Inexpensive guaranteed and efficient upper bounds on the algebraic error in finite element discretizations

Jan Papež, Martin Vohralík

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Abstract

We present new constructions of (approximate) $\mathbf{H}(\mathrm{div}, \Omega)$-liftings of the algebraic residual leading to estimators of the algebraic error in h and p finite element discretizations of a model diffusion problem. The estimators provide guaranteed bounds without any uncomputable constants and they are globally efficient, similarly to some recent developments, but the cost of their construction is significantly reduced. We provide a set of numerical experiments to assess the performance of the new estimators.

Key words: finite element method, iterative algebraic solver, algebraic error, a posteriori error estimate, guaranteed upper bound, hierarchical splitting

1 Introduction

Numerical discretizations of partial differential equations typically give rise to large sparse systems of linear algebraic equations, which are often solved by iterative solvers. Then two natural questions arise: 1) what is the algebraic error on iteration i? 2) when to stop the iterations of the algebraic solver?

Pioneering works on estimating and balancing the algebraic and discretization error components were proposes, e.g, by Brandt [8], Bank and Sherman [4], Bai and Brandt [3], Bank and Smith [5], Rüde [22, 23, 24], Oswald [19], Becker et al. [6], or more recently by, e.g., Arioli et al. [2, 1], Janssen and Kanschat [15], or Meidner et al. [17]. However, the available estimates often involve some unknown generic constants, which can limit their use in practice. Using so-called flux reconstruction techniques, a posteriori error estimates including algebraic error and avoiding uncomputable constants were derived, e.g., in Jiránek et al. [16], Ern and Vohralík [11], or Papež et al. [21], see also the references therein. However, these results still come with some limitations; namely additional iteration steps are required. This drawback was overcome by the estimates of [20] using a hierarchical lifting of the algebraic residual. Then guaranteed and fully computable bounds on the algebraic, discretization, and total errors in conforming finite element discretizations become available [20].

In this contribution, we elaborate on [20] and present new constructions of (approximate) $\mathbf{H}(\mathrm{div}, \Omega)$-liftings of the algebraic residual leading to estimators of the algebraic error in h and p finite element discretizations of a model diffusion problem. The estimators provide guaranteed bounds without any uncomputable constants and we show that they are globally efficient. In comparison to the construction of the lifting proposed in [20], the cost of the new construction is significantly reduced, while all the theoretical properties are maintained.

The rest of this contribution is organized as follows. We start in Sec. 2 with a warning example where some commonly used estimators on the algebraic error critically fail. In Sec. 3, we introduce the hierarchy of meshes and piecewise polynomial spaces that are used in the constructions of the liftings and the estimators. A cornerstone of our approach, a coarse grid solver, is introduced in Sec. 4. Then we recall the construction of the lifting from [20] in Sec. 5 and present two new constructions in Secs. 6 and 7. In Sec. 8 we finally construct an approximate lifting in the lowest-order polynomial space. The upper bounds on the algebraic error in the model Poisson problem are presented in Sec. 9 and their efficiency is proved in Sec. 10. Numerical performance of the estimates is then examined in Sec. 11. The paper ends with a concluding discussion in Sec. 12.

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2 Warning example

The purpose of this section is to present a simple example in which some commonly used estimators of the algebraic error fail importantly. In contrast, the rigorously justified upper bounds on the algebraic error developed in this manuscript behave without flaw.

Consider the Poisson model problem of finding $u : \Omega \to \mathbb{R}$ such that

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

Consider here the domain $\Omega = (0,1)^d$, $d = 1, 2, 3$, and choose the source term f such that the exact solutions in (9.1) are respectively given by $u_{d=1}(x) = x(1-x)$, $u_{d=2}(x,y) = x(1-x)y(1-y)$, and $u_{d=3}(x,y,z) = x(1-x)y(1-y)z(1-z)$. Consider a simplicial mesh with maximal element diameter H: uniform for $d = 1$, composed of congruent isosceles triangles for $d = 2$, and composed of tetrahedra generated using the FreeFEM [14] command `cube` for $d = 3$. Importantly, we also consider a regular refinement with mesh size $h = H/2$. We denote by u_H^{ex} and u_h^{ex} the associated Galerkin finite element solutions, obtained with continuous and piecewise affine polynomials with respect to the meshes $\mathcal{T}_H$ and $\mathcal{T}_h$. The associated coefficients in the standard nodal Lagrange basis on the mesh $\mathcal{T}_h$ are respectively denoted by U_h and U_h^{ex}, and the stiffness matrix on the mesh $\mathcal{T}_h$ is named $\mathbb{A}_h$, so that $\mathbb{A}_h \mathrm{U}_h^{\text{ex}} = \mathrm{F}_h$.

We elaborate on [7, Example 4.6], where the one-dimensional case is presented with a thorough discussion. The point is to view u_H^{ex} as an approximate algebraic solution to u_h^{ex}. There is no (iterative) algebraic solver here; we focus on estimating the energy norm of the algebraic error $\|\nabla(u_h^{\text{ex}} - u_H^{\text{ex}})\| = \|\mathrm{U}_h^{\text{ex}} - \mathrm{U}_h\|_{\mathbb{A}_h}$ when *varying the mesh size h*. We consider the probably most widespread algebraic error estimate given by the Euclidean norm $\|\mathrm{R}_h\|_2$ of the algebraic residual R_h associated with u_H^{ex}, $\mathrm{R}_h := \mathrm{F}_h - \mathbb{A}_h \mathrm{U}_h$. It is known that $\|\mathrm{R}_h\|_2$ can be largely disconnected from the error. For this reason, we also plot the value $\|\mathbb{A}_h^{-1}\|_2^{1/2}\|\mathrm{R}_h\|_2$, which provides a (worst-case) guaranteed upper bound on the algebraic error $\|\nabla(u_h^{\text{ex}} - u_H^{\text{ex}})\|$ (see, e.g., the discussion in [21, Sec. 3.1]). The results are given in Fig. 1.

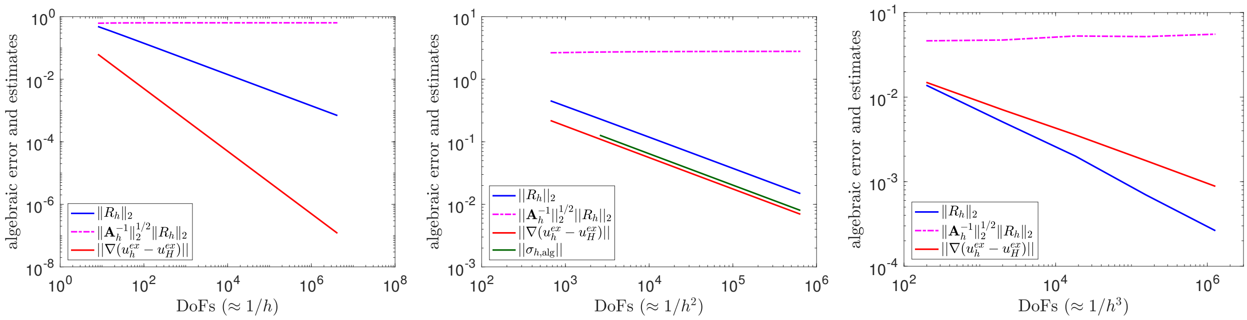


Figure 1: Unit cube problem with u_H^{ex} as an approximate algebraic solution to u_h^{ex}. The algebraic error $\|\nabla(u_h^{\text{ex}} - u_H^{\text{ex}})\|$, the norm $\|\mathrm{R}_h\|_2$ of the associated residual, and the guaranteed upper bound $\|\mathbb{A}_h^{-1}\|_2^{1/2}\|\mathrm{R}_h\|_2$ for the varying mesh size h in one- (left), two- (middle), and three-dimensions (right). For $d = 2$, we also plot the guaranteed upper bound $\eta_{6.2}^{p} = \|\boldsymbol{\sigma}_{h,\text{alg}}\|$ on the algebraic error from (9.9b)

We can see that the Euclidean norm of the algebraic residual $\|\mathrm{R}_h\|_2$ behaves with decreasing h *differently* for *each space dimension*. Though in two space dimensions, it happens to have the correct convergence rate and to be rather sharp, it dangerously over(under)estimates the algebraic error in one (three) space dimensions. The guaranteed upper bound $\|\mathbb{A}_h^{-1}\|_2^{1/2}\|\mathrm{R}_h\|_2$ provides in all dimensions misleading information on the energy norm of the error, further deteriorating with mesh refinement. Finally, in the two-dimensional case, the upper bound $\eta_{6.2}^{p} = \|\boldsymbol{\sigma}_{h,\text{alg}}\|$ on the algebraic error, proposed in Sec. 6 of this work is depicted to show that it tightly follows the algebraic error, in adequacy with the theory developed here.

3 Setting

This section introduces the piecewise polynomial spaces used in the construction of the liftings and estimators.

3.1 Hierarchy of meshes

Let $\Omega \subset \mathbb{R}^d$, $1 \le d \le 3$, be an open bounded polytope with a Lipschitz-continuous boundary. Let $\mathcal{T}_h$ be a simplicial mesh of Ω, matching in the sense that for two distinct elements K of $\mathcal{T}_h$, their intersection is either an empty set or a common vertex, edge, or face. Associated with $\mathcal{T}_h$, let there be a hierarchy of meshes $\{\mathcal{T}_j\}_{0\le j\le J}$. These are again matching simplicial partitions of the domain Ω, *nested* in the sense that $\mathcal{T}_j$ is a refinement of $\mathcal{T}_{j-1}$, $1 \le j \le J$, and satisfying $\mathcal{T}_h = \mathcal{T}_J$. The set of vertices of $\mathcal{T}_j$ is denoted by $\mathcal{V}_j$, and it is decomposed into interior vertices $\mathcal{V}_j^{\rm int}$ and boundary vertices $\mathcal{V}_j^{\rm ext}$. By $\psi_j^{\mathbf{a}}$ we denote the standard hat function associated with the vertex $\mathbf{a} \in \mathcal{V}_j$, $0 \le j \le J$, i.e., the function that is piecewise affine with respect to the j-th level mesh $\mathcal{T}_j$, taking the value 1 at the vertex $\mathbf{a}$ and zero at all other j-th level vertices of $\mathcal{V}_j$. The interior of the support of $\psi_j^{\mathbf{a}}$ is denoted by $\omega_j^{\mathbf{a}}$ and it corresponds to the patch of elements of $\mathcal{T}_j$ which share the vertex $\mathbf{a} \in \mathcal{V}_j$. For $\mathbf{a} \in \mathcal{V}_{j-1}$, $1 \le j \le J$, we denote $\mathcal{V}_j^{\mathbf{a}} := \mathcal{V}_j \cap \overline{\omega_{j-1}^{\mathbf{a}}}$, the set of vertices on a finer level that lie in the patch $\omega_{j-1}^{\mathbf{a}}$ (including its boundary). Finally, we identify $\omega_h^{\mathbf{a}} := \omega_J^{\mathbf{a}}$ and $\mathcal{V}_h := \mathcal{V}_J$, $\mathcal{V}_h^{\rm int} := \mathcal{V}_J^{\rm int}$, $\mathcal{V}_h^{\rm ext} := \mathcal{V}_J^{\rm ext}$.

