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► **To cite this version:**

M Antoñana, Philippe Chartier, J Makazaga, A Murua. Global time-regularization of the gravitational N -body problem. 2020. hal-02431607

HAL Id: hal-02431607

<https://hal.inria.fr/hal-02431607>

Preprint submitted on 8 Jan 2020

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Global time-regularization of the gravitational N -body problem

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January 8, 2020

Abstract

This work considers the *gravitational* N -body problem and introduces time-reparametrization *functions* that allow to define globally solutions of the N -body equations. First, a lower bound of the radius of convergence of the solution to the original equations is derived, which suggests an appropriate time-reparametrization. In the new fictitious time τ , it is then proved that any solution exists for all $\tau \in \mathbb{R}$, and that it is uniquely extended as a holomorphic function to a strip of fixed width. As a by-product, a global power series representation of the solutions of the N -body problem is obtained. Noteworthy, our global time-regularization remain valid in the limit when one of the masses vanishes. Finally, numerical experiments show the efficiency of the new time-regularization functions for some N -problems with close encounters.

1 Introduction

In this paper, we are concerned with the solution of the so-called N -body problem, which considers N masses m_i , $i = 1, \dots, N$ in a three-dimensional space, moving under the influence of gravitational forces. Each mass m_i will be described by its *position* $q_i \in \mathbb{R}^3$ and its *velocity* $v_i \in \mathbb{R}^3$. Following Newton, the product of mass and acceleration $m_i \frac{d^2 q_i}{dt^2}$ is equal to the sum of the forces applied on this mass and the gravitational force exerted on mass m_i by a single mass m_j is given by

$$F_{ij} = \frac{Gm_i m_j}{\|q_j - q_i\|^3} (q_j - q_i),$$

where G is the gravitational constant and $\|q_j - q_i\|$ is the distance between q_i and q_j (in the euclidean norm). The N -body equations, defined for positions q outside the set

$$\Delta = \{(q_1, \dots, q_N) \in \mathbb{R}^{3N} : q_i = q_j \text{ for some } i \neq j\},$$

are then obtained by summing over all masses

$$m_i \frac{d^2 q_i}{dt^2} = \sum_{j \neq i} \frac{G m_i m_j}{\|q_j - q_i\|^3} (q_j - q_i). \quad (1)$$

In physics and in celestial mechanics in particular, the N -body problem is of great importance for the prediction of the dynamics of objects interacting with each other. At first, the motivation for considering this problem was the desire to understand how the sun, the moon and other planets and stars are moving. Later on, other questions like the *stability* of the Solar System became subjects of much inquiry in astronomy. For instance, Jacques Laskar of the *Bureau des Longitudes* in Paris published in 1989 the results of numerical integrations of the Solar System over 200 million years [6] and showed that the earth's orbit is chaotic: a mere error of 15 meters in its position today makes it impossible to predict its motion over 100 million years.

In fact, the problem of finding the general solution of the N -body problem was already considered very important and challenging in the late 19th century when King Oscar II of Sweden established a prize for anyone who could represent the solution of the N -body problem in the form of a convergent series (in a variable that is some known function of time), under the assumption that no two points ever collide. Although no one could manage to solve the original problem, the prize was awarded to Henri Poincaré for his seminal contribution [14]. In 1909, the original problem was finally solved for $N = 3$ by Karl Fritiof Sundman [12], a Finnish mathematician, who was awarded a prize by the French Academy of Science for this work, and in the 1990s, Qiu-Dong Wang obtained a generalization of Sundman's solution for the $N \geq 3$ -case¹ [15]. As an intermediate step to obtain their globally convergent series expansion, they considered the solution of the N -body problem as a function of a new independent variable τ related to the physical time t by

$$\frac{d\tau}{dt} = s(q(t))^{-1}, \quad \tau(0) = 0, \quad (2)$$

with some appropriate *time-regularization function* $s(q)$ depending on the positions $q = (q_1, \dots, q_N) \in \mathbb{R}^{3N}$. In Sundman's work (resp. in Qiu-Dong's approach for the $N > 3$ case), the solution is shown to be uniquely determined as a holomorphic function of the complex variable τ in a strip along the real axis (resp. in a less simple neighborhood of the real axis). Finally, a conformal mapping is used to define the solution as a holomorphic function in the unit disk, which immediately leads to the existence of a globally convergent expansion of the solution as a power series of powers. Despite the theoretical interest of the power series expansions considered by Sundman and Wang, they are useless as practical methods to solve the N -body equations due to the extremely slow convergence of the series expansions of the solutions for the values of t corresponding to values of σ near the boundary of the unit disk. As already invoked, highly accurate numerical integration schemes are required instead.

