recognize here that $1 - B^\alpha_{\lambda,\theta}(\pi)$ is bounded by an Incomplete beta function $I_{1-\pi}(\lambda-\theta+1,\theta)$. It follows that

$$U(\lambda) = \lambda I_{1-\pi}(\lambda-\theta-\alpha\lambda\pi, \theta+\alpha\lambda\pi+1).$$

We next recall the following estimates: if $k \le n\pi$, then

$$I_{1-\pi}(n-k,k+1) \le \exp\left(-\frac{(n\pi-k)^2}{2\pi n}\right).$$

Since $\alpha < 1$, there exist λ_0 such that when $\lambda > \lambda_0$, we have $\theta + \alpha\lambda\pi \le \lambda\pi$. This gives

$$V(\lambda) \le U(\lambda) \le \lambda\exp\left(-\frac{(\lambda\pi-\theta-\alpha\lambda\pi)^2}{2\pi\lambda}\right) = \lambda\exp\left(-\frac{(1-\alpha)^2}{2}\lambda\pi + \frac{(1-\alpha)}{2}\theta - \frac{\theta^2}{2\lambda\pi}\right)$$

The righthand side tends to 0 when $\lambda \to +\infty$. Thus the maximizer exists and is finite. □

Remark 8. *The quantity U gives the value function for the same optimization problem for a modified system in which we allow failed nodes to receive rewards from their linkages and even recover, so it is intuitive that $V(\lambda) \le U(\lambda)$. We formally check that $V(\lambda) \le U(\lambda)$ by using the inequality $\beta^\alpha_{\lambda,\theta,\ell}(\pi) \le \binom{\lambda}{\ell}(1-\pi)^{\lambda-\ell}\pi^\ell$.*

We now impose the equilibrium condition that the anticipated global failure probability coincides with the global failure probability given by Theorem 4 when nodes are at their optimal connectivity.

Definition 9 (Equilibrium)**.** *We call $\Big(\pi^*, (\lambda^*_\theta)_{\theta\ge0}\Big)$ a rational expectations equilibrium if*

- *given π^*,*

$$\lambda^*_\theta \in \arg\max_\lambda \lambda\big(1 - B^\alpha_{\lambda,\theta}(\pi^*)\big), \quad \textit{for each } \theta; \tag{8}$$

- *π^* is the smallest solution of the fixed point equation :*

$$\pi^* = \sum_\theta \frac{\lambda^*_\theta\mu_\theta}{\sum_\theta \lambda^*_\theta\mu_\theta} B^\alpha_{\lambda^*_\theta,\theta}(\pi^*). \tag{9}$$

The fixed point equation in the equilibrium definition is derived from the fixed point equation in Theorem 4, where the connectivity is set to λ^*_θ and we let $\mu_\theta = \mu_{\lambda^*_\theta,\theta}$. Combining the two conditions above, we see that in equilibrium the failure probability of a link π^* is the smallest fixed point of the map

$$\phi^\alpha(\pi) := \sum_\theta \frac{\lambda^*_\theta(\pi)\mu_\theta}{\sum_\theta \lambda^*_\theta(\pi)\mu_\theta} \cdot B^\alpha_{\lambda^*_\theta(\pi),\theta}(\pi), \tag{10}$$

where $\lambda^*_\theta(\pi)$ is defined in (7).

Now we study the existence of the equilibrium.

Proposition 10 (Existence of equilibrium)**.** *When $\lambda^*_\theta(\pi)$ is continuous in $\pi \in [0,1)$, the function ϕ^α admits at least one fixed point π^*. When $\lambda^*_\theta(\pi)$ is not continuous in π, the map ϕ^α admits a relaxed fixed point π^* defined as*

$$\pi^* := \min\{\pi \in [0,1) \mid \phi^\alpha(\pi) \le \pi\}.$$

For both cases, we let $\lambda^*_\theta = \lambda^*_\theta(\pi^*)$.

Proof. The proof is immediate: when $\lambda^*_\theta(\pi)$ is continuous in $\pi \in [0,1)$ then the function ϕ^α is continuous on $[0,1)$. It thus admits at least one fixed point since

$$\lim_{\pi\to 1} \phi^\alpha(\pi) \le \sum_\theta \mu_\theta \frac{\lambda^*_\theta(\pi)}{\sum_\theta \lambda^*_\theta(\pi)\mu_\theta} = 1$$

and $\lim_{p\to 0} \phi^\alpha(0) \ge 0$. When $\lambda^*_\theta(\pi)$ is not continuous, the relaxed fixed point always exists since $\phi^\alpha(0) \ge 0$ and $\phi^\alpha(1) \le 1$. □

Proposition 11. *The continuity of the map $\pi \to \lambda^*_\theta(\pi)$ holds under uniqueness of the optimal connectivity $\lambda^*_\theta(\pi)$.*

Proof. Suppose uniqueness holds. For a fixed θ, if $\pi_i \to \pi$, let

$$\lambda_i = \arg\max_\lambda \lambda\big(1 - B^\alpha_{\lambda,\theta}(\pi_i)\big).$$

We may assume $\lambda_i \le K$ for some constant K. Now suppose λ^* is an accumulation point of λ_i, that is, $\lambda_{(i)} \to \lambda$ over some subsequence. Since $\lambda\big(1 - B^\alpha_{\lambda,\theta}(\pi_i)\big)$ is continuous in λ and π_i, we have

$$\lambda^* = \arg\max_\lambda \lambda\big(1 - B^\alpha_{\lambda,\theta}(\pi)\big).$$

In particular, any accumulation point of the sequence π_i is an optimizer. Since the optimizer is unique, the only accumulation point of the sequence λ_i is the optimizer λ^*. Thus we have proved that any subsequence of λ_i has a further subsequence that converges to λ^*, which indicates $\lambda_i \to \lambda^*$. This gives $\lambda(\pi_i) \to \lambda(\pi)$, so the map $\pi \to \lambda^*_\theta(\pi)$ is continuous. □

Remark 12 (Asymptotic Nash equilibrium)**.** *We can relate the equilibrium of Definition 9 to a Nash equilibrium of the following game. Any player with threshold θ_0, given the connectivity $\{\lambda_\theta\}_{\theta\neq\theta_0}$ of the other players, computes their optimal connectivity*

$$\lambda^*_{\theta_0}(\pi) \in \arg\max_\lambda \lambda\big(1 - B^\alpha_{\lambda,\theta_0}(\pi)\big),$$

under the constraint that

$$\sum_{\theta\neq\theta_0}\mu_\theta\frac{\lambda_\theta}{\sum\limits_{\theta\neq\theta_0}\mu_\theta\lambda_\theta+\mu_{\theta_0}\lambda}B^\alpha_{\lambda_\theta,\theta}(\pi)+\mu_{\theta_0}\frac{\lambda}{\sum\limits_{\theta\neq\theta_0}\mu_\theta\lambda_\theta+\mu_{\theta_0}\lambda}B^\alpha_{\lambda,\theta_0}(\pi)=\pi.$$

We can rewrite the above constraint as $\pi = J(\{\lambda^*_\theta\}_{\theta\neq\theta_0},\lambda)$ *for some function* J. *Thus, the optimization criterion can be rewritten as :*

$$\lambda^*_{\theta_0}\in\arg\max_\lambda \lambda\left(1-B^\alpha_{\lambda,\theta_0}(J(\{\lambda^*_\theta\}_{\theta\neq\theta_0},\lambda))\right).$$

In this sense, $(\lambda^*_\theta,\theta\geq 0)$ *is a Nash equilibrium.*

The following theorem establishes that the equilibrium in the network of size n converges to the asymptotic equilibrium.

Theorem 13. *For the network of size* n*, define a rational expectations equilibrium* $\left(T^{*(n)},(\lambda^{*(n)}_\theta,\theta\geq 0)\right)$ *as follows:*

- *Given* $T^{*(n)}$
$$\lambda^{*(n)}_\theta\in\arg\max_\lambda\lambda\left(1-\frac{D^{(n)}_{\lambda,\theta}(T^{*(n)})}{n}\right),\quad \textit{for each } \theta; \tag{11}$$

- $T^{*(n)}$ *is the smallest solution of the fixed point equation :*
$$T^{*(n)}=\sum_\theta\lambda^{*(n)}_\theta\mu^{(n)}_\theta\cdot D^{(n)}_{\lambda^{*(n)},\theta}(T^{*(n)}).$$

Suppose $\lambda^{*(n)}_\theta$ *is unique in (11). If we define* $\pi^{*(n)}:=\frac{T^{*(n)}}{\sum_\theta\lambda^{*(n)}_\theta\mu^{(n)}_{\lambda,\theta}n}$*, then we have (as* $n\to\infty$*)*

$$\pi^{*(n)}\longrightarrow\pi^*\quad \textit{and}\quad \lambda^{*(n)}_\theta\longrightarrow\lambda^*_\theta. \tag{12}$$

Proof. Firstly we notice that if $\left(T^{*(n)},(\lambda^{*(n)}_\theta,\theta\geq 0)\right)$ is the standard equilibrium of (11), then $\pi^{*(n)}$ is the fixed point of the map

$$\phi^{(n)}(\pi):=\sum_\theta\frac{\lambda^{*(n)}_\theta(\pi)\mu^{(n)}_\theta}{\sum_\theta\lambda^{*(n)}_\theta(\pi)\mu^{(n)}_{\lambda,\theta}}\cdot\frac{D^{(n)}_{\lambda^{*(n)}_\theta,\theta}\left(\lambda^{*(n)}_\theta(\pi)\mu^{(n)}_\theta n\pi\right)}{n},\quad \text{for } \pi\in[0,1]. \tag{13}$$

By the continuity of arg max, we have $\lambda^{*(n)}_\theta(\pi)\longrightarrow\lambda^*_\theta(\pi)$ as $n\to\infty$. By the uniform convergence results (30), we obtain

$$\frac{D^{(n)}_{\lambda^{*(n)}_\theta,\theta}\left(\lambda^{*(n)}_\theta(\pi)\mu^{(n)}_\theta n\pi\right)}{n}\xrightarrow{p}B^\alpha_{\lambda^*_\theta,\theta}(\pi).$$

These give $\phi^{(n)}(\pi)\xrightarrow{p}\phi(\pi)$ as $n\to\infty$ for $\pi\in[0,1]$. Obviously we have $|\phi^{(n)}(\pi)|\leq 1$ for each n. If we furthermore assume the sequence $\{\phi^{(n)}\}_{n\in\mathbb{N}}$ is equicontinuous, i.e. for each $\epsilon>0$ there is a $\delta>0$ such that

$$|\phi^{(n)}(\pi_1)-\phi^{(n)}(\pi_2)|<\epsilon \tag{14}$$

whenever $|\pi_1 - \pi_2| < \delta$ for all functions $\phi^{(n)}$ in the sequence. Then by Arzela-Ascoli theorem, the sequence $\{\phi^{(n)}\}_{n\in\mathbb{N}}$ contains a uniformly convergent subsequence $\{\phi^{(n_k)}\}_{k\in\mathbb{N}}$. Now suppose $\{\pi^{(n_k)}\}_{k\in\mathbb{N}}$ are the fixed points of the maps $\{\phi^{(n_k)}\}_{k\in\mathbb{N}}$. i.e. $\phi^{(n_k)}(\pi^{(n_k)}) = 0$. Since $\{\pi^{(n_k)}\}_{k\in\mathbb{N}}$ are bounded in $[0,1]$, π^* is an accumulation point of $\pi^{(n_k)}$, that is, $\pi^{(n_k)}_{(i)} \to \pi^*$ over some subsequence. Taking limit of both sides on subsequence, since $\phi^{(n)}$ is continuous in π, we obtain $\phi(\pi^*) = 0$. In particular, any accumulation point of the sequence $\pi^{(n_k)}_{(i)}$ is a zero point. Since the zero point is unique, the only accumulation point of the sequence $\pi^{(n_k)}$ is the optimizer π^*. Thus we have proved that any subsequence of $\pi^{(n_k)}$ has a further subsequence that converge to π^*, which implies $\pi^{(n_k)} \to \pi^*$. □

Numerical analysis of equilibrium. While the equilibrium connectivity cannot be given in closed form, it can be efficiently investigated numerically as it benefits of the simple closed form equations for the probability of survival and the fixed point equation for the global link failure probability. We now investigate how the resulting equilibrium depends on the initial distribution of the threshold θ and on the growth rate α. We consider the global link failure probability emerging in equilibrium. We assume that nodes initial thresholds θ are randomly distributed over a given range $[0, \theta_{max}]$. We keep the mean constant and we change the variance of this distribution. We assume a Gaussian distribution with mean 15 and standard deviation $\sigma \in [1, 6]$ (we then condition on the interval $[0, 30]$ and take the integer part). As we vary σ from 1 to 6 we have more heterogeneity in the initial threshold. We find that for both the cases with growth ($\alpha > 0$) and the cases without growth ($\alpha = 0$) the link failure probability in equilibrium decreases as thresholds become more heterogeneous, see Figure 2. As agents become more dissimilar in terms of their thresholds, the system is more diversified and turns out more robust. This is true even as larger standard deviation in the initial distribution of θ give raise to larger average connectivities in equilibrium.