3.2 Finite element approximation spaces

The pth-order conforming finite element space associated with the mesh $\mathcal{T}_j$ is denoted by V_j^p and defined as

$$\begin{aligned}
\mathbb{P}^q(\mathcal{T}_j) &:= \{v_h \in L^2(\Omega), v_h|_K \in \mathbb{P}^q(K) \quad \forall K \in \mathcal{T}_j\}, && 0 \le q,\ 0 \le j \le J,\\
V_j^p &:= \mathbb{P}^p(\mathcal{T}_j) \cap H_0^1(\Omega), && 1 \le p,\ 0 \le j \le J,
\end{aligned}$$

In the following, the polynomial degree p will be fixed and we use the notation $V_h^p := V_J^p$. By $(\cdot,\cdot)$ we denote the $L^2(\Omega)$ or $[L^2(\Omega)]^d$ scalar product and by Π_j^q the (L^2)-orthogonal projection onto $\mathbb{P}^q(\mathcal{T}_j)$.

3.3 Finite element spaces for $\mathbf{H}(\mathrm{div},\Omega)$-lifting

For our estimates, we construct $\mathbf{H}(\mathrm{div},\Omega)$-conforming vector-valued liftings in Raviart–Thomas (–Nédélec in three space dimensions)(RTN) finite element spaces $\mathbf{V}_j^q \subset \mathbf{H}(\mathrm{div},\Omega)$, $q \ge 0$. These contain vector-valued piecewise polynomials with continuous normal trace and are given by $\mathbf{V}_j^q := \{\mathbf{v}_j \in \mathbf{H}(\mathrm{div},\Omega); \mathbf{v}_j|_K \in [\mathbb{P}^q(K)]^d + \mathbb{P}^q(K)\mathbf{x} \quad \forall K \in \mathcal{T}_j\}$, see, e.g., Brezzi and Fortin [9]. We denote $\mathbf{V}_h^q := \mathbf{V}_J^q$ for a fixed polynomial degree q that we choose as either $q = p$ or $q = 0$, where, recall, p is the polynomial degree of the finite element approximation space V_h^p.

The constructions below use also the local subspaces

$$\begin{aligned}
\mathbf{V}_j^q(\omega_k^{\mathbf{a}}) &:= \{\mathbf{v}_j \in \mathbf{V}_j^q|_{\omega_k^{\mathbf{a}}};\ \mathbf{v}_j\cdot\mathbf{n}_{\omega_k^{\mathbf{a}}} = 0 \text{ on } \partial\omega_k^{\mathbf{a}}\},\\
Q_j^q(\omega_k^{\mathbf{a}}) &:= \{q_j \in \mathbb{P}^q(\mathcal{T}_j)|_{\omega_k^{\mathbf{a}}};\ (q_j,1)_{\omega_k^{\mathbf{a}}} = 0\},
\end{aligned} \qquad \mathbf{a} \in \mathcal{V}_k^{\rm int}, \tag{3.1a}$$

$$\begin{aligned}
\mathbf{V}_j^q(\omega_k^{\mathbf{a}}) &:= \{\mathbf{v}_j \in \mathbf{V}_j^q|_{\omega_k^{\mathbf{a}}};\ \mathbf{v}_j\cdot\mathbf{n}_{\omega_k^{\mathbf{a}}} = 0 \text{ on } \partial\omega_k^{\mathbf{a}} \setminus \partial\Omega\},\\
Q_j^q(\omega_k^{\mathbf{a}}) &:= \{q_j \in \mathbb{P}^p(\mathcal{T}_j)|_{\omega_k^{\mathbf{a}}}\},
\end{aligned} \qquad \mathbf{a} \in \mathcal{V}_k^{\rm ext}, \tag{3.1b}$$

where k is equal to either j or $j-1$. The degrees of freedom of the space $\mathbf{V}_j^1(\omega_{j-1}^{\mathbf{a}})$ are illustrated in Fig. 2.

4 Coarsest-level Riesz representer

Let $r_h \in \mathbb{P}^p(\mathcal{T}_h)$ and $\mathbf{A} \in [\mathbb{P}^0(\mathcal{T}_j)]^{d\times d}$. We will use below as a crucial step in the construction of the liftings a piecewise affine *Riesz representer* $\rho_{0,\rm alg} \in V_0^1$ of r_h given by

$$(\mathbf{A}\nabla\rho_{0,\rm alg}, \nabla v_0) = (r_h, v_0) \qquad \forall v_0 \in V_0^1. \tag{4.1}$$

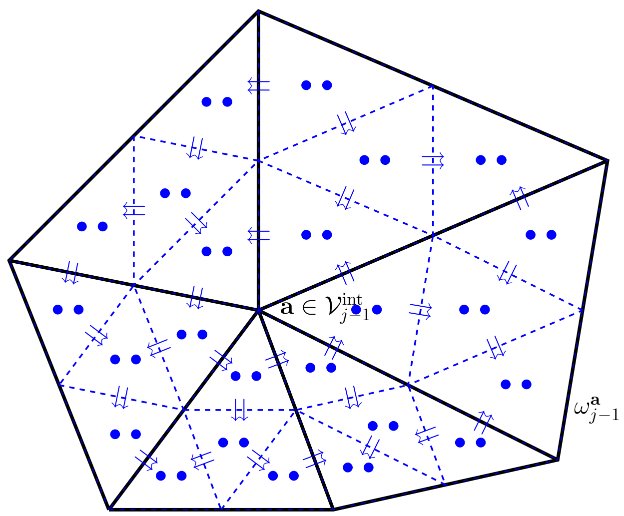
This procedure is familiar from the coarsest-grid residual solve in multigrid methods; see, e.g., [13, Section 2.5]. The piecewise affine Riesz representer was also used for estimating the algebraic error in hierarchical setting, e.g., in [18].

5 $\mathbf{H}(\mathrm{div},\Omega)$-lifting of a given piecewise polynomial with large patch solves

In [20, Def. 4.3], a $\mathbf{H}(\mathrm{div},\Omega)$-lifting of the given piecewise polynomial $r_h \in \mathbb{P}^p(\mathcal{T}_h)$ was introduced such that

$$\boldsymbol{\sigma}_{h,\rm alg} \in \mathbf{V}_J^p, \qquad \nabla\cdot\boldsymbol{\sigma}_{h,\rm alg} = r_h, \text{ where } \|\boldsymbol{\sigma}_{h,\rm alg}\| \text{ is locally minimized.} \tag{5.1}$$

This is the basis of our developments. We now recall it here,



coarse patch subdomain $\omega_{j-1}^{\mathbf{a}}$ for $\mathbf{a} \in \mathcal{V}_{j-1}^{\text{int}}$
coarse mesh $\mathcal{T}_{j-1}$ of $\omega_{j-1}^{\mathbf{a}}$ (full line)
fine mesh $\mathcal{T}_j$ of $\omega_{j-1}^{\mathbf{a}}$ (dashed line)
degrees of freedom of $\mathbf{V}_j^{p'}(\omega_{j-1}^{\mathbf{a}})$ for the $\mathbf{H}(\text{div}, \Omega)$ Neumann solve on large patch subdomains ($p' = 1$, arrows and bullets)

Figure 2: $\mathbf{H}(\text{div}, \Omega)$-lifting of Construction 5.1 by the large patch solve; degrees of freedom of local RTN space $\mathbf{V}_j^1(\omega_{j-1}^{\mathbf{a}})$ from (3.1)

Construction 5.1 ($\mathbf{H}(\text{div}, \Omega)$-lifting with large patch solves)**.** *Given* $r_h \in \mathbb{P}^p(\mathcal{T}_h)$, $\mathbf{A} \in [\mathbb{P}^0(\mathcal{T}_j)]^{d \times d}$, *and the corresponding* $\rho_{0,\text{alg}} \in V_0^1$ *given by* (4.1), *construct* $\boldsymbol{\sigma}_{h,\text{alg}} \in \mathbf{V}_J^p$ *as a hierarchical sum of local contributions*

$$\boldsymbol{\sigma}_{h,\text{alg}} := \sum_{j=1}^{J} \sum_{\mathbf{a} \in \mathcal{V}_{j-1}} \boldsymbol{\sigma}_{j,\text{alg}}^{\mathbf{a}}, \tag{5.2}$$

where the local contributions $\boldsymbol{\sigma}_{j,\text{alg}}^{\mathbf{a}} \in \mathbf{V}_j^p(\omega_{j-1}^{\mathbf{a}})$ *solve the mixed finite element problems: find* $(\boldsymbol{\sigma}_{j,\text{alg}}^{\mathbf{a}}, \gamma_j) \in \mathbf{V}_j^p(\omega_{j-1}^{\mathbf{a}}) \times Q_j^p(\omega_{j-1}^{\mathbf{a}})$ *such that*

$$(\mathbf{A}^{-1}\boldsymbol{\sigma}_{j,\text{alg}}^{\mathbf{a}}, \mathbf{v}_j^{\mathbf{a}})_{\omega_{j-1}^{\mathbf{a}}} - (\gamma_j, \nabla \cdot \mathbf{v}_j^{\mathbf{a}})_{\omega_{j-1}^{\mathbf{a}}} = 0 \qquad \forall \mathbf{v}_j^{\mathbf{a}} \in \mathbf{V}_j^p(\omega_{j-1}^{\mathbf{a}}), \tag{5.3a}$$

$$(\nabla \cdot \boldsymbol{\sigma}_{j,\text{alg}}^{\mathbf{a}}, q_j)_{\omega_{j-1}^{\mathbf{a}}} = (g^{\mathbf{a},j}, q_j)_{\omega_{j-1}^{\mathbf{a}}} \qquad \forall q_j \in Q_j^p(\omega_{j-1}^{\mathbf{a}}), \tag{5.3b}$$

with

$$g^{\mathbf{a},j} := \begin{cases} r_h \psi_{j-1}^{\mathbf{a}} - \mathbf{A}\nabla\rho_{0,\text{alg}} \cdot \nabla\psi_{j-1}^{\mathbf{a}} & j = 1, \\ (\text{Id} - \Pi_j^p)(r_h \psi_{j-1}^{\mathbf{a}}) & 1 < 1 \leq J. \end{cases} \tag{5.4}$$

We note that the equivalent formulation of the problem (5.3) is

$$\boldsymbol{\sigma}_{j,\text{alg}}^{\mathbf{a}} = \underset{\substack{\mathbf{v}_j \in \mathbf{V}_j^p(\omega_{j-1}^{\mathbf{a}}) \\ \nabla \cdot \mathbf{v}_j = \Pi_j^p g^{\mathbf{a},j}}}{\arg\min} \|\mathbf{A}^{-\frac{1}{2}}\mathbf{v}_j\|_{\omega_{j-1}^{\mathbf{a}}}. \tag{5.5}$$

The drawback of Construction 5.1 is the computational cost of the local solves in (5.3) (recall the degrees of freedom of $\mathbf{V}_j^1(\omega_{j-1}^{\mathbf{a}})$ for $p = 1$ in Fig. 2). Some possibilities of computational simplifications while not changing Construction 5.1 are presented in [20, Sec. 8], but all these possibilities still involve solving at least some problems of the form (5.3). In the following sections, we present three ways how to replace the local solve (5.3) or (5.5) by a much cheaper procedure, while still maintaining all important theoretical properties.

