A problematic issue in the Walton-Marshall method for some neutral delay systems

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Le Ha Vy Nguyen[1], Catherine Bonnet[1], Islam Boussaada[2], and Marianne Souaiby[1]

***Abstract*— This paper considers delay systems with characteristic equation being a quasi-polynomial with one delay and polynomials of degree one. It is shown that for a subclass of systems which have a chain of poles clustering the imaginary axis by the left, the procedure of Walton and Marshall fails: we prove the existence, for an infinitesimally small delay, of a positive real pole at infinity. This real pole is then proved to be the unique pole of the system in the closed right half-plane for all values of the delay. Some numerical examples illustrate the results.**

I. INTRODUCTION

The stability properties of linear delay systems have been widely studied since the sixties. A crucial step in this context is the determination of the location of the poles of the system (that is the zeros of a quasi-polynomial) in the complex plane. After the seminal work of Bellman and Cooke locating the chain of poles appearing as soon as a delay is present in the state (or derived state), many studies have been devoted to the characterization, via sufficient or necessary and sufficient conditions, of the different types of stability (asymptotic, exponential or H_∞) of a system as well as the development of practical methods to help verify these conditions and indeed decide on the stability of a given system. One of such practical methods has been given by Walton and Marshall in [1], and is referred as a direct method in comparison with methods requiring a substitution (e.g. the well-known Rekasius substitution), see [2], [3] for example.

Relying on the well-known continuity property of the roots of a quasi-polynomial with respect to a strictly positive delay, the Walton-Marshall method relies on three steps : (1) determine the location of the roots of the quasi-polynomial when the delay is equal to zero, (2) locate where in the complex plane the infinite number of roots appear when the delay becomes infinitesimally small, (3) find the crossings on the imaginary axis when the delay varies.

Step 2 is of particular importance: in the case of neutral systems, it may happen that an infinite number of poles appear in the closed right half-plane or near the imaginary axis (clustering the imaginary axis). Such neutral systems are difficult to analyze and most practical methods deal with systems for which it is guaranteed that the infinite number of poles appear in $\{s \in \mathbb{R} : \Re s < -a, a > 0\}$.

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In their step 2, Walton and Marshall claim that they can determine if the infinite number of poles appear in the left or right half-plane. Although they capture most of the poles, they miss in some cases a real positive pole at infinity.

In this paper we consider delay systems with characteristic equation being a quasi-polynomial with one delay and polynomials of degree one.

Our aim is to characterize systems for which there exists a positive real pole at infinity in step 2 and prove that this pole is the unique pole appearing in the closed right-half plane for an infinitesimally small delay.

The paper is organized as follows. Section II gives some preliminaries on the location of poles of neutral systems. Section III considers neutral delay systems which have a chain of poles in the open left half-plane and determines a subclass of systems which admit a real positive pole at infinity in step 2 of Walton-Marshall method. In Section IV we prove that this pole is unique in the closed right half-plane. Some examples illustrate those results in Section V.

II. PRELIMINARIES

We consider the quasi-polynomial

$$f(s) := p(s) + q(s)e^{-sh}, \tag{1}$$

where $p(s) = s + b$ and $q(s) = cs + d$ with $b, c, d, h \in \mathbb{R}$, $c \neq 0$.

As in [Bellman1963], invoking Rouché's Theorem on small circles of radius $1/h$ around the points $\frac{1}{h}(j2\pi n - \ln\left(-\frac{1}{c}\right))$, the roots of large modulus of $f(s)$ are asymptotic to those of

$$1 + ce^{-sh} = 0$$

and can be approximated by

$$s_n h = \lambda_n + o(1),$$

where

$$\lambda_n = j2\pi n - \ln\left(-\frac{1}{c}\right) = j2\pi n - \mathrm{Ln}\left|-\frac{1}{c}\right| + j\,\mathrm{Arg}\left(-\frac{1}{c}\right).$$

Hence, these roots are asymptotic to the vertical line

$$\Re s = -\frac{\mathrm{Ln}\left|-\frac{1}{c}\right|}{h} = \frac{\mathrm{Ln}\,|c|}{h}.$$

If $c = \pm 1$, this vertical line is the imaginary axis.