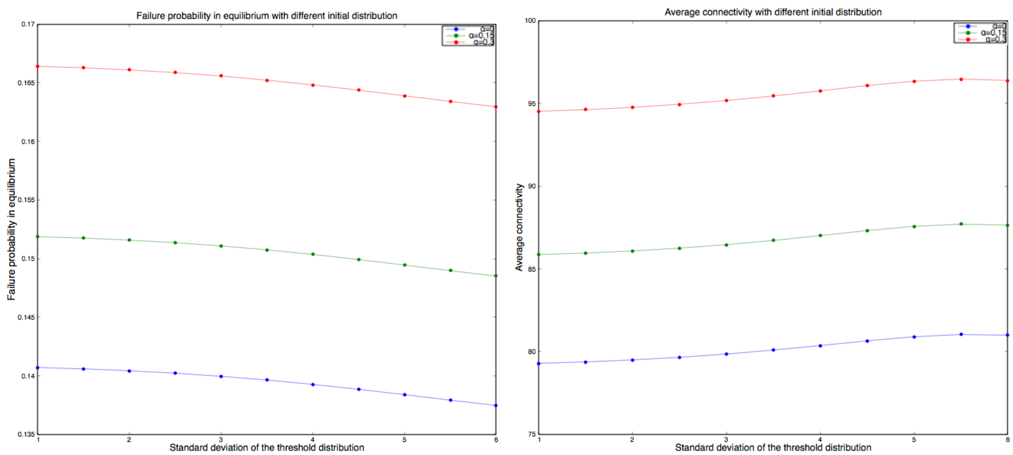

Figure 2: Failure probability and average connectivity in equilibrium with different initial distribution of θ on $[0, 30]$. As the standard deviation of θ increases, the failure probability in equilibrium drops.

We now fix the variance of the initial threshold distribution to $\sigma = 4$ and we vary the growth rate $\alpha \in [0, 0.35]$. We find the link failure probability in equilibrium and final fraction of failed nodes in equilibrium (see Theorem 4(ii)). Both increase as the growth rate increases, see Figure 3. This unintuitive result can be explained as follows. When the system has higher growth, agents may engage in over-lending (the equilibrium connectivity λ^*_θ increases). The growth effect on the thresholds can be outweighed by the increase in connectivity throughout the system, and more instability can ensue.

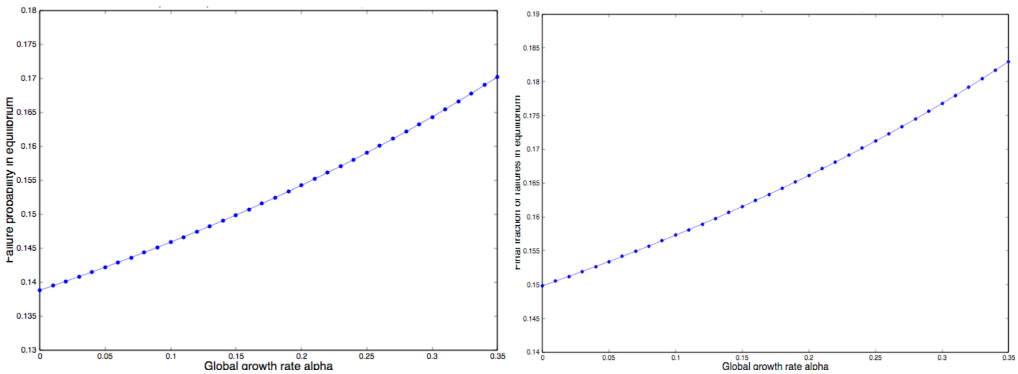

Figure 3: Failure probability and final fraction of failed nodes in equilibrium with different global growth rates. As α increases, the failure probability and the final fraction of failures in equilibrium increases.

4 Proofs and asymptotic results

In this section, we present the proof of Theorem 4. In [2], the authors extend the differential equation method of [40] to show that as the network size increases, the rescaled Markov chains that describe the contagion converge in probability to a limit described by a system of ordinary differential equations. The case solved there corresponds to the zero growth case $\alpha = 0$ of this paper. Here, with non-zero growth, the size of the state space also increases with time as the threshold increases. We show similar convergence results to a limit described by a more involved system of ordinary differential equations for the case $\alpha > 0$ and we obtain an analytical result on the final fraction of failed nodes in the network. The convergence result is shown in section 4.3.

4.1 Markov chain transitions

The set of survived nodes in different states $\mathbf{S}_n(k) = \left(S^{(n)}_{\lambda,\theta,\ell}(k)\right)_{\lambda, 0\le\theta\le\lambda, 0\le\ell\le\lambda}$ represents a Markov chain whose transition probabilities are as in [2]. The key difference is that the number of possible solvent states changes with time as nodes' threshold grows, in particular it is possible the have surviving nodes with ℓ failed links for $0 < \ell < \theta + \frac{\alpha}{\bar{\lambda}^{(n)}} \cdot \lambda \cdot T^{(n)}_k$.

Formally, we let $\Delta^{(n)}_k$ be the difference operator: $\Delta^{(n)}_k S := S^{(n)}(k+1) - S^{(n)}(k)$. We let $\mathcal{F}^{(n)}_k$ the natural filtration of the Markov chain. We obtain the following equations for the expectation of $\mathbf{S}_n(k+1)$, conditional on $\mathcal{F}^{(n)}_k$, by averaging over the possible transitions:

$$\mathbb{E}[\Delta^{(n)}_k S^{(n)}_{\lambda,\theta,0} \mid \mathcal{F}^{(n)}_k] = -\left(\frac{\lambda}{\bar{\lambda}^{(n)} n - k}\right) S^{(n)}_{\lambda,\theta,0}(k),$$

$$\mathbb{E}[\Delta^{(n)}_k S^{(n)}_{\lambda,\theta,\ell}(k) \mid \mathcal{F}^{(n)}_k] = \left(\frac{\lambda-\ell+1}{\bar{\lambda}^{(n)} n - k}\right) S^{(n)}_{\lambda,\theta,\ell-1}(k) - \left(\frac{\lambda-\ell}{\bar{\lambda}^{(n)} n - k}\right) S^{(n)}_{\lambda,\theta,\ell}(k), \quad 0 < \ell < \theta + \frac{\alpha}{\bar{\lambda}^{(n)}} \cdot \lambda \cdot T^{(n)}_k,$$

$$\mathbb{E}[\Delta^{(n)}_k S^{(n)}_{\lambda,\theta,\ell}(k) \mid \mathcal{F}^{(n)}_k] = \left(\frac{\lambda-\ell+1}{\bar{\lambda}^{(n)} n - k}\right) S^{(n)}_{\lambda,\theta,\ell-1}(k), \quad \ell = \theta + \frac{\alpha}{\bar{\lambda}^{(n)}} \cdot \lambda \cdot T^{(n)}_k,$$

$$S^{(n)}_{\lambda,\theta,\ell} = 0, \quad \ell \ge \theta + \frac{\alpha}{\bar{\lambda}^{(n)}} \cdot \lambda \cdot T^{(n)}_k.$$

The initial condition is

$$S^{(n)}_{\lambda,\theta,\ell}(0) = \mu^{(n)}_{\lambda,\theta} n \mathbb{1}(\ell = 0)\mathbb{1}(0 < \theta \le \lambda).$$

The last two equations capture the feature that failed nodes will not benefit from the threshold growth. Recall that the number of failed nodes among those with connectivity λ and initial threshold θ is

$$D^{(n)}_{\lambda,\theta}(k) = n\mu^{(n)}_{\lambda,\theta} - \sum_{0\le\ell<\lceil\theta+\alpha\frac{\lambda}{\bar\lambda^{(n)}}T^{(n)}_k\rceil} S^{(n)}_{\lambda,\theta,\ell}(k),$$

and contagion stops at a time when all failed links have been revealed

$$k^{(n)}_{\text{stop}} = \inf\{k = 0, 1, \ldots, n\bar\lambda^{(n)} : \sum_{\lambda,\theta} \lambda D^{(n)}_{\lambda,\theta}(k) = k\}. \tag{15}$$

Indeed, the number of (alive) outgoing half-edges belonging to failed nodes at time k is

$$D^{(n)}_{\text{out}} = \sum_{\lambda,\theta} \lambda D^{(n)}_{\lambda,\theta}(k) - k$$

and $k^{(n)}_{\text{stop}}$ is the first time that $D^{(n)}_{\text{out}} = 0$. The number of failed nodes at the end of the contagion process is thus given by

$$|\mathcal{D}^{(n)}_f| = \sum_{\lambda,\theta} D^{(n)}_{\lambda,\theta}(k^{(n)}_{\text{stop}}).$$

We introduce (for fixed $\lambda, \theta, \ell \le \lambda$)

$$\hat{k}^{(n)}_{\lambda,\theta,\ell} := \inf\left\{k \mid \ell \le \left(\theta + \frac{\alpha\lambda}{\bar\lambda_n} \cdot T^{(n)}_k\right)\right\} \wedge \lambda \tag{16}$$

as the first interaction time when a node starting with threshold θ has accumulated enough growth to withstand $\ell \le \lambda$ failed links. Clearly, for $\ell \le \theta$, the node can withstand the failed links using only the initial threshold, so we have $\hat{k}^{(n)}_{\lambda,\theta,\ell} = 0$ for $\ell \le \theta$.

The above transition probabilities equations can be rewritten as following

$$\begin{cases} \mathbb{E}[\Delta^{(n)}_k S^{(n)}_{\lambda,\theta,0} \mid \mathcal{F}^{(n)}_k] = -\left(\frac{\lambda}{\bar\lambda^{(n)}n-k}\right) S^{(n)}_{\lambda,\theta,0}(k), & \\ \mathbb{E}[\Delta^{(n)}_k S^{(n)}_{\lambda,\theta,\ell} \mid \mathcal{F}^{(n)}_k] = \left(\frac{\lambda-\ell+1}{\bar\lambda^{(n)}n-k}\right) S^{(n)}_{\lambda,\theta,\ell-1}(k) - \left(\frac{\lambda-\ell}{\bar\lambda^{(n)}n-k}\right) S^{(n)}_{\lambda,\theta,\ell}(k) & \text{for } k \ge \hat{k}^{(n)}_{\lambda,\theta,\ell}, \\ S^{(n)}_{\lambda,\theta,\ell}(k) = 0 & \text{for } k < \hat{k}^{(n)}_{\lambda,\theta,\ell}, \end{cases} \tag{SDE}$$

with initial condition

$$S^{(n)}_{\lambda,\theta,\ell}(0) = \mu^{(n)}_{\lambda,\theta} n \mathbb{1}(\ell = 0)\mathbb{1}(0 < \theta \le \lambda).$$

4.2 Fluid limit

We show that the Markov chain admits a fluid limit, namely that rescaled by the size of the network n it is close to the solution of the ordinary differential equations stated below.