6 $\mathbf{H}(\text{div}, \Omega)$-lifting of a given piecewise polynomial with small patch solves

The construction presented in this section ensures

$$\boldsymbol{\sigma}_{h,\text{alg}} \in \mathbf{V}_J^p, \qquad \nabla \cdot \boldsymbol{\sigma}_{h,\text{alg}} = r_h, \tag{6.1}$$

as in (5.1), but differs from Construction 5.1 in two key features. First, the contributions on the intermediate levels $j = 1, \dots, J-1$ are computed in the *lowest degree* Raviart–Thomas(–Nédelec) space $\mathbf{V}_j^0$. Second, the minimizations (5.3) on all levels $j = 1, \dots, J$ and over every patch $\omega_{j-1}^{\mathbf{a}}$, $\mathbf{a} \in \mathcal{V}_{j-1}$, are replaced by a single piecewise affine conforming FEM problem on the large patch $\omega_{j-1}^{\mathbf{a}}$ and several smaller problems posed on small patch subdomains $\omega^{\mathbf{a},\mathbf{a}'}$, $\mathbf{a}' \in \mathcal{V}_j^{\mathbf{a}}$, where $\mathcal{V}_j^{\mathbf{a}}$ denotes the set of vertices of $\mathcal{V}_j$ that lie in $\overline{\omega_{j-1}^{\mathbf{a}}}$ and

$$\omega^{\mathbf{a},\mathbf{a}'} := \omega_j^{\mathbf{a}'} \cap \omega_{j-1}^{\mathbf{a}}, \quad \mathbf{a}' \in \mathcal{V}_j^{\mathbf{a}};$$

see Figure 3. This significantly reduces the size of the associated local linear algebraic systems. The construction is illustrated in Fig. 3 and described in full detail below. Here we denote

$$H_\star^1(\omega_{j-1}^{\mathbf{a}}) := \{v \in H^1(\omega_{j-1}^{\mathbf{a}}), (v,1)_{\omega_{j-1}^{\mathbf{a}}} = 0\} \qquad \mathbf{a} \in \mathcal{V}_{j-1}^{\mathrm{int}}, \tag{6.2a}$$

$$H_\star^1(\omega_{j-1}^{\mathbf{a}}) := \{v \in H^1(\omega_{j-1}^{\mathbf{a}}), v = 0 \text{ on } \partial\Omega\} \qquad \mathbf{a} \in \mathcal{V}_{j-1}^{\mathrm{ext}}. \tag{6.2b}$$

and (cf. (3.1))

$$\mathbf{V}_j^{p'}(\omega^{\mathbf{a},\mathbf{a}'}) := \mathbf{V}_j^{p'}(\omega_j^{\mathbf{a}'})|_{\omega^{\mathbf{a},\mathbf{a}'}}, \qquad Q_j^{p'}(\omega^{\mathbf{a},\mathbf{a}'}) := Q_j^{p'}(\omega_j^{\mathbf{a}'})|_{\omega^{\mathbf{a},\mathbf{a}'}}.$$

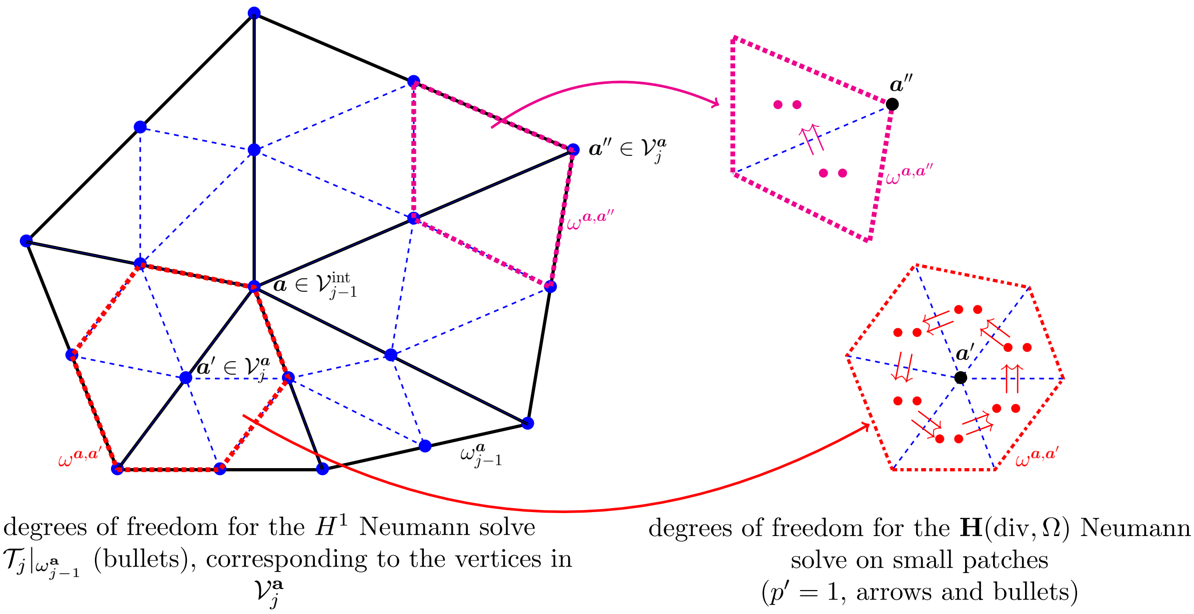


Figure 3: $\mathbf{H}(\mathrm{div},\Omega)$-lifting of Construction 6.1 and Construction 6.2 by piecewise affine conforming FEM problem on the large patch subdomain $\omega_{j-1}^{\mathbf{a}}$ and smaller problems posed on small patch subdomains $\omega^{\mathbf{a},\mathbf{a}'}$, $\mathbf{a}' \in \mathcal{V}_j^{\mathbf{a}}$

Construction 6.1 ($\mathbf{H}(\mathrm{div},\Omega)$-lifting eliminating large patch RTN solves)**.** *Given* $r_h \in \mathbb{P}^p(\mathcal{T}_h)$, $\mathbf{A} \in [\mathbb{P}^0(\mathcal{T}_j)]^{d\times d}$, *and the corresponding* $\rho_{0,\mathrm{alg}} \in V_0^1$ *given by* (4.1)*, set*

$$p' := 0 \quad \text{for } 1 \le j < J \quad \text{and} \quad p' := p \quad \text{for } j = J. \tag{6.3}$$

Then, for each level j, $j = 1, \dots, J$ *:*

1. *For each vertex* $\mathbf{a} \in \mathcal{V}_{j-1}$*, find* $\iota_j^{\mathbf{a}} \in \mathbb{P}^1(\mathcal{T}_j)|_{\omega_{j-1}^{\mathbf{a}}} \cap H_\star^1(\omega_{j-1}^{\mathbf{a}})$*, a continuous piecewise affine polynomial (with zero mean value on* $\omega_{j-1}^{\mathbf{a}}$ *if* $\mathbf{a} \in \mathcal{V}_{j-1}^{\mathrm{int}}$ *and zero on* $\partial\Omega$ *if* $\mathbf{a} \in \mathcal{V}_{j-1}^{\mathrm{ext}}$*, see* (6.2)*), such that*

$$(\nabla \iota_j^{\mathbf{a}}, \nabla v_j)_{\omega_{j-1}^{\mathbf{a}}} = (g^{I,\mathbf{a},j}, v_j)_{\omega_{j-1}^{\mathbf{a}}} \qquad \forall v_j \in \mathbb{P}^1(\mathcal{T}_j)|_{\omega_{j-1}^{\mathbf{a}}} \cap H_\star^1(\omega_{j-1}^{\mathbf{a}}). \tag{6.4}$$

2. *Run through all vertices* $\mathbf{a}' \in \mathcal{V}_j^{\mathbf{a}}$ *(including those on the boundary of* $\omega_{j-1}^{\mathbf{a}}$*) and on each small patch subdomain* $\omega^{\mathbf{a},\mathbf{a}'}$ *construct* $\boldsymbol{\sigma}_{j,\mathrm{alg}}^{\mathbf{a},\mathbf{a}'} \in \mathbf{V}_j^{p'}(\omega^{\mathbf{a},\mathbf{a}'})$*, such that*

$$\nabla \cdot \boldsymbol{\sigma}_{j,\mathrm{alg}}^{\mathbf{a},\mathbf{a}'} = g^{II,\mathbf{a},\mathbf{a}',j}. \tag{6.5}$$

Finally,

$$\boldsymbol{\sigma}_{j,\mathrm{alg}}^{\mathbf{a}} := \sum_{\mathbf{a}'\in\mathcal{V}_j^{\mathbf{a}}} \boldsymbol{\sigma}_{j,\mathrm{alg}}^{\mathbf{a},\mathbf{a}'}, \qquad \boldsymbol{\sigma}_{h,\mathrm{alg}} := \sum_{j=1}^{J} \sum_{\mathbf{a}\in\mathcal{V}_{j-1}} \boldsymbol{\sigma}_{j,\mathrm{alg}}^{\mathbf{a}}. \tag{6.6}$$

The right-hand sides in (6.4) *and* (6.5) *are respectively*

$$g^{I,\mathbf{a},j} := \begin{cases} \Pi_1^0(r_h\psi_0^{\mathbf{a}}) - \mathbf{A}\nabla\rho_{0,\mathrm{alg}}\cdot\nabla\psi_0^{\mathbf{a}} & j=1,\\ \Pi_j^{p'}(\mathrm{Id}-\Pi_{j-1}^0)(r_h\psi_{j-1}^{\mathbf{a}}) & j=2,\dots,J,\end{cases} \tag{6.7}$$

and

$$g^{II,\mathbf{a},\mathbf{a}',j} := \begin{cases} \Pi_1^0\big[[\Pi_1^0(r_h\psi_0^{\mathbf{a}}) - \nabla\rho_{0,\mathrm{alg}}\cdot\nabla\psi_0^{\mathbf{a}}]\,\psi_1^{\mathbf{a}'}\big] - \nabla\iota_1^{\mathbf{a}}\cdot\nabla\psi_1^{\mathbf{a}'} & j=1,\\ \Pi_j^{p'}[\Pi_j^{p'}(\mathrm{Id}-\Pi_{j-1}^0)(r_h\psi_{j-1}^{\mathbf{a}})\,\psi_j^{\mathbf{a}'}] - \nabla\iota_j^{\mathbf{a}}\cdot\nabla\psi_j^{\mathbf{a}'} & j=2,\dots,J.\end{cases} \tag{6.8}$$

It remains to detail the construction of $\boldsymbol{\sigma}_{j,\mathrm{alg}}^{\mathbf{a},\mathbf{a}'} \in \mathbf{V}_j^{p'}(\omega^{\mathbf{a},\mathbf{a}'})$ in (6.5). The first construction is similar to Construction 5.1, but the local minimization is done over a small patch, and it is therefore significantly cheaper; cf. Figs 2 and 3.