In the presence of close encounters, numerical integration has to be used either with an efficient strategy to adapt the step-size or in combination with some time-regularization. In the later case, one aims at determining a real analytic function

¹L. K. Babadzanjanz claims to have obtained the same generalization [1] in 1979.

$s(q)$ that relates the physical time t with a new independent variable τ by (2) that allows one to use constant time-step (in τ) without degrading the accuracy of the computed trajectories. In this paper, we consider more general time-regularization functions $s(q, v)$ that depend on both positions $q = (q_1, \dots, q_N) \in \mathbb{R}^{3N}$ and velocities $v = (v_1, \dots, v_N) \in \mathbb{R}^{3N}$.

In this work, we introduce new time-regularization functions by analyzing the holomorphic extension of maximal solutions of the N -body problem

$$\frac{dq_i}{dt} = v_i, \quad i = 1, \dots, N, \quad (3)$$

$$\frac{dv_i}{dt} = \sum_{\substack{j=1 \\ j \neq i}}^N \frac{Gm_j}{\|q_i - q_j\|^3} (q_j - q_i), \quad i = 1, \dots, N, \quad (4)$$

to the complex domain. The rationale of our approach is to obtain domains of existence as large as possible. Accordingly, we shall define *global time-regularization* functions as follows and use the parameter β as a measure of their quality.

Definition 1 *Let $s(q, v)$ be a real-analytic function defined on $\mathbb{R}^{3N} \setminus \Delta \times \mathbb{R}^{3N}$. We will say that $s(q, v)$ determines a global time-regularization of (3)–(4) if there exists $\beta > 0$ such that, for arbitrary $q^0 \in \mathbb{R}^{3N} \setminus \Delta$, $v^0 \in \mathbb{R}^{3N}$,*

- *the maximal solution $(q(t), v(t))$ of (3)–(4) supplemented with initial conditions*

$$q(0) = q^0 \in \mathbb{R}^{3N}, \quad v(0) = v^0 \in \mathbb{R}^{3N} \quad (5)$$

is transformed by the reparametrization of time

$$\tau = \theta(t) = \int_0^t s(q(\sigma), v(\sigma))^{-1} d\sigma \quad (6)$$

into $(\hat{q}(\tau), \hat{v}(\tau))$ defined on \mathbb{R} ,

- *the solution $(\hat{q}(\tau), \hat{v}(\tau))$ can be extended analytically in the strip*

$$\{\tau \in \mathbb{C} : |\text{Im}(\tau)| \leq \beta\}.$$

Our main contribution in this paper is to show that such a global time-regularization function exists with $\beta = 0.0444443$.

In practice, if $s(q, v)$ is a global time-regularization function with parameter β , then one can accurately integrate the transformed equations with a high-order integrator with a constant step-size $\Delta\tau$ chosen as a moderate fraction of β . Although standard adaptive step-size implementations may deal with the numerical integration of N -body problems involving close encounters, applying a suitable time-regularization and numerically integrating the resulting equations with constant step-size is often more efficient. This is particularly so for long-time numerical integration: constant step-size geometric integrators [5] applied to time-regularized equations is indeed expected to outperform standard adaptive step-size integrators. In addition, time-regularization is particularly

useful for approximating periodic solutions of N -body problems as truncated Fourier series. Indeed, the coefficients of the Fourier series expansion of periodic solutions $q(t)$ with close encounters decay (exponentially) at a very low rate, while the exponential decay of the Fourier coefficients of $(\hat{q}(\tau), \hat{v}(\tau))$ is bounded from below by the quotient of β with the period in τ . A similar situation occurs with the Chebyshev series expansion of solutions in a prescribed time interval $[0, T]$. In this case, the decay of the Chebyshev coefficients is determined by the size of the complex domain of analyticity (the size of the largest ellipse with foci in 0 and T).

We conclude this introductory section with the outline of the article.

In Section 2 we will state the main results: In Subsection 2.1, we treat the extension of the solutions of (3)–(4) as holomorphic functions of the complex time; a global time-regularization function (in the sense of Definition 1) is presented in Subsection 2.2, and a global power series representation of the solutions of (3)–(4) (similar to those of Sundman and Qiu-Dong) is presented as a by-product; the section is closed providing alternative time-regularization functions in Subsection 2.3. In Section 3 we give technical details of the proofs of the main results of Section 2. Section 4 is devoted to present some illustrative numerical experiments. Finally, some concluding remarks are presented in Section 5.

2 Main results

In this section, we present the main contributions of the paper. First, a lower bound of the radius of convergence of the solution to the original equations is derived (Theorem 2). That result motivates our choice of time-reparametrization function, which we show (Theorem 3) that is a global time-regularization function in the sense of Definition 1.