We define the following set of differential equations:

$$\begin{cases} \frac{ds_{\lambda,\theta,0}}{dt}(t) = -(\frac{\lambda}{\bar\lambda-t}) s_{\lambda,\theta,0}(t), & \\ \frac{ds_{\lambda,\theta,\ell}}{dt}(t) = (\frac{\lambda-\ell+1}{\bar\lambda-u}) s_{\lambda,\theta,\ell-1}(t) - (\frac{\lambda-\ell}{\bar\lambda-t}) s_{\lambda,\theta,\ell}(t) & \text{for } t \ge \hat{t}_{\lambda,\theta,\ell}, \\ s_{\lambda,\theta,\ell}(t) = 0 & \text{for } t < \hat{t}_{\lambda,\theta,\ell}, \end{cases} \tag{DE}$$

with initial conditions

$$s_{\lambda,\theta,\ell}(0) = \mu_{\lambda,\theta}\mathbb{1}(\ell = 0)\mathbb{1}(0 < \theta \le \lambda),$$

where for $\theta \le \ell \le \lambda$,

$$\hat{t}_{\lambda,\theta,\ell} = \frac{(\ell - \theta)\bar{\lambda}}{\alpha\lambda}, \tag{17}$$

and $\hat{t}_{\lambda,\theta,\ell} = 0$ for $0 \le \ell \le \theta$.

Lemma 14. *The system of ordinary differential equations (DE) admits the unique solution*

$$\mathbf{s}(t) = (s_{\lambda,\theta,\ell}(t))_{\lambda,0\le\theta\le\lambda,0\le\ell\le\lambda}$$

in the interval $0 \le t < \bar{\lambda}$,*with*

- *for* $\ell = 0, 1, \ldots, \theta$*:*
$$s_{\lambda,\theta,\ell}(t) = \mu_{\lambda,\theta}\binom{\lambda}{\ell}\left(\frac{t}{\bar{\lambda}}\right)^{\ell}\left(1 - \frac{t}{\bar{\lambda}}\right)^{\lambda-\ell}. \tag{18}$$

- *for* $\ell = \theta + 1, \ldots, \lambda$*:*
$$s_{\lambda,\theta,\ell}(t) = \mu_{\lambda,\theta}\binom{\lambda}{\ell}\left(\frac{t}{\bar{\lambda}}\right)^{\ell}\left(1 - \frac{t}{\bar{\lambda}}\right)^{\lambda-\ell}\mathbb{P}\left(U^t_{(\theta+1)} > \hat{t}_{\lambda,\theta,\theta+1}, U^t_{(\theta+2)} > \hat{t}_{\lambda,\theta,\theta+2}, \ldots, U^t_{(\ell)} > \hat{t}_{\lambda,\theta,\ell}\right),$$
where $U^t_{(1)} \le U^t_{(2)} \le \cdots \le U^t_{(\ell)}$ *denotes the order statistics of* ℓ *i.i.d. uniformly distributed random variables on* $[0, t]$*.*

The solution of the corresponding DE without growth is given by [2, Lemma 5.8]. Here the solution is significantly more involved, and we have to proceed piecewise and setting the initial condition at successive times $\hat{t}_{\lambda,\theta,\theta+k}$ (the first time when the threshold has grown from θ to $\theta + k$).

Proof. Let $\tau = \tau(t) = -\ln(\bar{\lambda} - t)$. Then $\tau(0) = -\ln(\bar{\lambda})$, τ is strictly increasing and so is the inverse function $t = t(\tau)$. We write the system of differential equations (DE) with respect to τ:

$$\begin{aligned}
s'_{\lambda,\theta,0}(\tau) &= -\lambda s_{\lambda,\theta,0}(\tau), \\
s'_{\lambda,\theta,\ell}(\tau) &= (\lambda - \ell + 1)s_{\lambda,\theta,\ell-1}(\tau) - (\lambda - \ell)s_{\lambda,\theta,\ell}(\tau) && \text{for } \tau \ge \hat{\tau}_{\lambda,\theta,\ell}, \\
s_{\lambda,\theta,\ell}(\tau) &= 0 && \text{for } \tau < \hat{\tau}_{\lambda,\theta,\ell},
\end{aligned}$$

where for $\theta + 1 \le \ell \le \lambda$,

$$\hat{\tau}_{\lambda,\theta,\ell} = -\ln\left(\bar{\lambda} - \frac{(\ell - \theta)\bar{\lambda}}{\alpha\lambda}\right),$$

and $\hat{\tau}_{\lambda,\theta,\ell} = \tau(0) = -\ln(\bar{\lambda})$ for $0 \le \ell \le \theta$. Then we have (for $\ell \ge 1$)

$$\frac{d}{d\tau}\left(e^{(\lambda-\ell)(\tau-\tau(0))}s_{\lambda,\theta,\ell}(\tau)\right) = (\lambda - \ell)e^{(\lambda-\ell)(\tau-\tau(0))}s_{\lambda,\theta,\ell-1}(\tau)\mathbb{1}(\tau \ge \hat{\tau}_{\lambda,\theta,\ell}). \tag{19}$$

(i) For the case $0 \leq \ell \leq \theta$, similar to [2, Lemma 5.8], by induction we find

$$s_{\lambda,\theta,\ell}(\tau) = e^{-(\lambda-\ell)(\tau-\tau(0))} \sum_{r=0}^{\ell-1} \binom{\lambda-r}{\ell-r} \left(1 - e^{-(\tau-\tau(0))}\right)^{\ell-r} s_{\lambda,\theta,r}(\tau(0)). \tag{20}$$

By going back to t, we have

$$s_{\lambda,\theta,\ell}(t) = \left(1 - \frac{t}{\bar{\lambda}}\right)^{\lambda-\ell} \sum_{r=0}^{\ell-1} s_{\lambda,\theta,r}(0) \binom{\lambda-r}{\ell-r} \left(\frac{t}{\bar{\lambda}}\right)^{\ell-r}.$$

Then, by using the initial conditions, we find (for $\theta > 0$)

$$s_{\lambda,\theta,\ell}(t) = \mu_{\lambda,\theta} \binom{\lambda}{\ell} \left(\frac{t}{\bar{\lambda}}\right)^{\ell} \left(1 - \frac{t}{\bar{\lambda}}\right)^{\lambda-\ell}.$$

(ii) Consider the case $\ell = \theta + k$ for $k \geq 1$. We proceed the proof by induction on k. For $k = 1$, when $\ell = \theta + 1$, by Equation 19, we have

$$\frac{d}{d\tau}\left(e^{(\lambda-\theta-1)(\tau-\tau(0))} s_{\lambda,\theta,\theta+1}(\tau)\right) = (\lambda-\theta-1) e^{(\lambda-\theta-1)(\tau-\tau(0))} s_{\lambda,\theta,\theta}(\tau) \mathbb{1}(\tau \geq \hat{\tau}_{\lambda,\theta,\theta+1}).$$

For $\tau \geq \hat{\tau}_{\lambda,\theta,\theta+1}$, by setting $\hat{\tau}_{\lambda,\theta,\theta+1}$ as the initial time, we have (similar to Equation 20)

$$s_{\lambda,\theta,\theta+1}(\tau) = e^{-(\lambda-\theta-1)(\tau-\hat{\tau}_{\lambda,\theta,\theta+1})} \sum_{r=0}^{\theta} \binom{\lambda-r}{\ell-r} \left(1 - e^{-(\tau-\hat{\tau}_{\lambda,\theta,\ell})}\right)^{\ell-r} s_{\lambda,\theta,r}(\hat{\tau}_{\lambda,\theta,\theta+1}).$$

By going back to t, we find for $\ell = \theta + 1$ and $\hat{t}_{\lambda,\theta,\ell} \leq t < \bar{\lambda}$,

$$\begin{aligned}
s_{\lambda,\theta,\ell}(t) &= \left(\frac{\bar{\lambda}-t}{\bar{\lambda}-\hat{t}_{\lambda,\theta,\ell}}\right)^{\lambda-\ell} \sum_{r=0}^{\theta} \binom{\lambda-r}{\ell-r} \left(\frac{t-\hat{t}_{\lambda,\theta,\ell}}{\bar{\lambda}-t}\right)^{\ell-r} s_{\lambda,\theta,r}(\hat{t}_{\lambda,\theta,\ell}) \\
&= \left(\frac{\bar{\lambda}-t}{\bar{\lambda}-\hat{t}_{\lambda,\theta,\ell}}\right)^{\lambda-\ell} \sum_{r=0}^{\theta} \binom{\lambda-r}{\ell-r} \left(\frac{t-\hat{t}_{\lambda,\theta,\ell}}{\bar{\lambda}-\hat{t}_{\lambda,\theta,\ell}}\right)^{\ell-r} \mu_{\lambda,\theta} \binom{\lambda}{r} \left(\frac{\hat{t}_{\lambda,\theta,\ell}}{\bar{\lambda}}\right)^{r} \left(\frac{\bar{\lambda}-\hat{t}_{\lambda,\theta,\ell}}{\bar{\lambda}}\right)^{\lambda-r} \\
&= \mu_{\lambda,\theta} \binom{\lambda}{\ell} \left(\frac{t}{\bar{\lambda}}\right)^{\ell} \left(1-\frac{t}{\bar{\lambda}}\right)^{\lambda-\ell} \frac{1}{t^\ell} \sum_{r=0}^{\theta} \binom{\ell}{r} (t-\hat{t}_{\lambda,\theta,\ell})^{\ell-r} \hat{t}^{r}_{\lambda,\theta,\ell} \\
&= \mu_{\lambda,\theta} \binom{\lambda}{\ell} \left(\frac{t}{\bar{\lambda}}\right)^{\ell} \left(1-\frac{t}{\bar{\lambda}}\right)^{\lambda-\ell} \frac{t^\ell - \hat{t}^{\ell}_{\lambda,\theta,\ell}}{t^\ell} \\
&= \mu_{\lambda,\theta} \binom{\lambda}{\ell} \left(\frac{t}{\bar{\lambda}}\right)^{\ell} \left(1-\frac{t}{\bar{\lambda}}\right)^{\lambda-\ell} \mathbb{P}\left(U^t_{(\theta+1)} > \hat{t}_{\lambda,\theta,\theta+1}\right),
\end{aligned}$$

since (for $\ell = \theta + 1$)

$$\mathbb{P}\left(U^t_{(\ell)} > \hat{t}\right) = 1 - \mathbb{P}\left(U^t_1 \leq \hat{t}, U^t_2 \leq \hat{t}, , \ldots, U^t_\ell \leq \hat{t}\right) = 1 - \left(\frac{\hat{t}}{t}\right)^{\ell} = \frac{t^\ell - \hat{t}^\ell}{t^\ell}.$$

Suppose now that the statement is true up to $\ell = \theta + k$. We now proceed the proof for $\ell = \theta + k + 1$. Let us denote by

$$P_{\lambda,\theta,\ell}(t) := \mathbb{P}\left(U^t_{(\theta+1)} > \hat{t}_{\lambda,\theta,\theta+1}, U^t_{(\theta+2)} > \hat{t}_{\lambda,\theta,\theta+2}, \ldots, U^t_{(\ell)} > \hat{t}_{\lambda,\theta,\ell}\right), \tag{21}$$

for $\ell = \theta + 1, \ldots, \lambda$ and $P_{\lambda,\theta,\ell}(t) = 1$ for $\ell = 0, 1, \ldots, \theta$.