Construction 6.2 (Local lifting $\boldsymbol{\sigma}_{j,\mathrm{alg}}^{\mathbf{a},\mathbf{a}'}$ by a small patch RTN solve)**.** *Let $j=1,\dots,J$, $\mathbf{a}\in\mathcal{V}_{j-1}$, and $\mathbf{a}'\in\mathcal{V}_j^{\mathbf{a}}$ be given. Set $p':=0$ if $1\le j<J$ and $p':=p$ for $j=J$. Let $g^{II,\mathbf{a},\mathbf{a}',j}\in\mathbb{P}^{p'}(\mathcal{T}_j)|_{\omega^{\mathbf{a},\mathbf{a}'}}$ be given by* (6.8)*. Define $\boldsymbol{\sigma}_{j,\mathrm{alg}}^{\mathbf{a},\mathbf{a}'}\in\mathbf{V}_j^{p'}(\omega^{\mathbf{a},\mathbf{a}'})$ as the solution of the mixed finite element problem: find $(\boldsymbol{\sigma}_{j,\mathrm{alg}}^{\mathbf{a},\mathbf{a}'},\gamma_j)\in\mathbf{V}_j^{p'}(\omega^{\mathbf{a},\mathbf{a}'})\times Q_j^{p'}(\omega^{\mathbf{a},\mathbf{a}'})$ such that*

$$(\mathbf{A}^{-1}\boldsymbol{\sigma}_{j,\mathrm{alg}}^{\mathbf{a},\mathbf{a}'},\mathbf{v}_j^{\mathbf{a}'})_{\omega^{\mathbf{a},\mathbf{a}'}} - (\gamma_j,\nabla{\cdot}\mathbf{v}_j^{\mathbf{a}'})_{\omega^{\mathbf{a},\mathbf{a}'}} = 0 \qquad \forall\mathbf{v}_j^{\mathbf{a}'}\in\mathbf{V}_j^{p'}(\omega^{\mathbf{a},\mathbf{a}'}), \tag{6.9a}$$

$$(\nabla{\cdot}\boldsymbol{\sigma}_{j,\mathrm{alg}}^{\mathbf{a},\mathbf{a}'},q_j)_{\omega^{\mathbf{a},\mathbf{a}'}} = (g^{II,\mathbf{a},\mathbf{a}',j},q_j)_{\omega^{\mathbf{a},\mathbf{a}'}} \qquad \forall q_j\in Q_j^{p'}(\omega^{\mathbf{a},\mathbf{a}'}). \tag{6.9b}$$

Analogously to (5.3) and (5.5), an equivalent formulation of the problem (6.9) is

$$\boldsymbol{\sigma}_{j,\mathrm{alg}}^{\mathbf{a},\mathbf{a}'} = \mathop{\arg\min}_{\substack{\mathbf{v}_j\in\mathbf{V}_j^{p'}(\omega^{\mathbf{a},\mathbf{a}'})\\ \nabla\cdot\mathbf{v}_j=g^{II,\mathbf{a},\mathbf{a}',j}}} \|\mathbf{A}^{-\frac12}\mathbf{v}_j\|_{\omega}^{\mathbf{a},\mathbf{a}'}. \tag{6.10}$$

Lemma 6.3 (Properties of $\boldsymbol{\sigma}_{h,\mathrm{alg}}$)**.** *Any lifting $\boldsymbol{\sigma}_{h,\mathrm{alg}}$ of Construction 6.1 satisfies* (6.1)*.*

Proof. All the contributions $\boldsymbol{\sigma}_{j,\mathrm{alg}}^{\mathbf{a},\mathbf{a}'}$ and $\boldsymbol{\sigma}_{j,\mathrm{alg}}^{\mathbf{a}}$ extended by zero to Ω belong to $\mathbf{V}_j^0$, for $j<J$, and to $\mathbf{V}_j^p$, for $j=J$, so that $\boldsymbol{\sigma}_{h,\mathrm{alg}}\in\mathbf{V}_J^p$. For the divergence claim, let $\boldsymbol{\sigma}_{j,\mathrm{alg}} := \sum_{\mathbf{a}\in\mathcal{V}_{j-1}}\boldsymbol{\sigma}_{j,\mathrm{alg}}^{\mathbf{a}}$, $1\le j\le J$. We rely on the two partitions of unity

$$\sum_{\mathbf{a}'\in\mathcal{V}_j^{\mathbf{a}}}\psi_j^{\mathbf{a}'} = 1 \text{ on } \omega_{j-1}^{\mathbf{a}} \qquad \forall\mathbf{a}\in\mathcal{V}_{j-1},\ \forall j=1\dots J \tag{6.11}$$

and

$$\sum_{\mathbf{a}\in\mathcal{V}_{j-1}}\psi_{j-1}^{\mathbf{a}} = 1 \text{ on } \Omega, \qquad \forall j=1\dots J \tag{6.12}$$

and proceed in four steps.

Step 1: First, using the partition of unity (6.11) for any $2\le j<J$ and owing to the choice (6.3), $\boldsymbol{\sigma}_{j,\mathrm{alg}}^{\mathbf{a}}$ satisfying (6.5)–(6.6) fulfills

$$\nabla{\cdot}\boldsymbol{\sigma}_{j,\mathrm{alg}}^{\mathbf{a}} = \sum_{\mathbf{a}'\in\mathcal{V}_j^{\mathbf{a}}}\nabla{\cdot}\boldsymbol{\sigma}_{j,\mathrm{alg}}^{\mathbf{a},\mathbf{a}'} = \Pi_j^0[\Pi_j^0(\mathrm{Id}-\Pi_{j-1}^0)(r_h\psi_{j-1}^{\mathbf{a}})] = \Pi_j^0(\mathrm{Id}-\Pi_{j-1}^0)(r_h\psi_{j-1}^{\mathbf{a}}).$$

Consequently, using the partition of unity (6.12),

$$\nabla{\cdot}\boldsymbol{\sigma}_{j,\mathrm{alg}} = \sum_{\mathbf{a}\in\mathcal{V}_{j-1}}\nabla{\cdot}\boldsymbol{\sigma}_{j,\mathrm{alg}}^{\mathbf{a}} = \Pi_j^0(\mathrm{Id}-\Pi_{j-1}^0)r_h = \Pi_j^0 r_h - \Pi_{j-1}^0 r_h.$$

Step 2: On the first mesh $\mathcal{T}_1$, we similarly see that

$$\nabla\cdot\boldsymbol{\sigma}^{\mathbf{a}}_{1,\mathrm{alg}} = \sum_{\mathbf{a}'\in\mathcal{V}^{\mathbf{a}}_J} \nabla\cdot\boldsymbol{\sigma}^{\mathbf{a},\mathbf{a}'}_{1,\mathrm{alg}} = \Pi^0_1(r_h\psi^{\mathbf{a}}_0) - \nabla\rho_{0,\mathrm{alg}}\cdot\nabla\psi^{\mathbf{a}}_0 \quad \text{and} \quad \nabla\cdot\boldsymbol{\sigma}_{1,\mathrm{alg}} = \Pi^0_1 r_h.$$

Step 3: On the finest mesh $\mathcal{T}_J$,

$$\nabla\cdot\boldsymbol{\sigma}^{\mathbf{a}}_{J,\mathrm{alg}} = \sum_{\mathbf{a}'\in\mathcal{V}^{\mathbf{a}}_j} \nabla\cdot\boldsymbol{\sigma}^{\mathbf{a},\mathbf{a}'}_{J,\mathrm{alg}} = \Pi^p_J(\mathrm{Id}-\Pi^0_{j-1})(r_h\psi^{\mathbf{a}}_{j-1}) \quad \text{and} \quad \nabla\cdot\boldsymbol{\sigma}^{\mathbf{a}}_{J,\mathrm{alg}} = \Pi^p_J r_h - \Pi^0_{J-1} r_h = r_h - \Pi^0_{J-1} r_h. \tag{6.13}$$

Step 4: From (6.6) and the three above steps, we now conclude

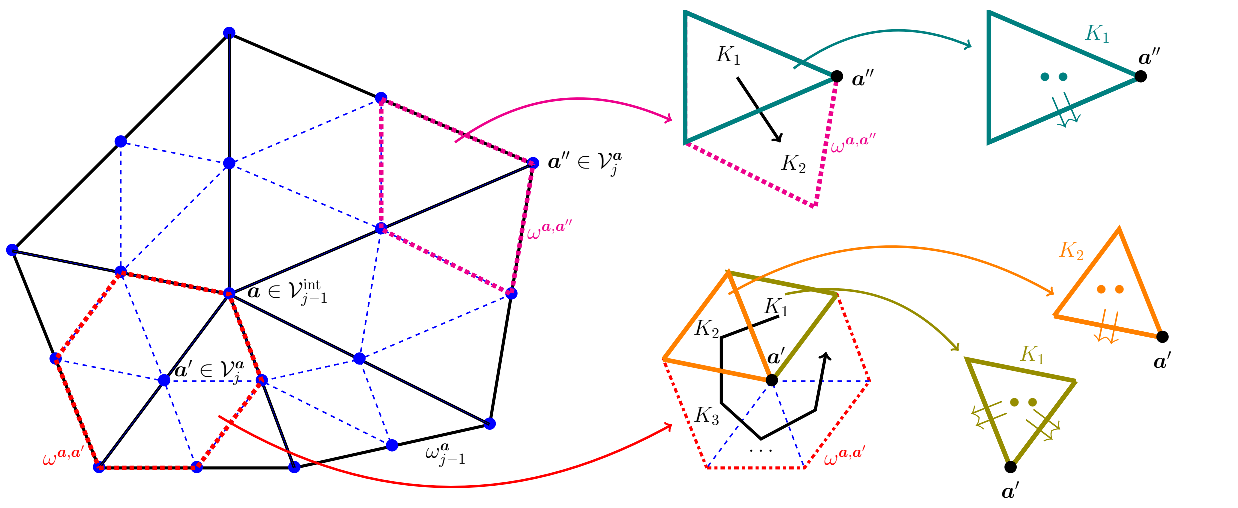
$$\nabla\cdot\boldsymbol{\sigma}_{h,\mathrm{alg}} = \sum_{j=1}^{J} \nabla\cdot\boldsymbol{\sigma}_{j,\mathrm{alg}} = \Pi^0_1 r_h + \sum_{j=2}^{J-1}(\Pi^0_j r_h - \Pi^0_{j-1} r_h) + r_h - \Pi^0_{J-1} r_h = r_h.$$

□

7 $\mathbf{H}(\mathrm{div},\Omega)$-lifting of a given piecewise polynomial with single element solves

This section presents a construction of local lifting $\boldsymbol{\sigma}^{\mathbf{a},\mathbf{a}'}_{j,\mathrm{alg}}$ satisfying (6.5) that is still cheaper than Construction 6.2. The local construction presented below follows [16, Sec. 7.3], [12, Sec. 5.1] and involves a sequential sweep through the elements of each small patch. As in (5.1) and (6.1), it leads to

$$\boldsymbol{\sigma}_{h,\mathrm{alg}} \in \mathbf{V}^p_J, \qquad \nabla\cdot\boldsymbol{\sigma}_{h,\mathrm{alg}} = r_h. \tag{7.1}$$



degrees of freedom for the H^1 Neumann solve on $\mathcal{T}_{j-1}|_{\omega^{\mathbf{a}}_{j-1}}$ (bullets) corresponding to the vertices in $\mathcal{V}^{\mathbf{a}}_j$

explicit run through small patch

degrees of freedom for the $\mathbf{H}(\mathrm{div},\Omega)$ Neumann solve on a single element ($p'=1$, arrows and bullets)

Figure 4: $\mathbf{H}(\mathrm{div},\Omega)$-lifting of Construction 6.1 and Construction 7.1 by piecewise affine conforming FEM problem on the large patch subdomain $\omega^{\mathbf{a}}_{j-1}$, a sequential sweep through the elements of small patches $\omega^{\mathbf{a},\mathbf{a}'}$, $\mathbf{a}'\in\mathcal{V}^{\mathbf{a}}_j$, and single element solve on $K_i \in \overline{\omega^{\mathbf{a},\mathbf{a}'}}$

Construction 7.1 (Local lifting $\boldsymbol{\sigma}^{\mathbf{a},\mathbf{a}'}_{j,\mathrm{alg}}$ by a sweep over small patch and single element solves)**.** *Let $j\in\{1,\dots,J\}$, $\mathbf{a}\in\mathcal{V}_{j-1}$, and $\mathbf{a}'\in\mathcal{V}^{\mathbf{a}}_j$ be given. Set $p' := 0$ for $1\le j<J$ and $p' := p$ for $j=J$. Let $g^{II,\mathbf{a},\mathbf{a}',j}\in\mathbb{P}^{p'}(\mathcal{T}_j)|_{\omega^{\mathbf{a},\mathbf{a}'}}$ be given by (6.8). Let the simplices in the patch subdomain $\omega^{\mathbf{a},\mathbf{a}'}$ be ordered as K_1, K_2, …, K_M such that K_i and K_{i-1} share a common $(d-1)$-dimensional face, $i=2,\dots,M$ (the fact that a suitable ordering exists is obvious in 1D and 2D; the proof for 3D is given in [12, Lemma B.1]). Then, for $i=1,\dots,M$:*