According to [4, Theorem 2.1], we develop $\frac{p(s)}{q(s)}$ as $|s| \to \infty$ as follows

$$\frac{p(s)}{q(s)} = \alpha + \frac{\beta}{s} + \frac{\gamma}{s^2} + \mathcal{O}(s^{-3}), \tag{2}$$

and with $c = \pm 1$ the asymptotic roots can be further approximated as follows

$$s_n = \frac{\lambda_n}{h} - \frac{\beta}{\alpha\lambda_n} + \frac{h}{\lambda_n^2}\left(\frac{\beta^2}{2} - \frac{\gamma}{\alpha}\right) + o\left(\frac{1}{n^2}\right).$$

If $\frac{\beta^2}{2} - \frac{\gamma}{\alpha} > 0$, then the asymptotic roots are on the left of the imaginary axis. If $\frac{\beta^2}{2} - \frac{\gamma}{\alpha} < 0$, then they are on the right.

III. EXISTENCE OF A POSITIVE ROOT

We are interested in quasi-polynomials of the form (1) whose chain of roots is in the open left half-plane. Hence, we restrict our attention to $0 < |c| \leq 1$ which implies that the asymptotic axis is either in the open left half-plane or is the imaginary axis.

Proposition 3.1: The quasi-polynomial (1) with $0 < |c| \leq 1$ has a large positive root for a small delay h if and only if $c = -1$ and $b + d < 0$. The root is then approximated by

$$s \approx \sqrt{-\frac{b+d}{h}}. \tag{3}$$

Proof: For a large positive root of $f(s)$, we have

$$e^{-sh} = -\frac{s+b}{cs+d} \tag{4}$$

with $\text{sign}(-\frac{s+b}{cs+d}) = \text{sign}\,\frac{-1}{c}$. Since $e^{-sh} > 0$, necessarily $c < 0$.

From (4) we get

$$-sh = \text{Ln}\left(1 - \frac{(c+1)s + b + d}{cs + d}\right).$$

If $c = -1$ we have $-sh = \text{Ln}\left(1 + \frac{b+d}{s-d}\right)$ and a solution s will exist if and only if $b + d < 0$.

Otherwise $-sh = \text{Ln}\left(1 - \frac{c+1}{c}(1 + \mathcal{O}(\frac{1}{s}))\right)$ which does not admit when $-1 < c < 0$.

■

IV. UNIQUENESS OF THE POSITIVE ROOT

From the previous section, we see that among systems whose asymptotic axis is in the closed right half-plane, only systems which satisfy $c = -1$, thus having a chain of roots clustering the imaginary axis, may have a large positive root appearing for $h = 0^+$. Among those, systems of interest are those which have no positive root when $h = 0$ and no right chain of roots. The following lemma characterizes such systems.

Lemma 4.1: The quasi-polynomial $f(s)$ given as in (1) with $c = -1$ satisfies the following statements:

1) $f(s)$ has no root in the closed right half-plane for $h = 0$ if and only if $b + d \neq 0$.
2) If $b^2 - d^2 > 0$, then $f(s)$ has infinitely many roots approaching the imaginary axis from the left.

Proof:

1) When $h = 0$, $f(s)$ becomes

$$b + d = 0,$$

which has no root in the closed right half-plane if and only if

$$b + d \neq 0.$$

2) To consider the existence of a left chain of roots when $h > 0$, we develop $\frac{p(s)}{q(s)}$ as $|s| \to \infty$ as in (2) and obtain $\alpha = -1$, $\beta = -(b + d)$, and $\gamma = -d(b + d)$. From [4, Theorem 2.1], $f(s)$ has roots of large modulus approaching the imaginary axis from the left if

$$\frac{\gamma}{\alpha} < \frac{\beta^2}{2},$$

which is equivalent to

$$b^2 - d^2 > 0.$$

■

According to Walton and Marshall [1, p. 25], quasi-polynomials satisfying the two conditions in Lemma 4.1 should not have roots in the closed right half-plane for small delay h. However, Proposition 3.1 has shown that a subset of these quasi-polynomials have a large positive root for $h = 0^+$. More precisely, this class is $f(s)$ with $b + d < 0$ and $|b| > |d|$.

Now, for such a quasi-polynomial, the question is whether there are other roots (they would be of large modulus) appearing in the closed right half-plane. In the next proposition, we prove the uniqueness of its large positive root.