Lemma 15. *For all $\ell \geq \theta+1$, the following recursive identity holds:*

$$P_{\lambda,\theta,\ell}(t) = \frac{1}{t^\ell}\sum_{r=0}^{\ell-1}\binom{\ell}{r}\left(t-\hat{t}_{\lambda,\theta,\ell}\right)^{\ell-r}\hat{t}^r_{\lambda,\theta,\ell}P_{\lambda,\theta,r}(\hat{t}_{\lambda,\theta,\ell}).$$

Proof. Let $U_1^t, U_2^t, \ldots, U_\ell^t$ be i.i.d. uniformly distributed random variables on $[0,t]$. Then $\mathbb{P}(U_i^t \leq \hat{t}_{\lambda,\theta,\ell}) = \frac{\hat{t}_{\lambda,\theta,\ell}}{t}$. We write $P_{\lambda,\theta,\ell}(t)$ by conditioning on the number of points in $[0,\hat{t}_{\lambda,\theta,\ell}]$. Let $\mathcal{A}_r$ be the event that there are exactly r points in $[0,\hat{t}_{\lambda,\theta,\ell}]$. Then we have (for $r=0,\ldots,\ell-1$)

$$\mathbb{P}(\mathcal{A}_r) = \binom{\ell}{r}\left(\frac{t-\hat{t}_{\lambda,\theta,\ell}}{t}\right)^{\ell-r}\left(\frac{\hat{t}_{\lambda,\theta,\ell}}{t}\right)^r.$$

Moreover, since $\hat{t}_{\lambda,\theta,\ell}$ is increasing in ℓ and $(U_i^t \mid U_i^t \leq \hat{t}_{\lambda,\theta,\ell})$ is uniformly distributed on $[0,\hat{t}_{\lambda,\theta,\ell}]$, given there are r points in $[0,\hat{t}_{\lambda,\theta,\ell}]$ we have

$$\mathbb{P}\left(U^t_{(\theta+1)} > \hat{t}_{\lambda,\theta,\theta+1}, U^t_{(\theta+2)} > \hat{t}_{\lambda,\theta,\theta+2}, \ldots, U^t_{(\ell)} > \hat{t}_{\lambda,\theta,\ell} \mid \mathcal{A}_r\right) = P_{\lambda,\theta,r}(\hat{t}_{\lambda,\theta,\ell}).$$

Thus we obtain the recursive equation

$$P_{\lambda,\theta,\ell}(t) = \sum_{r=0}^{\ell-1}\binom{\ell}{r}\left(\frac{t-\hat{t}_{\lambda,\theta,\ell}}{t}\right)^{\ell-r}\left(\frac{\hat{t}_{\lambda,\theta,\ell}}{t}\right)^r P_{\lambda,\theta,r}(\hat{t}_{\lambda,\theta,\ell}),$$

as desired. □

Using the above lemma and by following the same steps as the case $\ell = \theta+1$, we obtain for $\ell = \theta+k+1$ and $\hat{t}_{\lambda,\theta,\ell} \leq t < \bar{\lambda}$,

$$\begin{aligned}
s_{\lambda,\theta,\ell}(t) =& \left(\frac{\bar{\lambda}-t}{\bar{\lambda}-\hat{t}_{\lambda,\theta,\ell}}\right)^{\lambda-\ell}\sum_{r=0}^{\theta+k}\binom{\lambda-r}{\ell-r}\left(\frac{t-\hat{t}_{\lambda,\theta,\ell}}{\bar{\lambda}-t}\right)^{\ell-r} s_{\lambda,\theta,r}(\hat{t}_{\lambda,\theta,\ell})\\
=&\mu_{\lambda,\theta}\binom{\lambda}{\ell}\left(\frac{t}{\bar{\lambda}}\right)^\ell\left(1-\frac{t}{\bar{\lambda}}\right)^{\lambda-\ell}\frac{1}{t^\ell}\sum_{r=0}^{\theta+k}\binom{\ell}{r}\left(t-\hat{t}_{\lambda,\theta,\ell}\right)^{\ell-r}\hat{t}^r_{\lambda,\theta,\ell}P_{\lambda,\theta,r}(\hat{t}_{\lambda,\theta,\ell})\\
=&\mu_{\lambda,\theta}\binom{\lambda}{\ell}\left(\frac{t}{\bar{\lambda}}\right)^\ell\left(1-\frac{t}{\bar{\lambda}}\right)^{\lambda-\ell}P_{\lambda,\theta,\ell}(t),
\end{aligned}$$

which completes the proof.

□

A key idea to prove Theorem 4 is to approximate, following [40], the Markov chain by the solution of a system of differential equations in the large network limit. We summarize here the main result of [40].

For a set of variables $Y_1, \ldots, Y_b$ and for $\mathcal{D} \subseteq \mathbb{R}^{b+1}$, define the stopping time

$$T_\mathcal{D} = T_\mathcal{D}(Y_1,\ldots,Y_b) = \inf\{t \geq 1, (t/n; Y_1(t)/n, \ldots, Y_b(t)/n) \notin \mathcal{D}\}.$$

Lemma 16 ([38, 40])**.** *Given integers $b, n \geq 1$, a bounded domain $\mathcal{D} \subseteq \mathbb{R}^{b+1}$, functions $(f_\ell)_{1\leq\ell\leq b}$ with $f_\ell : \mathcal{D} \to \mathbb{R}$, and σ-fields $\mathcal{F}_{n,0} \subseteq \mathcal{F}_{n,1} \subseteq \ldots$, suppose that the random variables $\left(Y_\ell^{(n)}(t)\right)_{1\leq\ell\leq b}$ are $\mathcal{F}_{n,t}$-measurable for $t \geq 0$. Furthermore, assume that, for all $0 \leq t < T_\mathcal{D}$ and $1 \leq \ell \leq b$, the following conditions hold* *(i)* (Boundedness). $\max_{1\le\ell\le b}|Y_\ell^{(n)}(t+1) - Y_\ell^{(n)}(t)| \le \beta$,

(ii) (Trend-Lipschitz). $|\mathbb{E}[Y_\ell^{(n)}(t+1) - Y_\ell^{(n)}(t)|\mathcal{F}_{n,t}] - f_\ell(t/n, Y_1^{(n)}(t)/n, ..., Y_\ell^{(n)}(t)/n)| \le \delta$, *where the function* (f_ℓ) *is* L*-Lipschitz-continuous on* $\mathcal{D}$,

and that the following condition holds initially:

(iii) (Initial condition). $\max_{1\le\ell\le b}|Y_\ell^{(n)}(0) - \hat{y}_\ell n| \le \alpha n$, *for some* $(0, \hat{y}_1, \ldots, \hat{y}_b) \in \mathcal{D}$.

Then there are $R = R(\mathcal{D}, L) \in [1,\infty)$ *and* $C = C(\mathcal{D}) \in (0,\infty)$ *such that, whenever* $\alpha \ge \delta\min\{C, L^{-1}\} + R/n$, *with probability at least* $1 - 2be^{-n\alpha^2/(8C\beta^2)}$ *we have*

$$\max_{0\le t\le\sigma n}\max_{1\le\ell\le b}|Y_\ell^{(n)}(t) - y_\ell(t/n)n| < 3e^{CL}\alpha n,$$

where $(y_\ell(t))_{1\le\ell\le b}$ *is the unique solution to the system of differential equations*

$$\frac{dy_\ell(t)}{dt} = f_\ell(t, y_1, ..., y_b) \quad \text{with} \quad y_\ell(0) = \hat{y}_\ell, \quad \text{for} \quad \ell = 1, ..., b,$$

and $\sigma = \sigma(\hat{y}_1, \ldots, \hat{y}_b) \in [0, C]$ *is any choice of* $\sigma \ge 0$ *with the property that* $(t, y_1(t), ..., y_b(t))$ *has* ℓ^∞*-distance at least* $3e^{LC}\alpha$ *from the boundary of* $\mathcal{D}$ *for all* $t \in [0,\sigma)$.

In the next section, we apply Lemma 16 to the contagion model described in Section 4.1. Let us define, for $0 \le t \le \bar{\lambda}$

$$\begin{aligned}
\delta_{\lambda,\theta}(t) &:= \mu_{\lambda,\theta} - \sum_\ell s_{\lambda,\theta,\ell}(t),\\
\delta^-(t) &:= \sum_{\lambda,\theta}\lambda\delta_{\lambda,\theta}(t) - t, \quad \text{and}\\
\delta(t) &:= \sum_{\lambda,\theta}\delta_{\lambda,\theta}(t),
\end{aligned}$$

with $s_{\lambda,\theta,\ell}$ given in Lemma 14. Hence, we have

$$\delta_{\lambda,\theta}(t) = \mu_{\lambda,\theta}\left(1 - \sum_\ell \binom{\lambda}{\ell}\left(\frac{t}{\bar{\lambda}}\right)^\ell\left(1-\frac{t}{\bar{\lambda}}\right)^{\lambda-\ell}P_{\lambda,\theta,\ell}(t)\right) = \mu_{\lambda,\theta}\left(1 - B^\alpha_{\lambda,\theta}(\frac{t}{\bar{\lambda}})\right), \tag{22}$$

$$\delta^-(t) = \sum_{\lambda,\theta}\lambda\delta_{\lambda,\theta}(t) - t = \bar{\lambda}\left(J^\alpha(\frac{t}{\bar{\lambda}}) - \frac{t}{\bar{\lambda}}\right), \quad \text{and,} \tag{23}$$

$$\delta(\tau) = \sum_{\lambda,\theta}\mu_{\lambda,\theta}B^\alpha_{\lambda,\theta}(\frac{t}{\bar{\lambda}}). \tag{24}$$

4.3 Proof of Theorem 4

We now proceed to the proof of Theorem 4 whose aim is to approximate the value $\frac{D^{(n)}(T^{(n)}_{\text{stop}})}{n}$ as $n \to \infty$. We build on the techniques used in [2, Theorem 3.8]. In contrast to [2], the number of states of the Markov chain grows with time. We first prove the convergence for the rescaled number of nodes that have $\ell < \theta$ failed neighbors, i.e. which are guaranteed to survive. This part of the proof follows from [2, Theorem 3.8]. Next we consider $\ell \ge \theta$ and partition the time interval according to the possibility that growth is sufficient for survival. For every $\ell \ge \theta$ there is a minimal time $\hat{t}_{\lambda,\theta,\ell}$ such that a node with initial threshold θ can survive after $\hat{t}_{\lambda,\theta,\ell}$. As the induction initial step we have convergence on the entire time interval $[0,1]$ of the rescaled vector $S^{(n)}_{\lambda,\theta,\ell}$ for $\ell < \theta$. As induction step, we show that convergence on the interval $[\hat{t}_{\lambda,\theta,\tilde{\ell}}, 1]$ of $S^{(n)}_{\lambda,\theta,\ell}$ for $\ell < \tilde{\ell}$ implies convergence on the interval $[\hat{t}_{\lambda,\theta,\tilde{\ell}+1}, 1]$ of $S^{(n)}_{\lambda,\theta,\ell}$ for $\ell < \tilde{\ell}+1$.