For the sake of clarity, the proofs of some results stated in the present section are postponed to Section 3. With this in mind, consider the solution $(q(t), v(t))$ of the N -body problem (3)–(4) supplemented with initial conditions (5) such that $q^0 \notin \Delta$. Cauchy-Lipschitz theorem ensures the existence and uniqueness of a *maximal* solution $q(t) \in \mathbb{R}^{3N} \setminus \Delta$, $t \in (t_a, t_b)$.

Theorem 1 (Painlevé [9]) *If $t_a > -\infty$ (resp. $t_b < +\infty$), then $\min_{ij} \|q_i(t) - q_j(t)\|$ tends to 0 as $t \uparrow t_a$ (resp. $t \downarrow t_b$).*

Given that (4) is real analytic, it is well known that the maximal solution $q(t)$ is a real analytic function of t . For each $t^* \in (t_a, t_b)$, the Taylor expansion of $q(t)$ at $t = t^*$

$$q(t) = q(t^*) + (t - t^*)v(t^*) + \sum_{k \geq 2} \frac{(t - t^*)^k}{k!} q^{(k)}(t^*)$$

is converging for all complex times t such that $|t - t^*| < \rho(q^*, v^*)$, where the *radius of convergence* $\rho(q^*, v^*)$ is uniquely determined by $q(t^*) = q^* \in \mathbb{R}^{3N}$ and $v(t^*) = v^* \in \mathbb{R}^{3N}$. In particular, $q(t)$ can be analytically extended to the complex neighbourhood of the

real interval (t_a, t_b)

$$\mathcal{W} = \{t \in \mathbb{C} : |\operatorname{Im}(t)| < \rho(q(t^*), v(t^*)), t^* = \operatorname{Re}(t) \in (t_a, t_b)\}. \quad (7)$$

Remark 1 *Provided that $t_a > -\infty$ (resp. $t_b < +\infty$), $\rho(q(t^*), v(t^*)) < t^* - t_a$ (resp. $\rho(q(t^*), v(t^*)) < t_b - t^*$), and hence $\rho(q(t^*), v(t^*))$ shrinks to 0 as $t^* \downarrow t_a$ (resp. $t^* \uparrow t_b$). If a close encounter occurs around $t^* \in (t_a, t_b)$, then \mathcal{W} becomes very narrow at t^* .*

2.1 Holomorphic extension of the solution of the N -body problem

In the present subsection, we derive lower bounds of $\rho(q, v)$ for arbitrary $q \in \mathbb{R}^{3N} \setminus \Delta$ and $v \in \mathbb{R}^{3N}$ which define an explicit complex neighbourhood of (t_a, t_b) (included in (7)) where the solution $q(t)$ can be analytically extended. By means of Cauchy estimates, bounds of high-order derivatives of the components of $q(t)$ can be obtained straightforwardly.

We first rewrite (4) as $dv_i/dt = g_i(q)$, $i = 1, \dots, N$, where

$$g_i(q) = \sum_{\substack{j=1 \\ j \neq i}}^N \frac{Gm_j}{((q_i - q_j)^T(q_i - q_j))^{3/2}}(q_j - q_i). \quad (8)$$

Function g_i is now holomorphic in the open domain of $q \in \mathbb{C}^{3N}$ obtained by excluding its singularities, i.e. values of $q \in \mathbb{C}^{3N}$ such that

$$(q_i - q_j)^T(q_i - q_j) = 0$$

for some (i, j) , $1 \leq i < j \leq N$. Note that here, denoting $q_i - q_j = (x_{ij}, y_{ij}, z_{ij})$, $(q_i - q_j)^T(q_i - q_j) = x_{ij}^2 + y_{ij}^2 + z_{ij}^2 = 0$ does not imply $q_i = q_j$. In the sequel, the following lemma will prove of much use.

Lemma 1 *Let u and U be two vectors of \mathbb{C}^3 . The following two estimates hold true*

$$\begin{aligned} (i) \quad & |U^T U - u^T u| \leq 2 \|u\| \|U - u\| + \|U - u\|^2, \\ (ii) \quad & |U^T U| \geq |u^T u| - 2 \|u\| \|U - u\| - \|U - u\|^2. \end{aligned}$$

Proof of Lemma 1. In order to prove (i), we write $U^T U - u^T u$ as $(U - u)^T((U - u) + 2u)$ and use Cauchy-Schwartz inequality to get

$$|(U - u)^T((U - u) + 2u)| \leq \|U - u\| (\|U - u\| + 2\|u\|).$$

Point (ii) follows from $|u^T u| - |U^T U| \leq |U^T U - u^T u|$. ■

Now, given $q^0 \in \mathbb{R}^{3N} \setminus \Delta$, consider $q \in \mathbb{C}^{3N}$ such that

$$\|q_i - q_j - q_i^0 + q_j^0\| < (\sqrt{2} - 1) \|q_i^0 - q_j^0\|, \quad 1 \leq i < j \leq N. \quad (9)$$