Proposition 4.2: The quasi-polynomial $f(s)$ given as in (1) with $c = -1$, $|b| > |d|$, and $b + d < 0$ has a unique root in the closed right half-plane for all $h > 0$.

Proof: Recall that for $c = -1$, the roots of large modulus are approximated by

$$s_n = j\frac{2\pi n}{h} + j\frac{\beta}{\alpha 2\pi n} - \frac{h}{4\pi^2 n^2}\left(\frac{\beta^2}{2} - \frac{\gamma}{\alpha}\right) + o\left(\frac{1}{n^2}\right),$$

where $n \in \mathbb{Z}$, n large enough.

Let $R > 0$ and $R_1 = (2\pi n_1 - 1)/h < R$, $n_1 \in \mathbb{Z}_+$.

We consider the contour C which is a slightly modified right semicircle of radius R centered at the origin in the following way: On the upper half of the imaginary axis, from jR_1 onward, the contour goes about the points $j2\pi n/h$ by a right semicircle of radius $1/h$ (in the right half-plane). The same behavior is applied for the lower half of the imaginary axis. Figure 1 illustrates the contour C.

We can choose R_1 large enough to fulfill two conditions:

- For $|s| \geq R_1$ the roots of $f(s)$ are asymptotic to those of the equation $e^{-sh} = 1$.
  Note that, under conditions on the coefficients c, b and d, these roots are on the left of the imaginary axis. Therefore, within the semicircles of radius $1/h$, there is no root.
- $\Im f(s) > 0$ for s on the contour C satisfying $\Im s \geq R_1$ (see Lemma 6.1).

By abuse of notation, we denote $f(C)$ the image curve of C under the mapping $f(s)$.

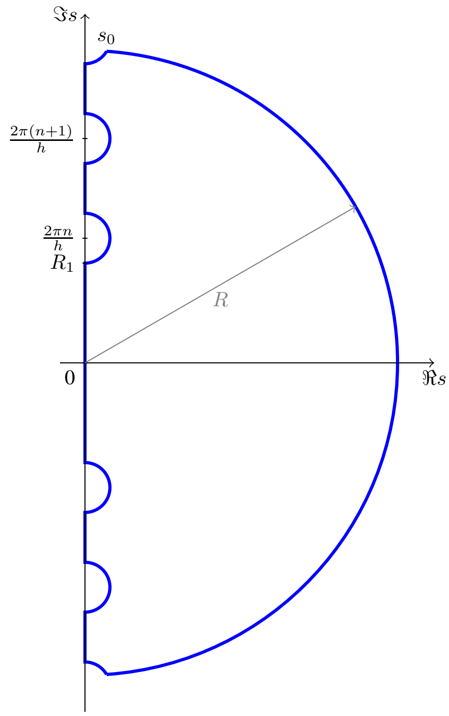

Fig. 1. The contour C

By the Argument Principle [5, p. 123] we have: $2j\pi N_f =$ winding number around the origin of $f(C)$ when R tends to infinity, where N_f is the number of zeros of f.

Before analyzing $f(s)$ on several parts of the contour C, note that $f(s)$ is symmetric with respect to the real axis since C is symmetric with respect to the real axis and $f(\bar{s}) = \overline{f(s)}$.

Let us consider the variation or argument of $f(s)$ along C_j, the straight segment of C which goes from jR_1 to $-jR_1$. Since $f(jR_1)$ is in the upper half-plane and $f(C_j)$ only cuts the negative axis (see Lemma 6.2), the argument of $f(s)$ increases by $2(\pi - \operatorname{Arg} f(jR_1))$ along C_j.

Let us consider the small semicircles about $j2\pi n/h$ and the straight segments on the imaginary axis that connect these semicircles. For $n > 0$, Lemma 6.1 shows that $\Im f(s) > 0$. Hence, on both the upper and lower parts, the argument of $f(s)$ changes by $2(\operatorname{Arg} f(jR_1) - \operatorname{Arg} f(s_0))$ in total, where s_0 is the intersection between a small semicircle or a segment and the upper part of the large semicircle of radius R (see Figure 1).