Let $k = \gamma\bar{\lambda}^{(n)}n + o(n)$ for some $\gamma \in [0,1]$. We first show that $T^{(n)}_k \xrightarrow{p} \gamma\bar{\lambda}$. Since

$$\mathbb{E}(T^{(n)}_k) = \sum_{i=1}^{k} \mathbb{E}(T^{(n)}_i - T^{(n)}_{i-1}) = \sum_{i=1}^{k} \frac{1}{n} = \frac{k}{n}, \tag{25}$$

and

$$\mathrm{Var}(T^{(n)}_k) = \sum_{i=1}^{k} \mathrm{Var}(T^{(n)}_i - T^{(n)}_{i-1}) = \sum_{i=1}^{k} \frac{1}{n^2} = \frac{k}{n^2} \sim O(\frac{1}{n}), \tag{26}$$

we have by Chebysev's inequality, as $n \to \infty$, in probability

$$T^{(n)}_k \xrightarrow{p} \gamma\bar{\lambda}.$$

This gives us that (for fixed $\lambda, \theta, \ell \leq \lambda$)

$$\frac{\hat{k}^{(n)}_{\lambda,\theta,\ell}}{n} = \hat{t}_{\lambda,\theta,\ell} + o_p(1). \tag{27}$$

We also need to bound the contribution of higher order terms in the infinite sums (23) and (24). Fix $\epsilon > 0$. By regularity conditions on the degree sequence, we know

$$\bar{\lambda} = \sum_{\lambda} \sum_{\theta=0}^{\lambda} \lambda\mu_{\lambda,\theta} \in (0,\infty).$$

Then, there exists an integer K_ϵ, such that

$$\sum_{\lambda \geq K_\epsilon} \sum_{\theta} \lambda\mu_{\lambda,\theta} < \epsilon.$$

It follows that for all $0 \leq t \leq \bar{\lambda}$,

$$\sum_{\lambda \geq K_\epsilon} \sum_{\theta} \lambda\delta_{\lambda,\theta}(t) = \sum_{\lambda \geq K_\epsilon} \sum_{\theta} \lambda\mu_{\lambda,\theta} \left(1 - B^{\alpha}_{\lambda,\theta}(\frac{t}{\bar{\lambda}})\right) < \epsilon. \tag{28}$$

The number of vertices with degree λ and initial threshold θ is $n\mu^{(n)}_{\lambda,\theta}$. Again, by regularity conditions,

$$\sum_{\lambda,\theta} \lambda\mu^{(n)}_{\lambda,\theta} \to \bar{\lambda} \in (0,\infty).$$

Therefore, for n large enough,

$$\sum_{\lambda \geq K_\epsilon} \sum_{\theta} \lambda\mu^{(n)}_{\lambda,\theta} < \epsilon,$$

which implies for all $0 \leq k \leq n\bar{\lambda}^{(n)}$,

$$\sum_{\lambda \geq K_\epsilon} \sum_{\theta} \lambda D^{(n)}_{\lambda,\theta}(k)/n < \epsilon. \tag{29}$$

For $K \geq 1$, we denote by

$$\mathbf{s}^K(t) := (s_{\lambda,\theta,\ell}(t))_{\lambda \leq K,\ 0 \leq \ell, \theta \leq \lambda} \text{ and}$$
$$\mathbf{S}_n^K(k) := \left(S^{(n)}_{\lambda,\theta,\ell}(k)\right)_{\lambda \leq K,\ 0 \leq \ell, \theta \leq \lambda},$$

both of dimension $b(K)$.

We now show by induction that

$$\sup_{0 \leq t \leq \sigma n} \left|\mathbf{S}_n^K(k)/n - \mathbf{s}^K(k/n)\right| \leq C\epsilon + o_p(1), \tag{30}$$

where $\sigma = \bar{\lambda} - \epsilon$. As the induction first step, we consider the subvector with $\ell < \theta$.

For $K \geq 1$, we denote this subvector where $\ell < \theta$ by

$$\mathbf{s}^{K,\theta} := (s_{\lambda,\theta,\ell}(t))_{\lambda \leq K,\ 0 \leq \ell < \theta \leq \lambda} \text{ and}$$
$$\mathbf{S}_n^{K,\theta} := \left(S^{(n)}_{\lambda,\theta,\ell}(k)\right)_{\lambda \leq K,\ 0 \leq \ell < \theta \leq \lambda},$$

where the superscript θ marks the upper bound on ℓ. For an arbitrary constant $\epsilon > 0$, we define the domain $\mathcal{D}_\epsilon^\theta$ as

$$\mathcal{D}_\epsilon^\theta = \{(\tau, s^K) \in \mathbb{R}^{K\theta+1} \ : \ -\epsilon < \tau < \bar{\lambda} - \epsilon\ ,\ -\epsilon < s_{\lambda,\theta,\ell} < 1\}$$

and we verify the conditions of Lemma 16 which shows that the fluid limit holds on the domain $\mathcal{D}_\epsilon^\theta$. We obtain that for a sufficiently large constant C

$$\sup_{0 \leq k \leq \sigma n} \left|\mathbf{S}_n^{K,\theta}(k)/n - \mathbf{s}^{K,\theta}(k/n)\right| \leq C\epsilon + o_p(1)$$

with $\sigma = \bar{\lambda} - \epsilon$.
For $\tilde{\ell} \geq \theta$ we denote the subvector $(s^{K,\tilde{\ell}}, S_n^{K,\tilde{\ell}})_{\lambda \leq K, \theta \leq \lambda, 0 \leq \ell < \tilde{\ell}}$ with index $\ell < \tilde{\ell}$ by

$$\mathbf{s}^{K,\tilde{\ell}} := (s_{\lambda,\theta,\ell}(t))_{\lambda \leq K, \theta \leq \lambda, 0 \leq \ell < \tilde{\ell}} \text{ and}$$
$$\mathbf{S}_n^{K,\tilde{\ell}} := \left(S^{(n)}_{\lambda,\theta,\ell}(k)\right)_{\lambda \leq K, \theta \leq \lambda, 0 \leq \ell < \tilde{\ell}}.$$

Now consider on the domain

$$\mathcal{D}_\epsilon^{\tilde{\ell}} = \{(t, s^K) \in \mathbb{R}^{K\tilde{\ell}+1} \ : \ -\epsilon < t < \bar{\lambda} - \epsilon\ ,\ -\epsilon < s_{\lambda,\theta,\ell} < 1\}.$$

We prove by induction on $\tilde{\ell}$ that the fluid limit holds on the domain $\mathcal{D}_\epsilon^{\tilde{\ell}}$

$$\sup_{0 \leq k \leq \sigma n} \left|\mathbf{S}_n^{K,\tilde{\ell}}(k)/n - \mathbf{s}^{K,\tilde{\ell}}(k/n)\right| \leq C^{\tilde{\ell}}\epsilon + o_p(1). \tag{31}$$

We have seen that the fluid limit holds for $\tilde{\ell} \leq \theta - 1$, so we use this as the initial induction step. We now suppose that it holds for an $\tilde{\ell} \geq \theta - 1$ and we need to show that it holds for $\tilde{\ell} + 1$. We consider the ODEs solution $\tilde{\mathbf{s}}^{K,\theta} := (\tilde{s}_{\lambda,\theta,\ell}(t))_{\lambda < K, \theta \leq \lambda, 0 \leq \ell < \tilde{\ell}+1}$ on the domain

$$\mathcal{D}_\epsilon^{\tilde{\ell}+1} = \{(t, s^K) \in \mathbb{R}^{K(\tilde{\ell}+1)+1} \ : \ -\epsilon \leq t < \bar{\lambda} - \epsilon\ ,\ -\epsilon < s_{\lambda,\theta,\ell} < 1\},$$

with initial condition $\tilde{\mathbf{s}}_n^{K,\tilde{\ell}+1}(\hat{k}^{(n)}_{\lambda,\theta,\tilde{\ell}+1}/n) = (\mathbf{S}_n^{K,\tilde{\ell}}(\hat{k}^{(n)}_{\lambda,\theta,\tilde{\ell}+1})/n, 0)$; namely we start the same system from the time $\hat{k}^{(n)}_{\lambda,\theta,\tilde{\ell}+1}/n$. At this time it is guaranteed that $S_n^{K,\tilde{\ell}+1} = 0$.

Then from Lemma 16, we have

$$\sup_{\hat{k}^{(n)}_{\lambda,\theta,\tilde{\ell}+1} \le k \le \sigma n} \left| \mathbf{S}_n^{K,\tilde{\ell}+1}(k)/n - \tilde{\mathbf{s}}^{K,\tilde{\ell}+1}(k/n) \right| \le C\epsilon + o_p(1). \tag{32}$$

By the induction hypothesis, namely (31), we have

$$\left| \mathbf{S}_n^{K,\tilde{\ell}}(\hat{k}^{(n)}_{\lambda,\theta,\tilde{\ell}+1})/n - \mathbf{s}^{K,\tilde{\ell}}(\hat{k}^{(n)}_{\lambda,\theta,\tilde{\ell}+1}/n) \right| \le C^{\tilde{\ell}}\epsilon + o_p(1),$$

and using that gives

$$\left| \tilde{\mathbf{s}}^{K,\tilde{\ell}}(\hat{k}^{(n)}_{\lambda,\theta,\tilde{\ell}+1}/n) - \mathbf{s}^{K,\tilde{\ell}}(\hat{k}^{(n)}_{\lambda,\theta,\tilde{\ell}+1}/n) \right| \le C^{\tilde{\ell}}\epsilon + o_p(1). \tag{33}$$

By definition

$$s_{\lambda,\theta,\tilde{\ell}+1}(t) = 0 \text{ for } t \le \hat{t}_{\lambda,\theta,\tilde{\ell}+1}$$

Thus by continuity property of ODEs,

$$\left| \tilde{s}_{\lambda,\theta,\tilde{\ell}+1}(\hat{k}^{(n)}_{\lambda,\theta,\tilde{\ell}+1}/n) - s_{\lambda,\theta,\tilde{\ell}+1}(\hat{k}^{(n)}_{\lambda,\theta,\tilde{\ell}+1}/n) \right| \le C\epsilon + o_p(1).$$

Combining with (33) we obtain

$$\left| \tilde{\mathbf{s}}^{K,\tilde{\ell}+1}(\hat{k}^{(n)}_{\lambda,\theta,\tilde{\ell}+1}/n) - \mathbf{s}^{K,\tilde{\ell}+1}(\hat{k}^{(n)}_{\lambda,\theta,\tilde{\ell}+1}/n) \right| \le C_1\epsilon + o_p(1).$$

Thus by the stability results of ODEs we have

$$\sup_{\hat{k}^{(n)}_{\lambda,\theta,\tilde{\ell}+1} \le k \le \sigma n} \left| \mathbf{s}_n^{K,\tilde{\ell}+1}(k/n) - \tilde{\mathbf{s}}^{K,\tilde{\ell}+1}(k/n) \right| \le C_2\epsilon + o_p(1)$$

Combined with (32) this gives

$$\sup_{\hat{k}^{(n)}_{\lambda,\theta,\tilde{\ell}+1} \le k \le \sigma n} \left| \mathbf{S}_n^{K,\tilde{\ell}+1}(k)/n - \mathbf{s}^{K,\tilde{\ell}+1}(k/n) \right| \le (C_2 + C)\epsilon + o_p(1) \tag{34}$$

By definition

$$S^{(n)}_{\lambda,\theta,\tilde{\ell}+1}(k) = 0 \text{ for } k \le \hat{k}^{(n)}_{\lambda,\theta,\tilde{\ell}+1}$$

and

$$s_{\lambda,\theta,\tilde{\ell}+1}(t) = 0 \text{ for } t \le \hat{t}_{\lambda,\theta,\tilde{\ell}+1}.$$

Combining with (31), it gives

$$\sup_{0 \le k \le \min\{\hat{t}_{\lambda,\theta,\tilde{\ell}+1} n, \hat{k}^{(n)}_{\lambda,\theta,\tilde{\ell}+1}\}} \left| \mathbf{S}_n^{K,\tilde{\ell}+1}(k)/n - \mathbf{s}^{K,\tilde{\ell}+1}(k/n) \right| = 0. \tag{35}$$

Thus by the continuity of solution $\left(s_{\lambda,\theta,\tilde{\ell}+1}(t)\right)_{t\ge 0}$ between $\min\{\hat{t}_{\lambda,\theta,\tilde{\ell}+1} n, \hat{k}^{(n)}_{\lambda,\theta,\tilde{\ell}+1}\}$ and $\hat{k}^{(n)}_{\lambda,\theta,\tilde{\ell}+1}$, combining with (31), (34) and (35), we obtain

$$\sup_{0 \le k \le \sigma n} \left| \mathbf{S}_n^{K,\tilde{\ell}+1}(k)/n - \mathbf{s}^{K,\tilde{\ell}+1}(k/n) \right| \le (C_2 + C)\epsilon + o_p(1).$$

Thus by mathematical induction, (30) holds.