Lemma 1 then implies that $(q_i - q_j)^T(q_i - q_j) > 0$ for all i, j , so that (8) is holomorphic in the open neighborhood of q^0

$$\mathcal{U}_\lambda(q^0) = \{q \in \mathbb{C}^{3N} : \|q_i - q_j - q_i^0 + q_j^0\| \leq \lambda \|q_i^0 - q_j^0\|, \quad 1 \leq i < j \leq N\} \quad (10)$$

for $\lambda = \sqrt{2} - 1$. A further application of Lemma 1 allows to bound (8) as in next proposition, whose proof is straightforward and thus omitted.

Proposition 1 *Let $\lambda \in (0, \sqrt{2} - 1)$. For all $q \in \mathcal{U}_\lambda(q^0)$, one has*

$$\|g_i(q)\| \leq \eta(\lambda) K_i(q),$$

where

$$K_i(q) = \sum_{\substack{j=1 \\ j \neq i}}^N \frac{G m_j}{\|q_i - q_j\|^2}, \quad \eta(\lambda) = \frac{1 + \lambda}{(1 - 2\lambda - \lambda^2)^{3/2}}. \quad (11)$$

We are now in position to state the following ‘‘existence and uniqueness’’ theorem.

Theorem 2 *Under the assumptions of Proposition 1, the solution $(q(t), v(t))$ of (3)–(5) is uniquely determined as a holomorphic function of t in the closed disk*

$$D_\lambda(q^0, v^0) = \{t \in \mathbb{C} : |t| \leq L(q^0, v^0, \lambda)^{-1}\},$$

where $L(q, v, \lambda) = \max_{1 \leq i < j \leq N} L_{ij}(q, v, \lambda)$, with

$$L_{ij}(q, v, \lambda) = \frac{\|v_i - v_j\|}{2\lambda \|q_i - q_j\|} + \sqrt{\left(\frac{\|v_i - v_j\|}{2\lambda \|q_i - q_j\|}\right)^2 + \frac{M_{ij}(q, \lambda)}{2\lambda \|q_i - q_j\|}},$$

and

$$M_{ij}(q, \lambda) = \eta(\lambda) (K_i(q) + K_j(q)). \quad (12)$$

Furthermore, for all $t \in D_\lambda(q^0, v^0)$,

$$\|q_i(t) - q_j(t) - q_i^0 + q_j^0\| \leq |t| \|v_i^0 - v_j^0\| + \frac{|t|^2}{2} M_{ij}(q^0, \lambda), \quad (13)$$

$$\|v_i(t) - v_j(t) - v_i^0 + v_j^0\| \leq M_{ij}(q^0, \lambda) |t|, \quad (14)$$

which in particular implies that $q(t) \in \mathcal{U}_\lambda(q^0)$.

Remark 2 *It is well known that, if $(q(t), v(t))$ is a solution of (3)–(4) and $\nu > 0$, then $(Q(t), V(t)) = (\nu^{-2/3}q(\nu t), \nu^{1/3}v(\nu t))$ is also a solution of (3)–(4). This implies that $\rho(\nu^{-2/3}q, \nu^{1/3}v) \equiv \nu^{-1} \rho(q, v)$. The lower estimate of the the radius of convergence $\rho(q, v)$ given by Theorem 2 is consistent with that scale invariance property, that is, $L(\nu^{-2/3}q, \nu^{1/3}v, \lambda) \equiv \nu L(q, v, \lambda)$.*

Remark 3 *Theorem 2 provides, for each $\lambda \in (0, \sqrt{2} - 1)$, a lower bound $1/L(q^*, v^*, \lambda)$ of the radius of convergence $\rho(q^*, v^*)$ at $t = t^*$ of any solution $q(t)$ of (1). This means that the maximal real solution $q(t)$ of (3)–(5) defined for $t \in (t_a, t_b)$ can be uniquely extended to the closed complex neighbourhood*

$$\{t \in \mathbb{C} : |\operatorname{Im}(t)| \leq L(q(t^*), v(t^*), \lambda)^{-1}, t^* = \operatorname{Re}(t) \in (t_a, t_b)\}$$

of (t_a, t_b) , which is included in (7). In fact, Theorem 1 can be seen as a corollary of Theorem 2. As a matter of fact, according to Remark 1, if $t_a > -\infty$, then $L(q(t), v(t), \lambda) \rightarrow \infty$ as $t \downarrow t_a$. This implies that either $\min_{i,j} \|q_i(t) - q_j(t)\| \rightarrow 0$ or $\max_{i,j} \|v_i(t) - v_j(t)\| \rightarrow \infty$ as $t \downarrow t_a$, but by virtue of (14), the later implies the former. A similar argument holds when $t_b < +\infty$.