On the large semicircle, denoted C_R, we have

$$\begin{aligned}
f(Re^{j\theta}) &= (Re^{j\theta} + b) + (-Re^{j\theta} + d)e^{-Re^{j\theta}h} \\
&= (R\cos\theta + b + jR\sin\theta) + (-R\cos\theta + d \\
&\quad - jR\sin\theta)e^{-Rh\cos\theta}(\cos(Rh\sin\theta) \\
&\quad - j\sin(Rh\sin\theta)) \\
&= [R\cos\theta + b + d\cos(Rh\sin\theta)e^{-Rh\cos\theta} \\
&\quad - R\cos(\theta - Rh\sin\theta)e^{-Rh\cos\theta}] \\
&\quad + j[R\sin\theta - d\sin(Rh\sin\theta)e^{-Rh\cos\theta} \\
&\quad - R\sin(\theta - Rh\sin\theta)e^{-Rh\cos\theta}].
\end{aligned}$$

Consider the upper part of C_R, that is $\theta \in [0, \pi/2]$. We can choose R large enough such that:

a) $\Re f(Re^{j\theta}) \approx R\cos\theta > 0$ for $\theta \in \left[0, \frac{\pi}{8}\right]$,
b) $\Im f(Re^{j\theta}) \approx R\sin\theta > 0$ for $\theta \in \left(\frac{\pi}{8}, \frac{3\pi}{8}\right]$, and
c) $\Im f(s) > 0$ for s going from $Re^{j3\pi/8}$ to s_0 according to Lemma 6.1.

For c) to be true, we choose R large enough to satisfy $R\sin\left(\frac{3\pi}{8}\right) > R_1$ and $\operatorname{Arg}(s_0) > \frac{3\pi}{8}$. As implied by a), b), c), $f(s)$ evolves in the upper half-plane. Therefore, the argument of $f(s)$ increases by $2\operatorname{Arg} f(s_0)$ along C_R.

Hence, the argument of $f(s)$ increases by 2π along C and thus $f(s)$ has one root in the open region defined by C. Since there is no root within the right semicircles around $j2\pi n/h$, there is one root within the right semicircle with radius R centered at the origin.

Therefore, with $R \to \infty$, we conclude that $f(s)$ has one root in the open right half-plane. ■

Note that the uniqueness result is not restricted to $h = 0^+$ but is valid for $h > 0$. This can be explained by the fact that the condition to have a left chain of roots implies no root crossing the imaginary axis for $h > 0$ (see Lemma 6.3) and thus the number of roots in the closed right half-plane is conserved for all $h > 0$.

V. NUMERICAL EXAMPLES

$$f(s) = (s-2) + (-s+1/2)e^{-sh}$$

Even though this system satisfies all three conditions in Walton-Marshall method (see Lemmas 4.1 and 6.3), it still has a positive real root (see Figures 2, 3, and 4). Furthermore, the location of this root satisfies (3), i.e. for $h = 0.01$, 0.001, and 0.0001, the approximated values of the positive root are 12, 37, and 122 respectively. A negative root with approximately the same magnitude is also shown in the figures.

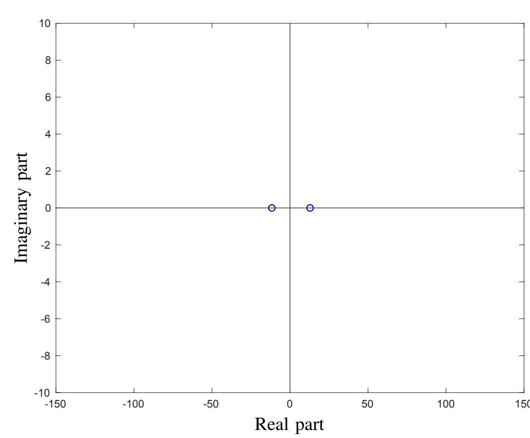

Fig. 2. Roots of $f(s)$ for $h = 0.01$

From Proposition 4.2, the positive real root is the unique root in the closed right half-plane. We illustrate several points of the proof of the proposition by examining the contours C