1. *Find* $\boldsymbol{\sigma}_i \in \mathbf{RTN}^{p'}(K_i) = \{\mathbf{v} \in [\mathbb{P}^{p'}(K_i)]^d + \mathbb{P}^{p'}(K_i)\mathbf{x}\}$ *such that*

$$\boldsymbol{\sigma}_i = \underset{\mathbf{v}_j \in \widetilde{\mathbf{RTN}}(K_i)}{\arg\min} \|\mathbf{A}^{-\frac{1}{2}}\mathbf{v}_j\|_{K_i},$$

where $\widetilde{\mathbf{RTN}}(K_i)$ *are functions of* $\mathbf{RTN}^{p'}(K_i)$ *such that*

$$\nabla{\cdot}\widetilde{\mathbf{v}_j} = g^{II,\mathbf{a},\mathbf{a}',j},$$

$$\widetilde{\mathbf{v}_j}{\cdot}\mathbf{n}_F = \begin{cases} \boldsymbol{\sigma}_{j,\mathrm{alg}}^{\mathbf{a},\mathbf{a}'}{\cdot}\mathbf{n}_F & \text{for } F \subset \partial K_i \cap \partial K_\ell, \quad \ell < i \\ 0 & \text{for } F \subset \partial K_i \cap \partial\omega^{\mathbf{a},\mathbf{a}'}, \quad \text{if } \mathbf{a} \in \mathcal{V}_{j-1}^{\mathrm{int}}, \\ 0 & \text{for } F \subset \partial K_i \cap (\partial\omega^{\mathbf{a},\mathbf{a}'} \setminus \partial\Omega), \quad \text{if } \mathbf{a} \in \mathcal{V}_{j-1}^{\mathrm{ext}} \wedge \mathbf{a}' \in \mathcal{V}_j^{\mathrm{ext}}. \end{cases}$$

Here F denotes a $(d-1)$-dimensional face and $\mathbf{n}_F$ the associated normal vector, whose orientation is chosen arbitrarily but fixed for interior faces and coinciding with the exterior normal of Ω for exterior faces. An illustration is given in Fig. 4 for $\mathbf{a} \in \mathcal{V}_{j-1}^{\mathrm{int}}$.

2. *Set* $\boldsymbol{\sigma}_{j,\mathrm{alg}}^{\mathbf{a},\mathbf{a}'}|_{K_i} := \boldsymbol{\sigma}_i$.

For $p' = 0$ in two space dimensions, the dimension of the local space $\widetilde{\mathbf{RTN}}(K_i)$ is equal to one on all but the first element K_1 and last element K_M; there, it is respectively equal to 2 and 0. Consequently, for the lowest-degree RTN space with $p' = 0$, where the degrees of freedom correspond to the flux through the faces of the elements, Construction 7.1 gives a formula that is explicit without any need for solving an algebraic system for all but the first element.

8 Approximate lifting in the lowest-order RTN space

Setting $p' = 0$ also for $j = J$ in Constructions 6.1 and 7.1 leads as in Lemma 6.3 to

$$\boldsymbol{\sigma}_{h,0,\mathrm{alg}} \in \mathbf{V}_J^0, \qquad \nabla{\cdot}\boldsymbol{\sigma}_{h,0,\mathrm{alg}} = \Pi_J^0 r_h. \tag{8.1}$$

This procedure is yet much cheaper since the lowest-order RTN functions are used also on the finest level J. The price to pay is that $\nabla{\cdot}\boldsymbol{\sigma}_{h,0,\mathrm{alg}}$ only equals to the mean value of r_h on each mesh element $K \in \mathcal{T}_h$, compare (8.1) with (7.1).

Remark 8.1 (Variants)**.** *It is also possible to construct the lowest-order approximate $\mathbf{H}(\mathrm{div},\Omega)$-liftings analogously to Construction 5.1 or Constructions 6.1 and 6.2. For the sake of simplicity, we do not consider such liftings in this manuscript.*

9 Guaranteed upper bounds on the algebraic error in the Poisson problem

We now consider a model Poisson problem discretization and the solution of the arising system of linear equations by an arbitrary algebraic solver. We present the notion of the algebraic residual and derive guaranteed upper bounds on the algebraic error that are based on the liftings presented in the previous sections.

9.1 Model problem

Given a source term $f \in L^2(\Omega)$, the model Poisson problem reads: find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega). \tag{9.1}$$

The Galerkin finite element approximation of (9.1) consists in finding $u_h^{\mathrm{ex}} \in V_h^p$ such that

$$(\nabla u_h^{\mathrm{ex}}, \nabla v_h) = (f, v_h) \qquad \forall v_h \in V_h^p. \tag{9.2}$$

Let ψ_h^l, $1 \le l \le N_h$, $N_h := \dim V_h^p$, form a basis of V_h^p. Then problem (9.2) is equivalent to solving a system of linear algebraic equations with a symmetric positive definite matrix: find $\mathrm{U}_h^{\mathrm{ex}} \in \mathbb{R}^{N_h}$ such that

$$\mathbb{A}_h \mathrm{U}_h^{\mathrm{ex}} = \mathrm{F}_h, \tag{9.3}$$

where $(\mathbb{A}_h)_{lm} := (\nabla\psi_h^m, \nabla\psi_h^l)$, $(\mathrm{F}_h)_l := (f, \psi_h^l)$, and $u_h^{\mathrm{ex}} = \sum_{m=1}^{N_h} (\mathrm{U}_h^{\mathrm{ex}})_m\, \psi_h^m$. We note that $\mathrm{U}_h^{\mathrm{ex}}$ depends on the choice of the basis of V_h^p while $u_h^{\mathrm{ex}} \in V_h^p$ does not.

9.2 Algebraic residual

Let $\mathrm{U}_h \in \mathbb{R}^{N_h}$ be an *arbitrary approximation* to the exact solution $\mathrm{U}_h^{\mathrm{ex}}$ of (9.3), corresponding to

$$u_h = \sum_{m=1}^{N_h} (\mathrm{U}_h)_m \, \psi_h^m \in V_h^p. \tag{9.4}$$

The *algebraic residual vector* is then

$$\mathrm{R}_h := \mathrm{F}_h - \mathbb{A}_h \mathrm{U}_h. \tag{9.5}$$

Following Papež *et al.* [21], we associate with R_h a *discontinuous elementwise polynomial* r_h of degree p, vanishing on the boundary of Ω, i.e., $r_h \in \mathbb{P}^p(\mathcal{T}_h)$, $r_h|_{\partial\Omega} = 0$. Denote by N_h^l the number of elements forming the support of the basis function ψ_h^l, $1 \le l \le N_h$. Then, for each fixed element $K \in \mathcal{T}_h$, we define $r_h|_K \in \mathbb{P}^p(K)$ by

$$(r_h, \psi_h^l)_K = \frac{(\mathrm{R}_h)_l}{N_h^l}, \quad r_h|_{\partial K \cap \partial\Omega} = 0, \tag{9.6}$$

for all basis functions ψ_h^l of the space V_h^p non-vanishing on K. Such r_h satisfies obviously $(\mathrm{R}_h)_l = (r_h, \psi_h^l)$, $1 \le l \le N_h$, and the algebraic relation (9.5) yields

$$(r_h, v_h) = (f, v_h) - (\nabla u_h, \nabla v_h) \qquad \forall v_h \in V_h^p. \tag{9.7}$$

We point out that although r_h is uniquely defined by (9.6), it is not the unique element in $\mathbb{P}^p(\mathcal{T}_h)$ which satisfies (9.7). It is also possible to define $r_h \in V_h^p$ directly by (9.7), as in, e.g., [6]. However, this requires the solution of a globally coupled mass system, see also [21, Sec. 5.1]. Thus we prefer to work with (9.6).

9.3 Bounds based on p-order lifting of the algebraic residual

Liftings of the algebraic residual presented in Secs. 5–7 can be used to bound the algebraic error as in [20]:

Theorem 9.1 (Guaranteed bounds on the algebraic error). *Let $u_h \in V_h^p$ be arbitrary, let $r_h \in \mathbb{P}^p(\mathcal{T}_h)$ satisfy (9.7), and let $\boldsymbol{\sigma}_{h,\mathrm{alg}} \in \mathbf{V}_J^p$ be given by any of the constructions of Sections 5–7. Then*

$$\|\nabla(u_h^{\mathrm{ex}} - u_h)\| \le \eta_{\mathrm{alg}}^p, \tag{9.8}$$

where

$$\eta_{\mathrm{alg}}^p = \eta_{5.1}^p := \|\boldsymbol{\sigma}_{h,\mathrm{alg}}\| \qquad \text{for } \boldsymbol{\sigma}_{h,\mathrm{alg}} \in \mathbf{V}_J^p \text{ given by Construction 5.1}, \tag{9.9a}$$

$$\eta_{\mathrm{alg}}^p = \eta_{6.2}^p := \|\boldsymbol{\sigma}_{h,\mathrm{alg}}\| \qquad \text{for } \boldsymbol{\sigma}_{h,\mathrm{alg}} \in \mathbf{V}_J^p \text{ given by Constructions 6.1 and 6.2}, \tag{9.9b}$$

$$\eta_{\mathrm{alg}}^p = \eta_{7.1}^p := \|\boldsymbol{\sigma}_{h,\mathrm{alg}}\| \qquad \text{for } \boldsymbol{\sigma}_{h,\mathrm{alg}} \in \mathbf{V}_J^p \text{ given by Constructions 6.1 and 7.1}. \tag{9.9c}$$

Proof. Using (5.1), (6.1), or (7.1), the Green theorem, and the Cauchy–Schwarz inequality

$$\|\nabla(u_h^{\mathrm{ex}} - u_h)\| = \sup_{v_h \in V_h^p,\, \|\nabla v_h\|=1} (r_h, v_h) = \sup_{v_h \in V_h^p,\, \|\nabla v_h\|=1} (\nabla\cdot\boldsymbol{\sigma}_{h,\mathrm{alg}}, v_h) = \sup_{v_h \in V_h^p,\, \|\nabla v_h\|=1} (\boldsymbol{\sigma}_{h,\mathrm{alg}}, \nabla v_h)$$

$$\le \sup_{v_h \in V_h^p,\, \|\nabla v_h\|=1} \|\boldsymbol{\sigma}_{h,\mathrm{alg}}\| \cdot \|\nabla v_h\| = \|\boldsymbol{\sigma}_{h,\mathrm{alg}}\|.$$

□

9.4 Bound based on the lowest-order approximate lifting of the algebraic residual

We now present an upper bound on the algebraic error using the lowest-order lifting as in Sec. 8. In order to bound the algebraic error, we have to in this case compensate for the fact that $\nabla\cdot\boldsymbol{\sigma}_{h,0,\mathrm{alg}} \neq r_h$, since $\nabla\cdot\boldsymbol{\sigma}_{h,0,\mathrm{alg}} = \Pi_J^0 r_h$ only. One can therefore expect that this bound may deteriorate for approximations using higher polynomial degrees p.