Remark 4 By virtue of Proposition 1, the second order derivative of the solution $q(t) = (q_1(t), \dots, q_N(t))$ of (3)–(5) for $t \in D_\lambda(q^0, v^0)$ can be bounded as

$$\|q_i^{(2)}(t)\| = \|g_i(q(t))\| \leq \eta(\lambda) K_i(q^0),$$

and Cauchy estimates give immediately

$$\|q_i^{(n)}(0)\| \leq \eta(\lambda) (n-2)! K_i(q^0) L(q^0, v^0, \lambda)^{n-2}, \quad n \geq 3.$$

2.2 Global time-regularization of N -body problems

Our aim in this subsection is to determine an appropriate real analytic function $s(q, v)$ that relates the physical time t in (3)–(4) to a fictitious time-variable τ by

$$\frac{d\tau}{dt} = s(q(t), v(t))^{-1}, \quad \tau(0) = 0, \quad (15)$$

such that equations (3)–(4) expressed in terms of τ read

$$\begin{aligned} \frac{d\hat{q}_i}{d\tau} &= s(\hat{q}, \hat{v}) \hat{v}_i, & i = 1, \dots, N, \\ \frac{d\hat{v}_i}{d\tau} &= s(\hat{q}, \hat{v}) \sum_{j \neq i} \frac{G m_j}{\|\hat{q}_i - \hat{q}_j\|^3} (\hat{q}_j - \hat{q}_i), & i = 1, \dots, N. \end{aligned} \quad (16)$$

As is well-known, solutions $(q(t), v(t))$ of (3)–(5) and $(\hat{q}(\tau), \hat{v}(\tau))$ of (16) with initial condition

$$\hat{q}(0) = q^0, \quad \hat{v}(0) = v^0, \quad (17)$$

are related by $q(t) = \hat{q}(\theta(t))$, $v(t) = \hat{v}(\theta(t))$, where $\theta : (t_a, t_b) \rightarrow \mathbb{R}$ is given by (6).

Before stating next proposition, recall from Theorem 2 that the solution $(q(t), v(t))$ of (3)–(5) is such that $(q(t), v(t)) \in \mathcal{U}_\lambda(q^0) \times \mathcal{V}_\lambda(q^0, v^0)$ for all $t \in D_\lambda(q^0, v^0)$, where

$$\mathcal{V}_\lambda(q^0, v^0) = \{(q, v) \in \mathbb{C}^{6N} : \|v_i - v_j - v_i^0 + v_j^0\| \leq M_{ij}(q^0, \lambda)/L(q^0, v^0, \lambda)\}. \quad (18)$$

Proposition 2 Let $s(q, v)$ be a positive real analytic function in $(\mathbb{R}^{3N} \setminus \Delta) \times \mathbb{R}^{3N}$. Given $q^0 \in \mathbb{R}^{3N} \setminus \Delta$, $v^0 \in \mathbb{R}^{3N}$, assume that there exists $\lambda \in (0, \sqrt{2} - 1)$ and $\delta \in (0, 1)$ such that $s(q, v)$ can be holomorphically extended to $\mathcal{U}_\lambda(q^0) \times \mathcal{V}_\lambda(q^0, v^0)$, and such that

$$\forall (q, v) \in \mathcal{U}_\lambda(q^0) \times \mathcal{V}_\lambda(q^0, v^0), \quad |s(q, v)^{-1} - s(q^0, v^0)^{-1}| \leq \delta s(q^0, v^0)^{-1}. \quad (19)$$

Then, the solution $(\hat{q}(\tau), \hat{v}(\tau))$ of (16)–(17) is uniquely defined as a holomorphic function of τ in the disk

$$\{\tau \in \mathbb{C} : |\tau| \leq (1 - \delta) s(q^0, v^0)^{-1} L(q^0, v^0, \lambda)^{-1}\}. \quad (20)$$

A positive real-analytic function $s(q, v)$ satisfying (19) for some λ and δ , and such that $s(q, v)^{-1}L(q, v, \lambda)^{-1}$ is bounded from below for all $(q, v) \in \mathbb{R}^{3N} \setminus \Delta \times \mathbb{R}^{3N}$ is thus a global time-regularization in the sense of Definition 1, with

$$\beta = (1 - \delta) \inf_{q \in \mathbb{R}^{3N} \setminus \Delta, v \in \mathbb{R}^{3N}} s(q, v)^{-1} L(q, v, \lambda)^{-1}.$$