When the solution reaches the boundary of $\mathcal{D}_\epsilon$, it violates the first constraint, determined by $\sigma = \bar{\lambda} - \epsilon$. By convergence of $\bar{\lambda}^{(n)}$ to λ, there is a value n_0 such that $\forall n \geq n_0$, $\bar{\lambda}^{(n)} > \lambda - \epsilon$, which ensures that $\sigma n \leq \bar{\lambda}^{(n)} n$. Using (28) and (29), we have, for $0 \leq k \leq \sigma n$ and $n \geq n_0$:

$$\begin{aligned} \left| D_{\text{out}}^{(n)}(k)/n - \delta^-(k/n) \right| &= |\sum_{\lambda} \sum_{\theta \leq \lambda} \lambda (D_{\lambda,\theta}^{(n)}(t)/n - \delta_{\lambda,\theta}(t/n))| \\ &\leq \sum_{\lambda} \sum_{\theta \leq \lambda} \lambda \left| D_{\lambda,\theta}^{(n)}(t)/n - \delta_{\lambda,\theta}(t/n) \right| \\ &\leq \sum_{\lambda \leq K_\epsilon} \sum_{\theta \leq \lambda} \lambda \left| D_{\lambda,\theta}^{(n)}(t)/n - \delta_{\lambda,\theta}(t/n) \right| + 2\epsilon, \end{aligned} \tag{36}$$

and

$$\left| D^{(n)}(k)/n - \delta(k/n) \right| \leq \sum_{\lambda \leq K_\epsilon} \sum_{\theta \leq \lambda} \left| D_{\lambda,\theta}^{(n)}(k)/n - \delta_{\lambda,\theta}(k/n) \right| + 2\epsilon. \tag{37}$$

We then obtain by (30) that

$$\sup_{0 \leq k \leq \sigma n} \left| D_{\text{out}}^{(n)}(k)/n - \delta^-(k/n) \right| \leq 2C\epsilon + o_p(1), \text{ and} \tag{38}$$

$$\sup_{0 \leq k \leq \sigma n} \left| D^{(n)}(k)/n - \delta(k/n) \right| \leq 2C\epsilon + o_p(1). \tag{39}$$

We now study the stopping time $k_{\text{stop}}^{(n)}$ defined in (15) and the size of the default cascade $D^{(n)}(k_{\text{stop}}^{(n)})$. First assume $J^\alpha(\pi) > \pi$ for all $\pi \in [0, 1)$, i.e., $\pi^* = 1$. Then we have for all $t < \sigma$,

$$\delta^-(t) = \sum_{\lambda,\theta} \lambda \delta_{\lambda,\theta}(t) - t > 0.$$

We have then that $k_{\text{stop}}^{(n)}/n = \sigma + O(\epsilon) + o_p(1)$ and from convergence (39), since $\delta(\sigma) = 1 - O(\epsilon)$, we obtain by tending ϵ to 0 that $|D^{(n)}(k_{\text{stop}}^{(n)})| = n - o_p(n)$. This proves the first part of the theorem.

Now consider the case $\pi^* < 1$, and furthermore π^* is a stable fixed point of $J^\alpha(\pi)$. Then by definition of π^* and by using the fact that $J^\alpha(1) \leq 1$, we have $J^\alpha(\pi) < \pi$ for some interval $(\pi^*, \pi^* + \tilde{\pi})$. Then $\delta^-(t)$ is negative in an interval $(t^*, t^* + \tau)$, with $t^* = \bar{\lambda}\pi^*$.

Let ϵ such that $2\epsilon < -\inf_{t \in (t^*, t^* + \tau)} \delta^-(t)$ and denote $\hat{\sigma}$ the first iteration at which it reaches the minimum. Since $\delta^-(\hat{\sigma}) < -2\epsilon$ it follows that with high probability $D_{\text{out}}^{(n)}(\hat{\sigma} n)/n < 0$, so $k_{\text{stop}}^{(n)}/n = t^* + O(\epsilon) + o_p(1)$. The conclusion follows by taking the limit $\epsilon \to 0$.

5 Further extensions

So far we have considered the baseline case where nodes have the same in- and out-degree (that we call connectivity) and where the relation between the interaction time and the calendar time is not dependent on the state of the system. In this section we show that both these assumptions can be relaxed to yield more realism to the model.

First, we consider the case when in- and out-degrees are allowed to differ. The in-degree is a channel by which one node is impacted by its neighbors, whereas the out-degree is the channel by which the node impacts others in the case of distress. Growth depends mainly on the in-degree: the higher the in-degree, the more a bank is exposed to others and in return it receives higher interest or fees. In particular, for a reinsurer its capital grows with the number of firms that it provides reinsurance to, as they pay premiums. However, growth can also be a function of the out-degree: banks' leverage and growth depends on how much debt they issue; reinsurers' business growth depends on how many other firms reinsure them.

Below we provide the asymptotic limit of the survival probability (under growth benefits). We leave it for future research the analysis of an equilibrium in which both in- and out-degrees are bank choices. Such equilibrium can be defined as in the baseline case, but with the additional constraint that average in-degree matches average out-degree.

Second, we consider the case when the calendar time between two interactions is an exponential whose parameter depends on the state of the system. In particular, the exponential time between interactions can depend on the current duration of the contagion process or on the number of unrevealed outgoing links from the failed nodes. The larger the number of unrevealed links, the higher the intensity with which we learn of affected counterparties. Growth is unlikely to stabilize the system if it happens uniformly over time. On the other hand, if the number of initial failures is small, then so is the intensity with which we learn the counterparties of failed linkages. This will slow contagion as there is a threshold growth in the meanwhile for the counterparties, and the system is likely to stabilize itself even without intervention.

In our last extension, we consider the case when growth is no longer uniform over time, but arrived with some prescribed intensity. As before, we give fluid limits for the Markov processes governing the number of surviving nodes in each category. Given this, one can find an optimal control strategy by introducing a tradeoff for the controller as in [6]: the controller can increase the growth intensity at a certain cost. Determining the optimal growth intensity as a state-dependent process is left for future research. This is interpreted as the optimal capital or liquidity injection by a government or a lender of last resort.

5.1 Different in/out-degree distribution

Let $\mu^{(n)}_{\lambda_+,\lambda_-,\theta}$ be the fraction of nodes with in-degree λ_+, out-degree λ_- and threshold θ. Assume the following regularity conditions $\mu^{(n)}_{\lambda_+,\lambda_-,\theta} \to \mu_{\lambda_+,\lambda_-,\theta}$, as $n \to \infty$, for some distribution $\mu : \mathbb{N}^3 \to [0,1]$. We also assume again that the average connectivity converges to a finite limit

$$\bar{\lambda}^{(n)} := \sum_{\lambda_+,\lambda_-,\theta} \lambda_+ \mu^{(n)}_{\lambda_+,\lambda_-,\theta} = \sum_{\lambda_+,\lambda_-,\theta} \lambda_- \mu^{(n)}_{\lambda_+,\lambda_-,\theta} \to \sum_{\lambda_+,\lambda_-,\theta} \lambda_+ \mu_{\lambda_+,\lambda_-,\theta} =: \bar{\lambda} \in (0,\infty). \tag{40}$$

Suppose that growth benefits arrive uniformly over time according to the "growth parameter" α and both the in- and out-degrees. Given a growth function g, $g(\alpha, \lambda_+, \lambda_-)$, one can define similarly to (17) the minimal time when the node could survive ℓ failed neighbors

$$\hat{t}_{\lambda_+,\lambda_-,\theta,\ell} = \frac{(\ell-\theta)\bar{\lambda}}{g(\alpha,\lambda_+,\lambda_-)}. \tag{41}$$

We let $P_{\lambda_+,\lambda_-,\theta,\ell}(\pi)$ as in (21) but with the new definition of $\hat{t}$.

Theorem 17. *Let π^* be the relaxed fixed point of the map J^α defined as*

$$\pi^* := \min\{\pi \in [0,1] \mid J^\alpha(\pi) \leq \pi\},$$

where

$$J^{\alpha}(\pi) := \sum_{\lambda_+,\lambda_-,\theta} \frac{\lambda_- \mu_{\lambda_+,\lambda_-,\theta}}{\bar{\lambda}} \cdot B^{\alpha}_{\lambda_+,\lambda_-,\theta}(\pi),$$

and

$$B^{\alpha}_{\lambda_+,\lambda_-,\theta}(\pi) := 1 - \sum_{\ell=0}^{\min\{\lceil \theta + g(\alpha,\lambda_+,\lambda_-)\pi \rceil - 1, \lambda_+\}} \binom{\lambda_+}{\ell} \pi^{\ell}(1-\pi)^{\lambda-\ell} P_{\lambda_+,\lambda_-,\theta,\ell}(\pi).$$

We have:

(i) If $\pi^* = 1$, i.e., if $J^{\alpha}(\pi) > \pi$ for all $\pi \in [0,1)$, then asymptotically (as $n \to \infty$) almost all nodes fail during the cascade.

(ii) If $\pi^* < 1$ and π^* is a stable fixed point of J^{α}, i.e., $J^{\alpha\prime}(\pi^*) < 1$, then the final fraction of failed nodes converges in probability to

$$\frac{|\mathcal{D}_f^{(n)}|}{n} \xrightarrow{p} \sum_{\lambda_+,\lambda_-,\theta} \mu_{\lambda_+,\lambda_-,\theta} B^{\alpha}_{\lambda_+,\theta}(\pi^*). \tag{42}$$

Furthermore, the asymptotic fraction of surviving nodes with degree λ and initial threshold θ converges in probability (as $n \to \infty$)

$$\frac{S_{\lambda_+,\lambda_-,\theta}(k^{(n)}_{\text{stop}})}{n} \xrightarrow{p} \mu_{\lambda_+,\lambda_-,\theta}\left(1 - B^{\alpha}_{\lambda_+,\theta}(\pi^*)\right).$$

5.2 Calendar time between interactions

So far, we have analyzed systemic risk under the assumption that the duration in calendar time between the two successive interactions follows an exponential distribution with parameter n, i.e. $\Delta_k^{(n)} = T_k^{(n)} - T_{k-1}^{(n)} \sim \text{Exp}(n)$. In order to allow the recovery intensity vary along time, we can extend the model by allowing the intensity of exponential distribution to be a general function of parameter n. This gives us the flexibility to control different recovery speeds depending on the state of the cascade. One interesting example is when the calendar time between two interactions starts to decrease as more and more unrevealed failed links are added while the duration increases when the total number of unrevealed failed links starts to reduce. The former stage may corresponding to the early stage of the cascade process while the latter case may corresponding to the later stage.