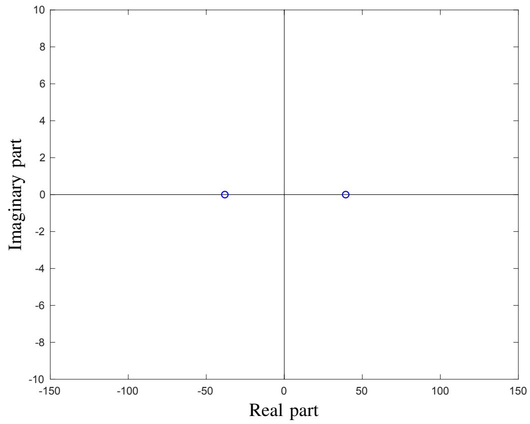

Fig. 3. Roots of $f(s)$ for $h = 0.001$

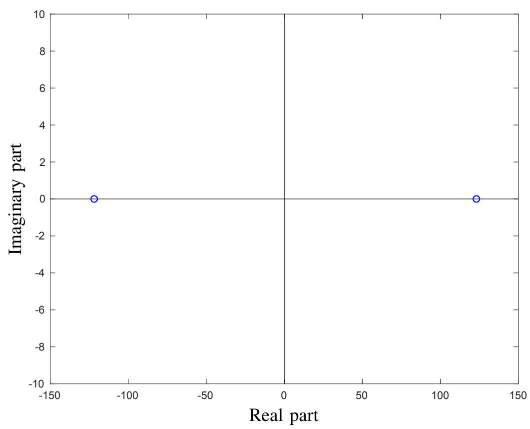

Fig. 4. Roots of $f(s)$ for $h = 0.0001$

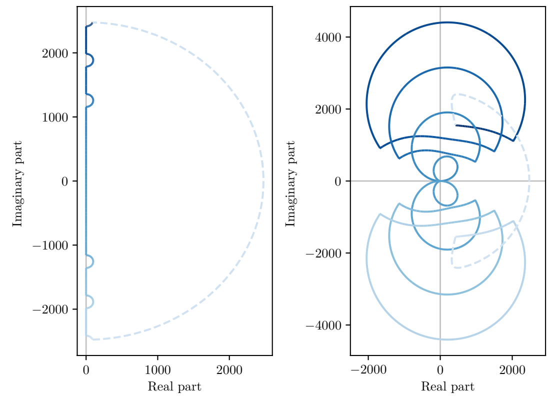

Fig. 5. The contour C and $f(C)$ for $h = 0.01$. The color intensity of C and $f(C)$ decreases when s varies starting from s_0.

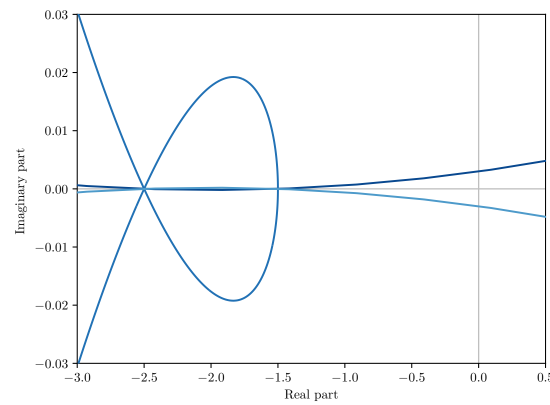

Fig. 6. Zoom on the contour $f(C)$ for $h = 0.01$

and $f(C)$ in the case $h = 0.01$ showed in Figure 5. Note that, the positive root is inside the contour C, which has $R \approx 2500$.

Figure 6, which is a zoom near the origin, shows that $f(C_j)$ cuts the real axis several times at two points -2.5 and -1.5 which conform with Lemma 6.2. Considering the segments on the imaginary axis with small semicircles that is above the origin (see Figure 5), its image curve under the mapping $f(s)$ is shown to be in the upper half-plane conforming with Lemma 6.1. Hence, $f(C)$ circles counter-clockwise once around the origin and thus the positive root is the unique one inside C.

VI. CONCLUSIONS

In this paper we have presented a counterexample to the second step in Walton-Marshall method, that is for quasi-polynomials of neutral type with one delay, when the delay appears, roots appear far from the real axis. We have proved the existence of a positive root for infinitesimally small delay for a class of quasi-polynomials with polynomials of degree one: the chain of roots clustering the imaginary axis and the degree of the delay-free polynomial decreases. Furthermore, for those whose chain of roots asymptotic to the imaginary axis from the left, the positive root has been proved to be unique in the closed right half-plane using the Argument principle. Note that stability criteria based on the Argument principle have been proposed both for retarded systems and neutral systems which do not have poles clustering the imaginary axis [6], [7], [8] but to the best of our knowledge never for neutral systems with poles clustering the imaginary axis.