Motivated by that, we thus seek a real-analytic function bounding $L(q, v, \lambda)$ from above for (q, v) in $(\mathbb{R}^{3N} \setminus \Delta) \times \mathbb{R}^{3N}$. Such a function can be obtained by first observing that

$$L(q, v, \lambda) = \max_{1 \leq i < j \leq N} L_{ij}(q, v, \lambda) \leq \sqrt{\sum_{1 \leq i < j \leq N} L_{ij}(q, v, \lambda)^2}, \quad (21)$$

and second noticing, as a consequence of the simple inequality

$$\forall (t_a, t_b) \in \mathbb{R}_+^2, \quad \frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 + \frac{b}{2}} \leq \sqrt{a^2 + b},$$

that

$$L(q, v, \lambda) \leq \lambda^{-1} \sqrt{\sum_{1 \leq i < j \leq N} \left(\frac{\|v_i - v_j\|}{\|q_i - q_j\|}\right)^2 + \lambda \eta(\lambda) \sum_{1 \leq i < j \leq N} \frac{K_i(q) + K_j(q)}{\|q_i - q_j\|}}. \quad (22)$$

It is thus natural to pick up the following candidate for $s(q, v)$

$$s(q, v) = \left(\sum_{1 \leq i < j \leq N} \frac{(v_i - v_j)^T (v_i - v_j)}{(q_i - q_j)^T (q_i - q_j)} + \sum_{1 \leq i < j \leq N} \frac{K_i(q) + K_j(q)}{\sqrt{(q_i - q_j)^T (q_i - q_j)}} \right)^{-1/2} \quad (23)$$

where for each $i = 1, \dots, N$,

$$K_i(q) = \sum_{\substack{k=1 \\ k \neq i}}^N \frac{G m_k}{(q_i - q_k)^T (q_i - q_k)}. \quad (24)$$

Let $\lambda_0 \approx 0.244204 < \sqrt{2} - 1$ be the unique positive zero of $\lambda \eta(\lambda) = 1$. Then, Clearly, we have that, if $(q^0, v^0) \in (\mathbb{R}^{3N} \setminus \Delta) \times \mathbb{R}^{3N}$, then

$$\forall 0 \leq \lambda \leq \lambda_0, \quad \forall (q^0, v^0) \in (\mathbb{R}^{3N} \setminus \Delta) \times \mathbb{R}^{3N}, \quad s(q^0, v^0)^{-1} L(q^0, v^0, \lambda)^{-1} \geq \lambda. \quad (25)$$

Proposition 3 *There exists $\lambda_* \in (0, \lambda_0)$ and $\delta : (0, \lambda_*) \rightarrow (0, 1)$ such that the assumptions of Proposition 2 hold for $s(q, v)$ given in (23) with arbitrary $\lambda \in (0, \lambda_*)$ and $\delta = \delta(\lambda)$.*

By combining Propositions 2 and 3 with (25), we finally obtain the following.

Theorem 3 *Consider $s(q, v)$ given by (23). For arbitrary $q^0 \in \mathbb{R}^{3N} \setminus \Delta$, $v^0 \in \mathbb{R}^{3N}$, the solution $(\hat{q}(\tau), \hat{v}(\tau))$ of (16)–(17) is uniquely defined as a holomorphic function of the complex variable τ in the strip*

$$\{\tau \in \mathbb{C} : |\operatorname{Im}(\tau)| \leq 0.0444443\}$$

In other words, $s(q, v)$ given by (23) is a time-regularization function for $\beta = 0.0444443$.

Remark 5 Proposition 2 holds with $\lambda_* = 0.0988424$ as it can be seen in its proof, given in Section 3 below. The constant $\beta = 0.0444443$ is obtained as the maximum (attained at $\lambda = 0.0694156$) of $(1 - \delta(\lambda)) \lambda$ subject to the constraint $0 < \lambda < \lambda_*$.

Remark 6 As a consequence of Theorem 3, we get a globally convergent series expansion in powers of a new variable σ , related to τ with the conformal mapping

$$\tau \mapsto \sigma = \frac{\exp(\frac{\pi}{2\beta}\tau) - 1}{\exp(\frac{\pi}{2\beta}\tau) + 1}.$$

that maps the strip $\{\tau \in \mathbb{C} : |\text{Im}(\tau)| \leq \beta\}$ into the unit disk. This is closely related to Sundman's result [12] for the 3-body problem as well as Qiu-Dong's results [15] for the general case of N -body problems. It is worth emphasizing that, contrary to Qiu-Dong's solution, our approach remains valid in the limit where $\min_{1 \leq i \leq N} m_i/M \rightarrow 0$ with $M = \sum_{1 \leq i \leq N} m_i$.