In particular, our results extend to the case when

$$\Delta_k^{(n)} = T_k^{(n)} - T_{k-1}^{(n)} \sim \text{Exp}(F_k^{(n)}),$$

where $F_k^{(n)}$ satisfies

$$F_k^{(n)} = nf(\frac{k}{n}) + o(1)$$

for some (squared) integrable function $1/f$ on $[0, \bar{\lambda} - \epsilon]$. The proof of Theorem 4 follows through as in Section 4.3. Indeed, we note that only two changes are necessary. Let $k = \gamma \bar{\lambda}^{(n)} n + o(n)$ for some $\gamma \in [0,1]$. First, we show that in this case $T_k^{(n)} \xrightarrow{p} \int_0^{\gamma\bar{\lambda}} \frac{1}{f(s)} ds$. Indeed, (25)-(26) become

$$\mathbb{E}(T_k^{(n)}) = \sum_{i=1}^{k} \frac{1}{F_i^{(n)}} = \sum_{i=1}^{k} (\frac{i}{n} - \frac{i-1}{n}) \frac{1}{\frac{F_i^{(n)}}{n}} \approx \int_0^{\gamma\bar{\lambda}} \frac{1}{f(s)} ds + o(1),$$

and,

$$\mathrm{Var}(T_k^{(n)}) = \sum_{i=1}^{k} \frac{1}{\left(F_i^{(n)}\right)^2} = \frac{1}{n}\sum_{i=1}^{k}(\frac{i}{n} - \frac{i-1}{n})\frac{1}{\left(\frac{F_i^{(n)}}{n}\right)^2} \approx \frac{1}{n}\int_0^{\gamma\bar{\lambda}} \frac{1}{f^2(s)}ds + o(\frac{1}{n}) = O(\frac{1}{n}).$$

Thus (again) by using Chebysev's inequality, we have as $n \to \infty$, in probability

$$T_k^{(n)} \xrightarrow{p} \int_0^{\gamma\bar{\lambda}} \frac{1}{f(s)}ds.$$

Second, let $V_1^\pi, V_2^\pi, \ldots, V_\ell^\pi$ be i.i.d. random variables on $[0, \pi\bar{\lambda}]$ with density

$$g(s) = \frac{\frac{1}{f(s)}}{\int_0^{\pi\bar{\lambda}} \frac{1}{f(s)}ds} \quad \text{for} \quad s \in [0, \pi\bar{\lambda}]$$

and the order statistics be

$$V_{(1)}^\pi \le V_{(2)}^\pi \le \cdots \le V_{(\ell)}^\pi.$$

In this case, we need to set

- for $\ell = 0, 1, \ldots, \theta$:

$$\beta_{\lambda,\theta,\ell}^\alpha(\pi) = \binom{\lambda}{\ell}\pi^\ell(1-\pi)^{\lambda-\ell}.$$

- for $\ell = \theta+1, \ldots, \lambda$:

$$\beta_{\lambda,\theta,\ell}^\alpha(\pi) = \binom{\lambda}{\ell}\pi^\ell(1-\pi)^{\lambda-\ell}\mathbb{P}\left(V_{(\theta+1)}^\pi > t_{\theta+1}, V_{(\theta+2)}^\pi > t_{\theta+2}, \ldots, V_{(\ell)}^\pi > t_{\lambda,\theta,\ell}\right),$$

  where

$$\hat{t}_u := \frac{u-\theta}{\alpha\lambda}\bar{\lambda}$$

  for all $u = \theta+1, \ldots, \lambda$.

Using these changes, Theorem 4 will be still valid.

We now consider a more concrete example where the intensity between the two successive interactions depends on the number of unrevealed links in the system, i.e.

$$\Delta_k^{(n)} = T_k^{(n)} - T_{k-1}^{(n)} \sim \mathrm{Exp}(D_{\mathrm{out}}^{(n)}(k)),$$

where $D_{\mathrm{out}}^{(n)}(k)$ denotes the number of unrevealed outgoing links belonging to failed agents at the k-th interaction time $T_k^{(n)}$. As shown in Section 4, for $k = \gamma\bar{\lambda}^{(n)}n + o(n)$ and $\gamma < \pi^*$,

$$\frac{D_{\mathrm{out}}^{(n)}(k)}{n} \xrightarrow{p} \bar{\lambda}\left(J^\alpha(\gamma) - \gamma\right).$$

Similarly, in this case we expect all our results go through, since

$$\mathbb{E}(T_k^{(n)}) \approx \int_0^\gamma \frac{1}{\bar{\lambda}\left(J^\alpha(s) - s\right)}ds + o(1)$$

and

$$\mathrm{Var}(T_k^{(n)}) \approx \frac{1}{n}\int_0^\gamma \frac{1}{\bar{\lambda}^2\left(J^\alpha(s) - s\right)^2}ds + o(\frac{1}{n}) \sim o(\frac{1}{n}).$$

5.3 Different growth attribution policies

So far we investigated the case when growth is linear over time and the total growth is distributed among nodes proportionally to their number of links. We now allow for different types of growth attribution. Instead of allowing growth to be continuous in time, we can incorporate both failure and growth jumps into Markov jump diffusion processes with different jump intensities. At each interaction time, the jump is due either to a failed link or to growth.

We assume during each interaction time, either a failed link is revealed or threshold growth happens with relative rate μ and μ_1. Moreover, if threshold growth happens, it is related to each institution's number of links.

In the network of size n, we let $S^{(n)}_{\lambda,\theta,k,\ell}(t)$ represent the number of agents with initial threshold θ, connectivity λ, k units of growth and ℓ failed links at time t. The conservation relation

$$\sum_{0\le k\le\lambda, 0\le\ell\le\lambda} S^{(n)}_{\lambda,\theta,k,\ell}(t) = S^{(n)}_{\lambda,\theta,0,0}(0)$$

gives that the Markov processes $(S^{(n)}_{\lambda,\theta,k,\ell}(t))_{0\le k\le\lambda,0\le\ell\le\lambda}$ has the Q-matrix (for fixed λ,θ)

$$\begin{cases} q^{(n)}(x, x-e_{(k,\ell)}+e_{(k,\ell+1)}) = \mu(\lambda-\ell)x_{(k,\ell)} & 0\le k\le\lambda, 0\le\ell\le\lambda-1, \\ q^{(n)}(x, x-e_{(k,\ell)}+e_{(k+1,\ell)}) = \mu_1\lambda x_{(k,\ell)} & \theta+k-\ell>0, 0\le k\le\lambda-1, 0\le\ell\le\lambda, \\ q^{(n)}(x, x-e_{(\lambda,\ell)}) = \mu_1\lambda x_{(\lambda,\ell)} & \theta+\lambda-\ell>0, 0\le\ell\le\lambda. \end{cases} \tag{43}$$

For a càdlàg (right continuous with left limits) function h on $\mathbb{R}$, we denote by $\mathcal{N}_h$ a point processes defined as follows

$$\mathcal{N}_h = \int_0^t \mathcal{P}([0,h(s-)]\times ds)$$

where $\mathcal{P}$ is a Poisson processe in $\mathbb{R}^2_+$ whose intensity is the Lebesgue measure on $\mathbb{R}^2_+$. When h is deterministic $\mathcal{N}_h$ is Poisson processes with intensity $(h(t-))$.
We can represent the Markov processes $(S^{(n)}_{\lambda,\theta,k,\ell}(t))_{0\le\ell\le\lambda,0\le k\le\lambda}$ in terms of jump components

$$\begin{aligned}
dS^{(n)}_{\lambda,\theta,k,0}(t) =& \mathcal{N}_{\mu_1\lambda S^{(n)}_{\lambda,\theta,k-1,0}}(dt) - \mathcal{N}_{\mu_1\lambda S^{(n)}_{\lambda,\theta,k,0}}(dt) - \mathcal{N}_{\mu(\lambda-\ell)S^{(n)}_{\lambda,\theta,k,0}}(dt), \\
dS^{(n)}_{\lambda,\theta,0,\ell}(t) =& -\mathcal{N}_{\mu_1\lambda S^{(n)}_{\lambda,\theta,0,\ell}}(dt) + \mathcal{N}_{\mu(\lambda-\ell+1)S^{(n)}_{\lambda,\theta,0,\ell-1}}(dt) - \mathcal{N}_{\mu(\lambda-\ell)S^{(n)}_{\lambda,\theta,0,\ell}}(dt) && \ell<\theta, \\
dS^{(n)}_{\lambda,\theta,0,\ell}(t) =& \mathcal{N}_{\mu(\lambda-\ell+1)S^{(n)}_{\lambda,\theta,0,\ell-1}}(dt) - \mathcal{N}_{\mu(\lambda-\ell)S^{(n)}_{\lambda,\theta,0,\ell}}(dt) && \ell\ge\theta, \\
dS^{(n)}_{\lambda,\theta,k,\ell}(t) =& \mathcal{N}_{\mu_1\lambda S^{(n)}_{\lambda,\theta,k-1,\ell}}(dt) - \mathcal{N}_{\mu_1\lambda S^{(n)}_{\lambda,\theta,k,\ell}}(dt) \\
&+ \mathcal{N}_{\mu(\lambda-\ell+1)S^{(n)}\lambda,\theta,k,\ell-1}(dt) - \mathcal{N}_{\mu(\lambda-\ell)S^{(n)}_{\lambda,\theta,k,\ell}}(dt) && \theta+k-\ell>0, \\
dS^{(n)}_{\lambda,\theta,k,\ell}(t) =& \mathcal{N}_{\mu(\lambda-\ell+1)S^{(n)}_{\lambda,\theta,k,\ell-1}}(dt) - \mathcal{N}_{\mu(\lambda-\ell)S^{(n)}_{\lambda,\theta,k,\ell}}(dt) && \theta+k-\ell\le 0.
\end{aligned}$$

We let the martingales associated to the jumps of these processes

$$\begin{aligned}
U^{(n)}_{\lambda,\theta,k,\ell}(t) :=& \int_0^t [\mathcal{N}_{\mu_1\lambda S^{(n)}_{\lambda,\theta,k-1,\ell}}(du) - \mu_1\lambda S^{(n)}_{\lambda,\theta,k-1,\ell}du] - \int_0^t [\mathcal{N}_{\mu_1\lambda S^{(n)}_{\lambda,\theta,k,\ell}}(du) - \mu_1\lambda S^{(n)}_{\lambda,\theta,k,\ell}du] \\
&+ \int_0^t [\mathcal{N}_{\mu(\lambda-\ell+1)S^{(n)}_{\lambda,\theta,k,\ell-1}}(du) - \mu(\lambda-\ell+1)S^{(n)}_{\lambda,\theta,k,\ell-1}du] \\
&- \int_0^t [\mathcal{N}_{\mu(\lambda-\ell)S^{(n)}_{\lambda,\theta,k,\ell}}(du) - \mu(\lambda-\ell)S^{(n)}_{\lambda,\theta,k,\ell}du],
\end{aligned}$$

and we denote by

$$W^{(n)}_{\lambda,\theta,k,\ell}(t) := \int_0^t \mu_1\lambda S^{(n)}_{\lambda,\theta,k-1,\ell}du - \int_0^t \mu_1\lambda S^{(n)}_{\lambda,\theta,k,\ell}du + \int_0^t \mu(\lambda-\ell+1)S^{(n)}_{\lambda,\theta,k,\ell-1}du - \int_0^t \mu(\lambda-\ell)S^{(n)}_{\lambda,\theta,k,\ell}du$$

its increasing part.