Our ongoing work considers quasi-polynomials with polynomials of higher degree and with several delays.

APPENDIX

The next lemma examines $f(s)$ given as in (1) on a horizontal band far from the real axis.

Lemma 6.1: Let $f(s)$ be given by (1) with $c = -1$. Consider $sh = j2\pi n + x + jy$ with $n > 0$, $x \geq 0$, $|y| \leq \pi$, $\sqrt{x^2 + y^2} \geq \delta$ where δ is a constant in $[1, \pi]$. Then $\Im f(s) > 0$ for n large enough.

Proof: We have

$$e^{-sh} = e^{-x-jy} = e^{-x}e^{-jy},$$

and thus

$$\Im f(s) = \frac{2\pi n + y}{h}\left(1 - e^{-x}\cos y\right) - de^{-x}\sin y. \quad (5)$$

Consider the first term in (5). We have $e^{-x} \leq 1$ with equality at $x = 0$ and $\cos y \leq 1$ with equality at $y = 0$. Because of the assumption $\sqrt{x^2 + y^2} \geq \delta$, we have $\max e^{-x}\cos y = \alpha < 1$, where α depends on δ. Hence, $1 - e^{-x}\cos y \geq 1 - \alpha > 0$ and thus the first term is positive.

Since the second term in (5) is bounded, we can choose n large enough such that $\Im f(s) > 0$. ■

Lemma 6.2: Let $f(s)$ be given in Proposition 4.2 and $C_j = \{jX : X \in [-R, R]\}$ with $R > 0$. Then $f(C_j)$ does not intersect the positive real axis.

Proof: For $s \in C_j$, we have

$$\begin{aligned} f(jX) &= (jX + b) + (-jX + d)e^{-jXh} \\ &= [b - X\sin(Xh) + d\cos(Xh)] + j[X - X\cos(Xh) \\ &\quad - d\sin(Xh)]. \end{aligned}$$

When $f(C_j)$ cuts the real axis, we have

$$\begin{aligned} & \Im f(jX) = 0 \\ \iff \quad & X(1 - \cos(Xh)) - d\sin(Xh) = 0 \\ \iff \quad & 2X\sin^2\frac{Xh}{2} - 2d\sin\frac{Xh}{2}\cos\frac{Xh}{2} = 0 \\ \iff & 2\sin\frac{Xh}{2}\left(X\sin\frac{Xh}{2} - d\cos\frac{Xh}{2}\right) = 0, \end{aligned}$$

which is equivalent to

$$\sin\frac{Xh}{2} = 0 \quad (6)$$

or

$$X\sin\frac{Xh}{2} - d\cos\frac{Xh}{2} = 0. \quad (7)$$

We can rewrite the real part of $f(jX)$ as follows

$$\begin{aligned} \Re f(jX) = b - 2X\sin\frac{Xh}{2}\cos\frac{Xh}{2} + d\Bigg(\cos^2\frac{Xh}{2} \\ - \sin^2\frac{Xh}{2}\Bigg). \end{aligned}$$

Hence, in the case (6) $\Re f(jX) = b + d$ and in the case (7) $\Re f(jX) = b - d$. By assumption, $b^2 - d^2 > 0$ and $b + d < 0$. Hence $b - d < 0$. Therefore $f(C_j)$ only cuts the negative axis. ■

Lemma 6.3: $f(s)$ given as in (1) with $c = -1$ has no root crossing the imaginary axis for $h > 0$ if and only if $b^2 - d^2 \neq 0$.

Proof: For $h > 0$, no crossing means $W(\omega) := |p(j\omega)|^2 - |q(j\omega)|^2 \neq 0 \ \forall \omega > 0$. We have

$$W(\omega) = b^2 - d^2.$$

Then $W(\omega) \neq 0 \ \forall \omega > 0$ if and only if

$$b^2 - d^2 \neq 0.$$

■

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