Theorem 9.2 (Guaranteed bound on the algebraic error). *Let $u_h \in V_h^p$ be arbitrary, let $r_h \in \mathbb{P}^p(\mathcal{T}_h)$ satisfy (9.7), and let $\boldsymbol{\sigma}_{h,0,\mathrm{alg}} \in \mathbf{V}_J^0$ be given by the construction in Section 8. Then*

$$\|\nabla(u_h^{\mathrm{ex}} - u_h)\| \le \eta_{\mathrm{alg}}^0, \tag{9.10}$$

where

$$\eta_{\rm alg}^0 := \left\{ \sum_{K\in\mathcal{T}_h} \left(\frac{h_K}{\pi} \|r_h - \Pi_J^0 r_h\|_K + \|\boldsymbol{\sigma}_{h,0,{\rm alg}}\|_K \right)^2 \right\}^{1/2}, \tag{9.11}$$

for $\boldsymbol{\sigma}_{h,0,{\rm alg}} \in \mathbf{V}_J^0$ given by Constructions 6.1 and 7.1 with $p' = 0$ for $j = J$.

Proof. For $v_h \in V_h^p$,

$$\begin{aligned}(r_h, v_h) &= (r_h - \Pi_J^0 r_h + \Pi_J^0 r_h, v_h)\\ &= (r_h - \Pi_J^0 r_h, v_h - \Pi_J^0 v_h) + (\nabla\cdot\boldsymbol{\sigma}_{h,0,{\rm alg}}, v_h)\\ &\le \sum_{K\in\mathcal{T}_h} \left(\|r_h - \Pi_J^0 r_h\|_K \cdot \frac{h_K}{\pi}\|\nabla v_h\|_K + \|\boldsymbol{\sigma}_{h,0,{\rm alg}}\|_K \|\nabla v_h\|_K \right)\\ &\le \left\{ \sum_{K\in\mathcal{T}_h} \left(\frac{h_K}{\pi} \|r_h - \Pi_J^0 r_h\|_K + \|\boldsymbol{\sigma}_{h,0,{\rm alg}}\|_K \right)^2 \right\}^{1/2} \|\nabla v_h\|,\end{aligned}$$

where we have used the Poincaré and Cauchy–Schwarz inequalities. Consequently, as in the proof of (9.8),

$$\|\nabla(u_h^{\rm ex} - u_h)\| = \sup_{v_h\in V_h,\, \|\nabla v_h\|=1} (r_h, v_h) \le \left\{ \sum_{K\in\mathcal{T}_h} \left(\frac{h_K}{\pi} \|r_h - \Pi_J^0 r_h\|_K + \|\boldsymbol{\sigma}_{h,0,{\rm alg}}\|_K \right)^2 \right\}^{1/2}.$$

□

10 Efficiency of the error bounds

In this section we prove the efficiency of the upper bounds on the algebraic error of Theorems 9.1 and 9.2, i.e., we show that the bounds do not significantly overestimate the algebraic error $\|\nabla(u_h^{\rm ex} - u_h)\|$. We start by recalling the efficiency of the upper bound $\eta_{5.1}^p$ proved in [20, Theorem 7.4].

Theorem 10.1 (Efficiency of the upper bound $\eta_{5.1}^p$ on the algebraic error, [20])**.** *Let $\eta_{5.1}^p$ be given by (9.9a). There holds*

$$\eta_{5.1}^p \le \overline{C}_{\rm alg}^{{\rm eff},p,5.1} \|\nabla(u_h^{\rm ex} - u_h)\|,$$

where $\overline{C}_{\rm alg}^{{\rm eff},p,5.1}$ is a generic constant only depending on the shape regularity of the mesh $\mathcal{T}_h$, the space dimension d, the polynomial degree p, and the number of mesh levels J.

Proof. For later use, we present here the key steps of the proof of [20, Theorem 7.4]. For ease of presentation below, C denotes a generic constant depending on the shape regularity of $\mathcal{T}_h$, and d, p, J.

Step 1: Bound the norm of the lifting by the local contributions

$$\|\boldsymbol{\sigma}_{h,{\rm alg}}\|^2 \le J(d+1) \sum_{j=1}^{J} \sum_{\mathbf{a}\in\mathcal{V}_{j-1}} \|\boldsymbol{\sigma}_{j,{\rm alg}}^{\mathbf{a}}\|_{\omega_{j-1}^{\mathbf{a}}}^2.$$

Step 2: Bound the local contributions using the norm of the algebraic residual

$$\begin{aligned}\|\boldsymbol{\sigma}_{1,{\rm alg}}^{\mathbf{a}}\|_{\omega_0^{\mathbf{a}}} &\le C h_{\omega_0^{\mathbf{a}}} \left\{ \|r_h\|_{\omega_0^{\mathbf{a}}} + \|\nabla\rho_{0,{\rm alg}}\cdot\nabla\psi_0^{\mathbf{a}}\|_{\omega_0^{\mathbf{a}}} \right\},\\ \|\boldsymbol{\sigma}_{j,{\rm alg}}^{\mathbf{a}}\|_{\omega_{j-1}^{\mathbf{a}}} &\le C h_{\omega_{j-1}^{\mathbf{a}}} \|r_h\|_{\omega_{j-1}^{\mathbf{a}}}, \qquad 1 < j \le J,\end{aligned}$$

where $h_{\omega_{j-1}^{\mathbf{a}}}$ denotes the diameter of the patch $\omega_{j-1}^{\mathbf{a}}$.

Step 3: Bound

$$h_K \|r_h\|_K \le C \|\nabla(u_h^{\rm ex} - u_h)\|_{\omega_K}, \qquad \forall K \in \mathcal{T}_h,$$

where $\omega_K := \bigcup_{\mathbf{a}\in\mathcal{V}_h, \mathbf{a}\subset\partial K} \omega_h^{\mathbf{a}}$ denotes the domain of all elements sharing a node with K.

Step 4: Combine Steps 2 and 3 to bound

$$\sum_{\mathbf{a}\in\mathcal{V}_{j-1}} \|\boldsymbol{\sigma}_{j,{\rm alg}}^{\mathbf{a}}\|_{\omega_{j-1}^{\mathbf{a}}}^2 \le C \|\nabla(u_h^{\rm ex} - u_h)\|^2, \qquad 1 < j \le J.$$

Step 5: Combine Steps 2 and 3 together with $\|\nabla \rho_{0,\mathrm{alg}}\| \le \|\nabla(u_h^{\mathrm{ex}} - u_h)\|$ to bound

$$\sum_{\mathbf{a}\in\mathcal{V}_0} \|\boldsymbol{\sigma}^{\mathbf{a}}_{1,\mathrm{alg}}\|^2_{\omega_0^{\mathbf{a}}} \le C\|\nabla(u_h^{\mathrm{ex}} - u_h)\|^2.$$

Step 6: Combine Steps 1,4, and 5 to prove the assertion of the theorem.

□

We now prove the efficiency for the upper algebraic error bounds based on the p-th order liftings of the algebraic residual derived in Sections 6 and 7.

Theorem 10.2 (Efficiency of the upper bounds $\eta^p_{6.2}$ and $\eta^p_{7.1}$ on the algebraic error)**.** *Let $\eta^p_{6.2}$ and $\eta^p_{7.1}$ be given by* (9.9b)–(9.9c)*. There holds*

$$\eta^p_{6.2} \le \overline{C}^{\mathrm{eff},p,6.2}_{\mathrm{alg}} \|\nabla(u_h^{\mathrm{ex}} - u_h)\|, \tag{10.1}$$

$$\eta^p_{7.1} \le \overline{C}^{\mathrm{eff},p,7.1}_{\mathrm{alg}} \|\nabla(u_h^{\mathrm{ex}} - u_h)\|, \tag{10.2}$$

where $\overline{C}^{\mathrm{eff},p,6.2}_{\mathrm{alg}}$, $\overline{C}^{\mathrm{eff},p,7.1}_{\mathrm{alg}}$ are generic constants only depending on the shape regularity of the mesh $\mathcal{T}_h$, the space dimension d, the polynomial degree p, and the number of mesh levels J.

Proof. We show that the norm of the local contributions $\boldsymbol{\sigma}^{\mathbf{a}}_{j,\mathrm{alg}}$ of Construction 6.1, with $\boldsymbol{\sigma}^{\mathbf{a},\mathbf{a}'}_{j,\mathrm{alg}}$ constructed either by Construction 6.2 or 7.1, can be bounded by the norm of $\boldsymbol{\sigma}^{\mathbf{a}}_{j,\mathrm{alg}}$ constructed by the local minimization of Construction 5.1. Then the proof of Theorem 10.2 follows using the proof of Theorem 10.1 above.

Efficiency of $\eta^p_{6.2}$: Given $j \in \{1,\dots,J\}$, $\mathbf{a}\in\mathcal{V}_{j-1}$, and $g^{\mathbf{a},j} \in \mathbb{P}^{p'}(\mathcal{T}_j)|_{\omega^{\mathbf{a}}_{j-1}}$ from (5.4), we denote $\boldsymbol{\sigma}^{\mathbf{a}}_{j,(5.3),\mathrm{alg}}$ the local flux constructed by (5.3) and $\boldsymbol{\sigma}^{\mathbf{a}}_{j,(6.6),\mathrm{alg}}$ the local flux given in (6.6) as a sum of small patch contributions of Construction 6.2. Then we use a modification of [10, Theorem 1.1] when setting $\Omega := \omega^{\mathbf{a}}_{j-1}$ (see, in particular, the construction in [10, Secs. 3 and 5]) to show that

$$\|\boldsymbol{\sigma}^{\mathbf{a}}_{j,(6.6),\mathrm{alg}}\|_{\omega^{\mathbf{a}}_{j-1}} \le C\|\boldsymbol{\sigma}^{\mathbf{a}}_{j,(5.3),\mathrm{alg}}\|_{\omega^{\mathbf{a}}_{j-1}}, \tag{10.3}$$

where the constant C is a generic constant depending on the shape regularity of $\mathcal{T}_h$, and d. From the definition of $\eta^p_{6.2}$,

$$\begin{aligned}(\eta^p_{6.2})^2 = \left\|\sum_{j=1}^{J}\sum_{\mathbf{a}\in\mathcal{V}_{j-1}} \boldsymbol{\sigma}^{\mathbf{a}}_{j,(6.6),\mathrm{alg}}\right\|^2 &\le J(d+1)\sum_{j=1}^{J}\sum_{\mathbf{a}\in\mathcal{V}_{j-1}} \|\boldsymbol{\sigma}^{\mathbf{a}}_{j,(6.6),\mathrm{alg}}\|^2_{\omega^{\mathbf{a}}_{j-1}} \\ &\le C^2 J(d+1)\sum_{j=1}^{J}\sum_{\mathbf{a}\in\mathcal{V}_{j-1}} \|\boldsymbol{\sigma}^{\mathbf{a}}_{j,(5.3),\mathrm{alg}}\|^2_{\omega^{\mathbf{a}}_{j-1}},\end{aligned} \tag{10.4}$$

and the efficiency of $\eta^p_{6.2}$ then follows from the proof of Theorem 10.1.