2.3 Alternative time-regularization functions

A computationally simpler alternative of (23) can be derived by observing that for each $1 \leq i < j \leq N$,

$$K_i(q) + K_j(q) \leq \sum_{1 \leq k < \ell \leq N} \frac{G(m_k + m_\ell)}{\|q_k - q_\ell\|^2}.$$

It is straightforward to check that the proofs of Proposition 3 and Theorem 3 are also valid for the time-regularization function

$$s(q, v) = \left(\sum_{1 \leq i < j \leq N} \left(\frac{\|v_i - v_j\|}{\|q_i - q_j\|} \right)^2 + A(q) \sum_{1 \leq i < j \leq N} \frac{G(m_i + m_j)}{\|q_i - q_j\|^2} \right)^{-1/2} \quad (26)$$

where

$$A(q) = \sum_{1 \leq i < j \leq N} \frac{1}{\|q_i - q_j\|}.$$

Another alternative time-regularization function $s(q, v)$ can be obtained as follows: It is apparent from the proof of Theorem 2 given in Section 3 that its conclusion remains true if $M_{ij}(q, \lambda)$ in (12) is replaced by any upper bound of

$$\sup_{q \in \mathcal{U}_\lambda(q^0)} \|g_i(q) - g_j(q)\| \quad (27)$$

in lieu of $\eta(\lambda)(K_i(q^0) + K_j(q^0))$. E.g., by writing for any $1 \leq i < j \leq N$ as

$$g_i(q) - g_j(q) = \frac{G(m_i + m_j)}{D(q_i, q_j)}(q_j - q_i) + \sum_{\substack{k=1 \\ k \neq i, j}} \left(\frac{G m_k}{D(q_i, q_k)}(q_k - q_i) - \frac{G m_k}{D(q_j, q_k)}(q_k - q_j) \right)$$

where $D(q_i, q_j) = ((q_i - q_j)^T(q_i - q_j))^{3/2}$ and accordingly for indices j and k , it can be proven that there exists $\lambda_0 > 0$ such that

$$\forall \lambda \in (0, \lambda_0), \quad \sup_{q \in \mathcal{U}_\lambda(q^0)} \|g_i(q) - g_j(q)\| \leq \tilde{M}_{ij}(q^0, \lambda),$$

where

$$\tilde{M}_{ij}(q^0, \lambda) = \eta(\lambda) \frac{G(m_i + m_j)}{\|q_i^0 - q_j^0\|^2} + \tilde{\eta}(\lambda) \sum_{\substack{k=1 \\ k \neq i, j}} \left(\frac{G m_k}{\|q_i - q_k\|^3} + \frac{G m_k}{\|q_j - q_k\|^3} \right) \|q_i^0 - q_j^0\|,$$

for some continuous function $\tilde{\eta}(\lambda)$ of λ in $[0, \lambda_0)$. This leads to consider \tilde{L} instead of L with

$$\tilde{L}(q, v, \lambda) = \max_{1 \leq i < j \leq N} \sqrt{\left(\frac{\|v_i - v_j\|}{\lambda \|q_i - q_j\|} \right)^2 + \frac{\tilde{M}_{ij}(q, \lambda)}{\lambda \|q_i - q_j\|}},$$

Then, noticing that

$$\tilde{M}_{ij}(q^0, \lambda) \leq \tilde{\eta}(\lambda) \sum_{1 \leq k < l \leq N} \frac{G(m_k + m_l)}{\|q_k - q_l\|^3},$$

this suggest to consider

$$s(q, v) = \left(\kappa \sum_{1 \leq i < j \leq N} \left(\frac{\|v_i - v_j\|}{\|q_i - q_j\|} \right)^2 + \sum_{1 \leq i < j \leq N} \frac{G(m_i + m_j)}{\|q_i - q_j\|^3} \right)^{-1/2} \quad (28)$$

for a suitable $\kappa > 0$ as an alternative time-regularization function. Indeed, it is possible to prove, in the vein as Theorem 3, that $s(q, v)$ given by (28) (for some $\kappa > 0$) is a time-regularization function for some value of $\beta > 0$ depending on $\kappa > 0$ (in the sense of Definition 1).