In the following we show that when $n\to\infty$ the sequence of processes

$$\Big(\frac{S^{(n)}_{\lambda,\theta,k,\ell}(t)}{n}, 0\le\ell\le\lambda, 0\le k\le\lambda, 0\le\theta\le\lambda\Big)$$

is tight and converges in distribution uniformly on compact sets to a continuous processes $(s_{\lambda,\theta,k,\ell}(t))$, see e.g.[13].

Since $0\le S^{(n)}_{\lambda,\theta,k,\ell-1}\le n$, the above relation gives the existence of a constant C_1 such that

$$\mathbb{E}\big(U^{(n)}_{\lambda,\theta,k,\ell}(t)^2\big) = \mathbb{E}\big(\langle U^{(n)}_{\lambda,\theta,k,\ell}(t)\rangle(t)\big) \le C_1 N t$$

We can apply Doob's Inequality and obtain

$$\mathbb{P}\Big(\sup_{0\le s\le t}\frac{U^{(n)}_{\lambda,\theta,k,\ell}(s)}{n}\ge\epsilon\Big) \le \frac{1}{(\epsilon N)^2}\mathbb{E}\big(U^{(n)}_{\lambda,\theta,k,\ell}((t)^2\big) \le \frac{C_1 t}{\epsilon^2 N}, \tag{44}$$

which shows that the martingale $\big(\frac{U^{(n)}_{\lambda,\theta,k,\ell}(s)}{n}\big)$ converges in probability to 0 uniformly on compact sets.

Tightness: For $T>0,\delta>0$, define $\omega_Z(\delta)$ as the modulus of continuity of the càdlàg functions on the interval $[0,T]$,

$$\omega_Z(\delta) = \sup_{0\le s\le t\le T, |t-s|\le\delta}|Z(t)-Z(s)|.$$

Then for any $\epsilon>0$ and $\eta>0$, there exists $\delta>0$ such that $\mathbb{P}(\omega_{S^{(n)}_{\lambda,\theta,k,\ell}/N}(\delta)\ge\eta)\le\epsilon$.

Recall that $\frac{S^{(n)}_{\lambda,\theta,k,\ell}}{n} = \frac{U^{(n)}_{\lambda,\theta,k,\ell}}{n} + \frac{W^{(n)}_{\lambda,\theta,k,\ell}}{n}$ (corresponds to the martingale part and the increasing part of $\frac{S^{(n)}_{\lambda,\theta,k,\ell}}{n}$. Thus we have

$$\mathbb{P}(\omega_{S^{(n)}_{\lambda,\theta,k,\ell}/N}(\delta)\ge\eta) \le \mathbb{P}(\omega_{U^{(n)}_{\lambda,\theta,k,\ell}/N}(\delta)\ge\frac{\eta}{2}) + \mathbb{P}(\omega_{W^{(n)}_{\lambda,\theta,k,\ell}/N}(\delta)\ge\frac{\eta}{2})$$

The results follow by the inequality $\mathbb{P}(\omega_{U^{(n)}_{\lambda,\theta,k,\ell}}(\delta)\ge\frac{\eta}{2}) \le \mathbb{P}(\sup_{0\le s\le t}|\frac{U^{(n)}_{\lambda,\theta,k,\ell}(s)}{n}|\ge\frac{\eta}{4})$.

The limit processes $(s_{\lambda,\theta,k,\ell}(t))$ satisfy

$$\begin{aligned}
&ds_{\lambda,\theta,k,0}(t) = \mu_1(js_{\lambda,\theta,k-1,0}-js_{\lambda,\theta,k,0})(dt) + \mu\{-(\lambda-\ell)s_{\lambda,\theta,k,0})\}(dt) && \\
&ds_{\lambda,\theta,0,\ell}(t) = \mu_1(-js_{\lambda,\theta,0,\ell})(dt) + \mu\{(\lambda-\ell+1)s_{\lambda,\theta,0,\ell-1}-(\lambda-\ell)s_{\lambda,\theta,0,\ell})\}(dt) && \ell<\theta,\\
&ds_{\lambda,\theta,0,\ell}(t) = \mu(\lambda-\ell+1)s_{\lambda,\theta,0,\ell-1}(dt) - \mu(\lambda-\ell)s_{\lambda,\theta,0,\ell}(dt) && \ell\ge\theta,\\
&ds_{\lambda,\theta,k,\ell}(t) = \mu_1(js_{\lambda,\theta,k-1,\ell}-js_{\lambda,\theta,k,\ell})(dt) + \mu\{(\lambda-\ell+1)s_{\lambda,\theta,k,\ell-1}-(\lambda-\ell)s_{\lambda,\theta,k,\ell})\}(dt) && \theta+k-\ell>0,\\
&ds_{\lambda,\theta,k,\ell}(t) = \mu(\lambda-\ell+1)s_{\lambda,\theta,k,\ell-1}(dt) - \mu(\lambda-\ell)s_{\lambda,\theta,k,\ell}(dt) && \theta+k-\ell\le 0.
\end{aligned}$$

This can be rewriten as

$$ds_{\lambda,\theta,k,\ell}(t) = \mu_1\lambda s_{\lambda,\theta,k-1,\ell}(dt) + \mu((\lambda-\ell+1)s_{\lambda,\theta,k,\ell-1}(dt) - \{\mu_1\lambda + \mu(\lambda-\ell)\}s_{\lambda,\theta,k,\ell}(dt) \quad \theta+k-\ell>0,$$
$$ds_{\lambda,\theta,k,\ell}(t) = \mu(\lambda-\ell+1)s_{\lambda,\theta,k,\ell-1}(dt) - \mu(\lambda-\ell)s_{\lambda,\theta,k,\ell}(dt) \quad \theta+k-\ell\leq 0,$$

and we obtain

$$\frac{d}{dt}(s_{\lambda,\theta,k,\ell}e^{(\mu_1\lambda+\mu(\lambda-\ell))t}) = \mu_1\lambda s_{\lambda,\theta,k-1,\ell} + \mu(\lambda-\ell+1)s_{\lambda,\theta,k,\ell-1}\}e^{(\mu_1\lambda+\mu(\lambda-\ell))t} \quad \theta+k-\ell>0,$$
$$\frac{d}{dt}(s_{\lambda,\theta,k,\ell}e^{(\mu(\lambda-\ell))t}) = \mu(\lambda-\ell+1)s_{\lambda,\theta,k,\ell-1}e^{\mu(\lambda-\ell)t} \quad \theta+k-\ell\leq 0.$$

The ODE system could be solved recursively:

If $\theta+k-\ell>0$,

$$s_{\lambda,\theta,k,\ell}(t) = e^{-(\mu_1\lambda+\mu(\lambda-\ell))t}\int_0^t \{\mu_1\lambda s_{\lambda,\theta,k-1,\ell}(u) + \mu(\lambda-\ell+1)s_{\lambda,\theta,k,\ell-1}(u)\}e^{(\mu_1\lambda+\mu(\lambda-\ell))u}du,$$

and if $\theta+k-\ell\leq 0$,

$$s_{\lambda,\theta,k,\ell}(t) = e^{-\mu(\lambda-\ell)t}\int_0^t \{\mu(\lambda-\ell+1)s_{\lambda,\theta,k,\ell-1}(u)\}e^{(\mu_1\lambda+\mu(\lambda-\ell))u}du$$

Recall that in the network of size n the number of failed nodes with connectivity λ and threshold θ is given by

$$D^{(n)}_{\lambda,\theta}(t) = n\mu^{(n)}_{\lambda,\theta} - \sum_{0\leq\ell<\theta+k} S_n^{j,\theta,k,\ell}(t),$$

the total number of failed nodes at time t is thus

$$D^{(n)}(t) = \sum_{j,\theta} D^{(n)}_{\lambda,\theta}(t)$$

and the number of unrevealed (outgoing) failed links is

$$D^{(n)}_{\text{out}}(t) = \sum_{j,\theta} j\cdot D^{(n)}_{\lambda,\theta}(t) - \mathcal{N}^{(n)}_t,$$

where $\mathcal{N}^{(n)}_t$ is the jump diffusion process associated to the link- revealing process. They are generalized Poisson processes with intensity $\int_0^t \lambda^{(n)}_s ds$ with $\lambda^{(n)}_t = \mu(m_n - \mathcal{N}^{(n)}_t)$.

Informally, we have

$$\frac{\mathcal{N}^{(n)}_t}{n} = \frac{\int_0^t \lambda^{(n)}_s ds}{n} + \frac{\mathcal{N}_t - \int_0^t \lambda^{(n)}_s dt}{n} \tag{45}$$

Let $\frac{\hat{\mathcal{N}}_t}{n}$ be the martingale $\frac{\mathcal{N}_t - \int_0^t \lambda^{(n)}_s ds}{n}$, then as before it convergences in distribution to 0. Letting

$$\kappa_t = \lim_{n\to\infty}\frac{\mathcal{N}^{(n)}_t}{n},$$

from equation (45),κ_t satisfies

$$\kappa_t = \mu(\lambda t - \int_0^t \kappa_s ds) \tag{46}$$

which gives $\kappa_t = \lambda(1 - e^{-\mu t})$.

The length of the cascade is given by

$$T^{(n)}_{\text{stop}} = \inf\{0 \le t \le \infty,\ D^{(n)}_{\text{out}}(t) = 0\}. \tag{47}$$

Since both $D^{(n)}(t)$ and $\mathcal{N}_t$ take value in $\mathbb{N}$, the above is equivalent to

$$T^{(n)}_{\text{stop}} = \inf\{\tau^{(n)}_i,\ 0 \le i \le \bar{\lambda}^{(n)} n,\ D^{(n)}_{\text{out}}(\tau_i) = 0\}, \tag{48}$$

where $\tau^{(n)}_i = \inf\{0 < t < \infty, \mathcal{N}^{(n)}_t = i\}$. We are interested in the proportion $\frac{D^{(n)}(T^{(n)}_{\text{stop}})}{n}$. In the limiting case ($n \to \infty$), this amounts to

$$t_{\text{stop}} = \inf\{0 \le t \le \infty, \sum_{\lambda,\theta} \lambda \delta_{\lambda,\theta}(t) = \lambda(1 - e^{-\mu t})\}.$$

Finally, the proportion of failed nodes in the limit case is given by $\delta(t_{\text{stop}})$.

Conclusions

We investigated the fluid limit for a network of Cramer-Lundberg processes interacting through a random graph. Nodes' receive growth benefits and suffer downward jumps occurring on exponential clocks after the failure of a neighbor. Using the limiting solution for the probability of failure, we define a game in which nodes choose their connectivity under the tradeoff that linkages provide income, and at the same time they bear the risk of contagion. In equilibrium, the risk of contagion depends on the choices of all nodes in the system. Our notion of equilibrium is similar to a mean field game: players take as given a mean-field, namely the conjectured failure probability of a link (which also gives the proportion of failed nodes at the end of a potential contagion process). They then decide on their own connectivity. This leads to an actual failure probability in the network and we check that a fixed point holds: the actual link failure probability is the same as the conjectured link failure probability.

Our results show that a higher heterogeneity in the initial distribution of the threshold (as captured by its standard deviation) implies a lower default probability in equilibrium even as it leads to a larger average connectivity in equilibrium. More importantly, systems with higher growth/recovery rates may lead to equilibria with higher failure probability as well as higher final fraction of failed agents. This result is surprising and gives new insights into potential policies that promote financial stability. In particular, this shows that even if equity is injected over time, the strategic agents will adapt and potentially take more risks in equilibrium as captured by increased connectivity. This means that any policies that promote growth must be accompanied by limiting connectivity and this must be targeted on agents which have higher initial thresholds. Otherwise, in anticipation of future growth agents would otherwise take too high risks. By limiting their connectivity, their thresholds will grow even larger and they will act as shock absorbers of the system.

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