Efficiency of $\eta^p_{7.1}$: Given $j \in \{1,\dots,J\}$, $\mathbf{a}\in\mathcal{V}_{j-1}$, $\mathbf{a}'\in\mathcal{V}^{\mathbf{a}}_j$, and $g^{II,\mathbf{a},\mathbf{a}',j} \in \mathbb{P}^{p'}(\mathcal{T}_j)|_{\omega^{\mathbf{a},\mathbf{a}'}}$ we denote $\boldsymbol{\sigma}^{\mathbf{a},\mathbf{a}'}_{j,6.2,\mathrm{alg}}$ the local contribution of Construction 6.2 and by $\boldsymbol{\sigma}^{\mathbf{a},\mathbf{a}'}_{j,7.1,\mathrm{alg}}$ the local contribution of Construction 7.1. Then, using constructive proofs in [12, Secs. 5.1, 6.3 and 7], we get

$$\|\boldsymbol{\sigma}^{\mathbf{a},\mathbf{a}'}_{j,7.1,\mathrm{alg}}\|_{\omega^{\mathbf{a},\mathbf{a}'}} \le C_{\mathrm{st}}\|\boldsymbol{\sigma}^{\mathbf{a},\mathbf{a}'}_{j,6.2,\mathrm{alg}}\|_{\omega^{\mathbf{a},\mathbf{a}'}},$$

where the constant C_{st} of [12, Theorems 2.3 and 2.5] only depends on the shape regularity of the elements in the patch $\omega^{\mathbf{a},\mathbf{a}'}$. Setting $\boldsymbol{\sigma}^{\mathbf{a}}_{j,7.1,\mathrm{alg}} := \sum_{\mathbf{a}'\in\mathcal{V}^{\mathbf{a}}_j} \boldsymbol{\sigma}^{\mathbf{a},\mathbf{a}'}_{j,7.1,\mathrm{alg}}$, we proceed as above in (10.3) to infer

$$\|\boldsymbol{\sigma}^{\mathbf{a}}_{j,7.1,\mathrm{alg}}\|^{\mathbf{a},\mathbf{a}'}_{\omega} \le C_{\mathrm{st}}C\|\boldsymbol{\sigma}^{\mathbf{a}}_{j,(5.3),\mathrm{alg}}\|^{\mathbf{a},\mathbf{a}'}_{\omega}.$$

The efficiency of $\eta^p_{7.1}$ then follows as in (10.4).

□

Finally, we prove the efficiency of the upper bound based on the lowest-order algebraic residual lifting.

Theorem 10.3 (Efficiency of the upper bound η_{alg}^0 on the algebraic error)**.** *Let η_{alg}^0 be given by* (9.11)*. There holds*

$$\eta_{\mathrm{alg}}^0 \le \overline{C}_{\mathrm{alg}}^{\mathrm{eff},0} \|\nabla(u_h^{\mathrm{ex}} - u_h)\|, \tag{10.5}$$

where $\overline{C}_{\mathrm{alg}}^{\mathrm{eff},0}$ is a generic constant only depending on the shape regularity of the mesh $\mathcal{T}_h$, the space dimension d, the polynomial degree p, and the number of mesh levels J.

Proof. First, we bound the norm of the local contributions $\|\boldsymbol{\sigma}_{j,0,\mathrm{alg}}^{\mathbf{a}}\|_{\omega_{j-1}^{\mathbf{a}}}$ by $\|\Pi_j^0 r_h\|_{\omega_{j-1}^{\mathbf{a}}}$ and $\|\nabla \rho_{0,\mathrm{alg}}\|_{\omega_{j-1}^{\mathbf{a}}}$ as in Theorem 10.2 and in Step 2 of Theorem 10.1. Then we use $\|\Pi_j^0 r_h\|_K \le \|r_h\|_K$ (since Π_j^0 is a projection) and $\|r_h - \Pi_j^0 r_h\|_K \le \|r_h\|_K$. Finally, the proof follows using the Steps 3–6 of Theorem 10.1 above. □

11 Numerical results

For the numerical experiments, we consider the same test problems and setting as in [20]. We consider the model problem (9.1) with three different choices of the domain $\Omega \in \mathbb{R}^2$ and of the exact solution u:

$$\begin{aligned} &\Omega := (-1,1)^2, && u(x,y) := \sin(2\pi x)\sin(2\pi y) && \text{sinus},\\ &\Omega := (0,1)^2, && u(x,y) := x(x-1)y(y-1)\exp\left(-100((x-\tfrac{1}{2})^2 - (y - \tfrac{117}{1000})^2)\right) && \text{peak},\\ &\Omega := (-1,1)^2 \setminus [0,1]\times[-1,0], && u(r,\theta) := r^{2/3}\sin(2\theta/3) && \text{L-shape}.\end{aligned}$$

In the last case, we impose an inhomogeneous Dirichlet boundary condition corresponding to the prescribed exact solution. We consider the finite element method (9.2) with the polynomial degrees $p = 1, \dots, 4$. For each test problem, we start from an initial Delaunay triangulation of the domain Ω and consider four uniform refinement steps, so that $J = 4$. We give in the tables and figures below the effectivity index defined as the ratio of the estimate (error upper bound) to the error,

$$\text{eff. index} := \frac{\text{estimate}}{\text{error}}.$$

11.1 Algebraic error bounds based on the p-degree polynomial lifting

First, we present the results for bounds from Theorem 9.1, based on the liftings constructed in p-degree RTN space $\mathbf{V}_J^p$ in Sections 5–7. We present the results in the same way as in [20], to allow for a test comparison. As an algebraic solver, we consider a geometric multigrid method (MG) with V(5,0)-cycles, i.e., employing 5 Gauss-Seidel pre-smoothing iterations and no post-smoothing. Second, we consider a full multigrid (FMG) method using a single V(3,3)-cycle on each level (i.e. with 3 Gauss-Seidel pre- and post-smoothing iterations). Our starting iterate is the zero vector, and in MG we iterate until the algebraic error drops below 10% of the discretization error.

As presented in [20, Theorem 7.1], there also exists an upper bound on the total error $\|\nabla(u - u_h)\|$ in the form

$$\|\nabla(u - u_h)\| \le \eta_{\mathrm{tot}}^p := \widetilde{\eta}_{\mathrm{dis}} + \eta_{\mathrm{osc}} + \eta_{\mathrm{alg}}^p, \tag{11.1}$$

where η_{alg}^p is given by (9.9). The term η_{osc} measures the oscillation in the data f and $\widetilde{\eta}_{\mathrm{dis}}$ can be used to approximate the discretization error $\|\nabla(u - u_h^{\mathrm{ex}})\|$. These two terms are independent of the construction of the algebraic residual lifting $\boldsymbol{\sigma}_{h,\mathrm{alg}}$. In the tables below, we show the energy norm of the total error and the efficiency of the upper bound determined by using in (11.1) the upper bounds on the algebraic error $\eta_{\mathrm{alg}}^p = \eta_{5.1}^p$, $\eta_{6.2}^p$, or $\eta_{7.1}^p$.

The experiments confirm excellent efficiency of $\eta_{5.1}^p$ (reported already in [20, Sec. 9]) and also of the new upper bound $\eta_{6.2}^p$; typically, $\eta_{6.2}^p$ is very close to $\eta_{5.1}^p$. The effectivity index of $\eta_{7.1}^p$ is slightly worse but still, in most of the experiments and iterations, its value is below 2, which we consider very satisfactory. We also note that the efficiency of all the bounds (based on the p-degree polynomial lifting) does not in the experiments deteriorate with the increasing polynomial degree p of the FEM approximation, though we could not prove this rigorously in Section 10.

p (N_h)	MG iter	algebraic error	eff. index η^p_{alg} $\eta^p_{5.1}$	$\eta^p_{6.2}$	$\eta^p_{7.1}$	total error	eff. index η^p_{tot} $\eta^p_{5.1}$	$\eta^p_{6.2}$	$\eta^p_{7.1}$	discretization error
1 (3.5×10^4)	1	1.2	1.11	1.11	1.24	1.3	1.46	1.47	1.59	2.4×10^{-1}
	2	8.0×10^{-2}	1.13	1.16	1.29	2.5×10^{-1}	1.35	1.36	1.40	
	3	5.1×10^{-3}	1.15	1.17	1.32	2.4×10^{-1}	1.06	1.06	1.07	
2 (1.4×10^5)	1	1.2	1.10	1.11	1.39	1.2	1.48	1.50	1.77	2.9×10^{-3}
	2	1.1×10^{-1}	1.18	1.20	1.58	1.1×10^{-1}	1.77	1.79	2.17	
	3	2.8×10^{-3}	1.18	1.20	1.59	4.1×10^{-3}	1.66	1.67	1.94	
	4	9.6×10^{-5}	1.20	1.25	1.61	2.9×10^{-3}	1.05	1.05	1.06	
3 (3.2×10^5)	1	5.9×10^{-1}	1.09	1.17	1.48	5.9×10^{-1}	1.33	1.41	1.73	2.2×10^{-5}
	3	2.2×10^{-3}	1.19	1.18	1.55	2.2×10^{-3}	1.75	1.74	2.11	
	5	1.0×10^{-5}	1.19	1.18	1.54	2.4×10^{-5}	1.44	1.44	1.59	
	6	1.4×10^{-6}	1.18	1.18	1.53	2.2×10^{-5}	1.08	1.08	1.10	
4 (5.6×10^5)	1	4.5×10^{-1}	1.08	1.19	1.64	4.5×10^{-1}	1.39	1.49	1.94	1.5×10^{-7}
	4	1.2×10^{-4}	1.13	1.14	1.49	1.2×10^{-4}	1.58	1.59	1.93	
	7	6.4×10^{-7}	1.11	1.12	1.46	6.5×10^{-7}	1.53	1.54	1.86	
	10	8.3×10^{-9}	1.12	1.12	1.45	1.5×10^{-7}	1.06	1.06	1.08	

Table 1: Sinus problem, multigrid V-cycles: efficiency of the error upper bounds of Theorem 9.1

p (N_h)	algebraic error	eff. index η^p_{alg} $\eta^p_{5.1}$	$\eta^p_{6.2}$	$\eta^p_{7.1}$	total error	eff. index η^p_{tot} $\eta^p_{5.1}$	$\eta^p_{6.2}$	$\eta^p_{7.1}$	discretization error
1 (3.5×10^4)	8.7×10^{-4}	1.03	1.07	1.25	2.4×10^{-1}	1.04	1.04	1.04	2.4×10^{-1}
2 (1.4×10^5)	1.9×10^{-5}	1.10	1.18	1.49	2.9×10^{-3}	1.02	1.02	1.02	2.9×10^{-3}
3 (3.2×10^5)	2.6×10^{-6}	1.08	1.06	1.53	2.2×10^{-5}	1.12	1.12	1.17	2.2×10^{-5}
4 (5.6×10^5)	5.1×10^{-8}	1.03	1.10	1.47	1.6×10^{-7}	1.29	1.31	1.43	1.5×10^{-7}

Table 2: Sinus problem, one full multigrid cycle: efficiency of the error upper bounds of Theorem 9.1

p (N_h)	MG iter	algebraic error	eff. index η^p_{alg} $\eta^p_{5.1}$	$\eta^p_{6.2}$	$\eta^p_{7.1}$	total error	eff. index η^p_{tot} $\eta^p_{5.1}$	$\eta^p_{6.2}$	$\eta^p_{7.1}$	discretization error
1 (9.3×10^3)	1	6.1×10^{-3}	1.13	1.13	1.22	6.9×10^{-3}	1.61	1.61	1.69	3.3×10^{-3}
	2	1.9×10^{-4}	1.13	1.15	1.28	3.3×10^{-3}	1.10	1.10	1.11	
2 (3.8×10^4)	1	7.5×10^{-3}	1.13	1.14	1.48	7.5×10^{-3}	1.61	1.62	1.97	1.1×10^{-4}
	2	4.5×10^{-4}	1.17	1.18	1.56	4.6×10^{-4}	1.76	1.77	2.14	
	3	8.1×10^{-6}	1.17	1.18	1.57	1.1×10^{-4}	1.10	1.10	1.12	
3 (8.5×10^4)	1	4.9×10^{-3}	1.10	1.17	1.62	4.9×10^{-3}	1.40	1.47	1.92	2.9×10^{-6}
	3	1.3×10^{-5}	1.18	1.18	1.57	1.3×10^{-5}	1.75	1.74	2.12	
	5	7.8×10^{-9}	1.17	1.16	1.59	2.9×10^{-6}	1.01	1.01	1.01	
4 (1.5×10^5)	1	4.4×10^{-3}	1.09	1.21	1.80	4.4×10^{-3}	1.44	1.56	2.15	6.3×10^{-8}
	3	1.8×10^{-5}	1.15	1.16	1.48	1.8×10^{-5}	1.67	1.68	1.99	
	5	2.4×10^{-8}	1.11	1.12	1.40	6.8×10^{-8}	1.34	1.34	1.44	
	6	1.1×10^{-9}	1.11	1.11	1.38	6.3×10^{-8}	1.02	1.02	1.03	

Table 3: Peak problem, multigrid V-cycles: efficiency of the error upper bounds of Theorem 9.1

p (N_h)	algebraic error	eff. index η^p_{alg} $\eta^p_{5.1}$	$\eta^p_{6.2}$	$\eta^p_{7.1}$	total error	eff. index η^p_{tot} $\eta^p_{5.1}$	$\eta^p_{6.2}$	$\eta^p_{7.1}$	discretization error
1 (9.3×10^3)	1.8×10^{-5}	1.02	1.09	1.33	3.3×10^{-3}	1.04	1.04	1.05	3.3×10^{-3}
2 (3.8×10^4)	1.9×10^{-7}	1.07	1.10	1.37	1.1×10^{-4}	1.01	1.01	1.01	1.1×10^{-4}
3 (8.5×10^4)	2.2×10^{-7}	1.08	1.06	1.55	2.9×10^{-6}	1.08	1.08	1.12	2.9×10^{-6}
4 (1.5×10^5)	9.1×10^{-9}	1.05	1.10	1.49	6.4×10^{-8}	1.14	1.15	1.21	6.3×10^{-8}

Table 4: Peak problem, one full multigrid cycle: efficiency of the error upper bounds of Theorem 9.1

p (N_h)	MG iter	algebraic error	eff. index η^p_{alg} $\eta^p_{5.1}$	$\eta^p_{6.2}$	$\eta^p_{7.1}$	total error	eff. index η^p_{tot} $\eta^p_{5.1}$	$\eta^p_{6.2}$	$\eta^p_{7.1}$	discretization error
1 (2.5×10^4)	1	1.4	1.14	1.18	1.49	1.4	1.60	1.64	1.95	2.2×10^{-2}
	2	6.7×10^{-2}	1.14	1.19	1.44	7.0×10^{-2}	1.61	1.65	1.90	
	3	4.3×10^{-3}	1.16	1.25	1.68	2.3×10^{-2}	1.37	1.39	1.47	
	4	4.1×10^{-4}	1.17	1.31	1.89	2.2×10^{-2}	1.22	1.22	1.23	
2 (1.0×10^5)	1	2.6	1.19	1.22	1.84	2.6	1.78	1.81	2.43	8.9×10^{-3}
	2	8.9×10^{-2}	1.19	1.20	1.63	8.9×10^{-2}	1.79	1.80	2.23	
	3	2.2×10^{-3}	1.18	1.21	1.61	9.2×10^{-3}	1.55	1.56	1.66	
	4	8.6×10^{-5}	1.19	1.25	1.68	8.9×10^{-3}	1.32	1.32	1.32	
3 (2.3×10^5)	1	2.4	1.19	1.20	1.71	2.4	1.72	1.74	2.25	5.3×10^{-3}
	2	1.1×10^{-1}	1.20	1.19	1.59	1.1×10^{-1}	1.76	1.76	2.16	
	3	3.6×10^{-3}	1.18	1.17	1.58	6.4×10^{-3}	1.89	1.88	2.11	
	4	1.8×10^{-4}	1.17	1.16	1.60	5.3×10^{-3}	1.48	1.48	1.49	
4 (4.0×10^5)	1	2.6	1.18	1.25	2.00	2.6	1.68	1.75	2.50	3.8×10^{-3}
	2	1.3×10^{-1}	1.18	1.18	1.57	1.3×10^{-1}	1.71	1.72	2.10	
	3	6.0×10^{-3}	1.16	1.15	1.46	7.1×10^{-3}	1.87	1.87	2.13	
	4	3.5×10^{-4}	1.13	1.13	1.44	3.8×10^{-3}	1.57	1.57	1.60	

Table 5: L-shape problem, multigrid V-cycles: efficiency of the error upper bounds of Theorem 9.1

p (N_h)	algebraic error	eff. index η^p_{alg} $\eta^p_{5.1}$	$\eta^p_{6.2}$	$\eta^p_{7.1}$	total error	eff. index η^p_{tot} $\eta^p_{5.1}$	$\eta^p_{6.2}$	$\eta^p_{7.1}$	discretization error
1 (2.5×10^4)	4.4×10^{-4}	1.11	1.27	2.02	2.2×10^{-2}	1.22	1.22	1.23	2.2×10^{-2}
2 (1.0×10^5)	8.0×10^{-5}	1.12	1.44	2.26	8.9×10^{-3}	1.32	1.32	1.33	8.9×10^{-3}
3 (2.3×10^5)	5.5×10^{-5}	1.09	1.21	1.95	5.3×10^{-3}	1.45	1.45	1.46	5.3×10^{-3}
4 (4.0×10^5)	7.2×10^{-5}	1.08	1.24	1.94	3.8×10^{-3}	1.49	1.49	1.51	3.8×10^{-3}

Table 6: L-shape problem, one full multigrid cycle: efficiency of the error upper bounds of Theorem 9.1

11.2 Algebraic error bound based on the approximate lowest-degree polynomial lifting

We now focus on the efficiency of the upper bound η^0_{alg} from Theorem 9.2, based on the algebraic residual lifting constructed in the lowest degree RTN space $\mathbf{V}^0_J$ in Sec. 8. As remarked in Sec. 9.4, we expect deterioration of the upper bound for increasing polynomial degree p of the FEM approximation. Here, we consider iterations of V-cycle multigrid solver with 5 pre-smoothing Gauss–Seidel iterations and no postsmoothing.

The observed efficiency is in particular satisfactory for $p = 1$, where it stays close to 2 in the peak, respectively, to 2.5 in the sinus and L-shape test problems. One can note a significant deterioration of the efficiency between $p = 1$ and $p = 2$, while the deterioration is less substantial between higher polynomial degrees. Nevertheless, the efficiency stays in all the numerical experiments bounded by $p + 2$. Taking into account the importantly reduced computational cost of constructing the lowest-order lifting of the algebraic residual, we believe that the guaranteed upper bound η^0_{alg} of Theorem 9.2 can be of interest in many applications.

12 Conclusions

In this contribution, we presented new constructions of (approximate) liftings of a piecewise polynomial function. As showed in [20], this can be used to recover mass balance for any problem, any numerical discretization, and any situation. Here we used these liftings to construct guaranteed upper bounds on the algebraic error in the higher-order conforming discretization of the model Poisson problem. Together with the results of [20, 18], one can also monitor the total and discretizations errors.

The presented constructions of the liftings are significantly cheaper w.r.t. those of [20] while still preserving the same theoretical properties, i.e., guaranteed upper bound on the algebraic error and efficiency. The numerical experiments confirm the theoretical expectations and lead to very satisfactory effectivity indices of the estimators. Moreover, for the liftings of Sections 6–7, of the same polynomial order p as the FEM

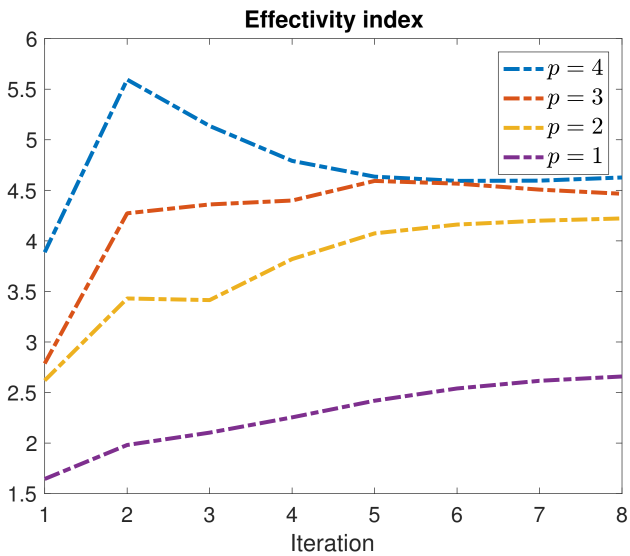


Figure 5: Sinus problem, multigrid V-cycles: efficiency of the bound $\eta_{\rm alg}^0$ of Theorem 9.2 for varying polynomial degree p of FEM approximation

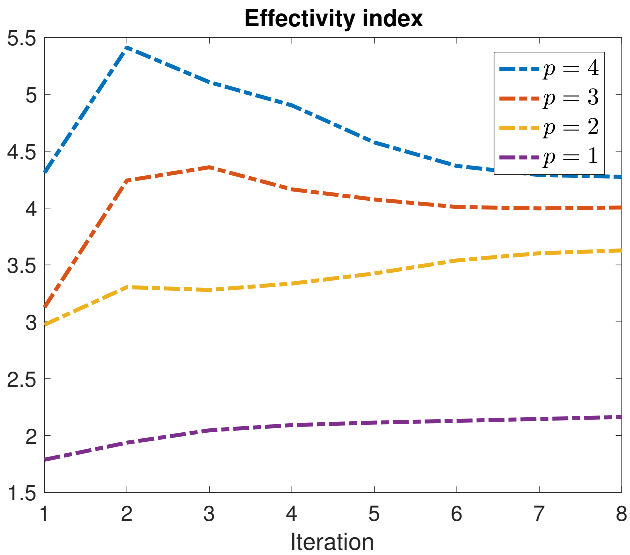


Figure 6: Peak problem, multigrid V-cycles: efficiency of the bound $\eta_{\rm alg}^0$ of Theorem 9.2 for varying polynomial degree p of FEM approximation

discretization has, we numerically observed p-robustness. This is not the case for the bounds based on the lowest-order approximate lifting of Section 8, which attained in our experiments effectivity indices below $p+2$. We however note that the lowest-order approximate lifting has the computational cost significantly further reduced and, therefore, it can be of interest in many applications.

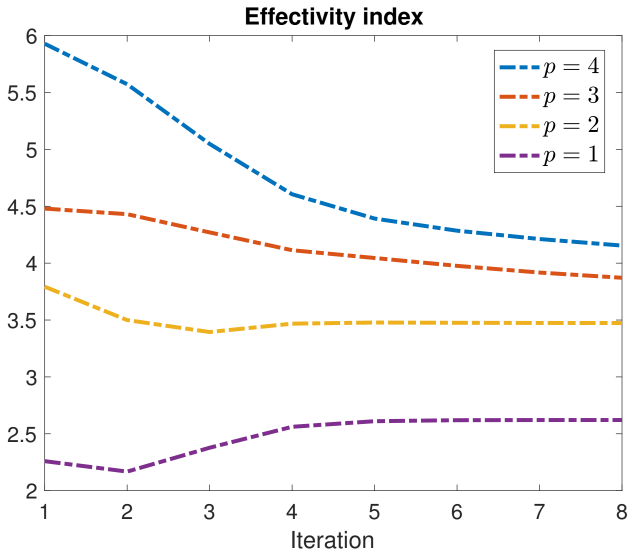


Figure 7: Lshape problem, multigrid V-cycles: efficiency of the bound η_{alg}^0 of Theorem 9.2 for varying polynomial degree p of FEM approximation

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