Remark 7 *Time-regularization function that do not depend on the velocities v_i are required for some numerical integrators. In that case, one can consider (26) with $\kappa = 0$ (already considered by K. Babadzanjanz [1] in 1979):*

$$s(q, v) = \left(\sum_{1 \leq i < j \leq N} \frac{G(m_i + m_j)}{\|q_i - q_j\|^3} \right)^{-1/2}. \quad (29)$$

Although (29) is not a global time-regularization function in the sense of Definition 1 (see second example in Section 4), one indeed expects from (28) a behaviour similar to (26) near a binary close encounter of elliptic or parabolic character. More specifically, when (28) is dominated by $\frac{G(m_i + m_j)}{\|q_i - q_j\|^3}$ for a particular pair of indices (i, j) , then the equations of motion of the relative position $q_i - q_j$ can be seen as a small perturbation of a two-body problem. If the close encounter corresponds to the periastron of a very eccentric elliptic orbit, then $\frac{G(m_i + m_j)}{\|q_i - q_j\|^3}$ and $\frac{1}{2} \left(\frac{\|v_i - v_j\|}{\|q_i - q_j\|} \right)^2$ will be of comparable magnitudes.

Remark 8 Some numerical integrators for time-regularized equations require that $s(q, v)$ only depend on the potential function [8, 10]

$$U(q) = \sum_{1 \leq i < j \leq N} \frac{G m_i m_j}{\|q_i - q_j\|}.$$

In that case, it make sense to choose ² $s(q) = U(q)^{3/2}$, which is expected to behave asymptotically as (29) near binary collisions. It should be noted (see [7]) that this may perform poorly for systems with small mass ratios.

Remark 9 All time-regularization functions introduced so far are scale invariant in the sense of [2, 3], that is, $s(\nu^{-2/3}q, \nu^{1/3}v) = \nu^{-1}s(q, v)$. This implies that, if $(\hat{q}(\tau), \hat{v}(\tau), t(\tau))$ is a solution of the system consisting on the equations (16) together with $\frac{d}{d\tau}t = s(\hat{q}, \hat{v})$, then, for each $\nu > 0$, $(\nu^{-2/3}\hat{q}(\tau), \nu^{1/3}\hat{v}(\tau), \nu t(\tau))$ is also a solution of that system.

3 Proofs of main results

Existence and uniqueness of solutions of initial value problems of ordinary differential equations in the complex domain can be found in the book of E. Hille [4], in particular, Theorem 2.3.1 (page 48) and its generalization to the vector case, referred to as Theorem 2.3.2 in [4]. The technique of proof in Theorem 2.3.1 (based on Picard iteration) is the one we adopted in the following.

Proof of Theorem 2. The solution $q(t)$ of equations 3)–(5), as long as it exists as a holomorphic function of the complex variable t , satisfies

$$q_i(t) = q_i^0 + t v_i^0 + \int_0^t \int_0^\sigma g_i(q(r)) dr d\sigma, \quad i = 1, \dots, N.$$

We define the Picard iteration for $q_i(t)$, $i = 1, \dots, N$, as follows:

$$\begin{aligned} n = 0 : \quad & q_i^{[0]}(t) = q_i^0 + t v_i^0, \\ n \geq 1 : \quad & q_i^{[n]}(t) = q_i^0 + t v_i^0 + \int_0^t \int_0^\sigma g_i(q^{[n-1]}(r)) dr d\sigma. \end{aligned}$$

We will prove by induction that for all $n \in \mathbb{N}$

- (i) $q^{[n]}$ is well defined and holomorphic on $D_\lambda(q^0, v^0) = \{t \in \mathbb{C}, |t| \leq L(q^0, v^0, \lambda)^{-1}\}$,
- (ii) for all $t \in D_\lambda(q^0, v^0)$, $q^{[n]}(t) \in \mathcal{U}_\lambda(q^0)$.

Note that, most importantly, $t \in D_\lambda(q^0, v^0)$ if and only if

$$|t| \|v_i^0 - v_j^0\| + \frac{|t|^2}{2} M_{ij}(q^0, \lambda) \leq \lambda \|q_i^0 - q_j^0\|, \quad 1 \leq i < j \leq N. \quad (30)$$

²This is the choice of time-reparametrization function adopted in [15] to change the time-variable, as an intermediate step to get the globally convergent series expansion of the solutions of the N -body problem

Given that statements (i) and (ii) hold for $n = 0$, assume that they hold for some $n - 1 \geq 0$. We have in particular that,

$$\forall 1 \leq i < j \leq N, \forall t \in D_\lambda(q^0, v^0), \quad \|g_i(q^{[n-1]}(t)) - g_j(q^{[n-1]}(t))\| \leq M_{ij}(q^0, \lambda).$$

Then, as a double integral of a holomorphic function, which is itself holomorphic as the composition of two holomorphic functions, $q^{[n]}(t)$ is also holomorphic on $D_\lambda(q^0, v^0)$. In order to prove (ii), we estimate for $i = 1, \dots, N$

$$\begin{aligned} \|q_i^{[n]}(t) - q_j^{[n]}(t) - (q_i^0 - q_j^0)\| &\leq |t| \|v_i^0 - v_j^0\| + \int_0^t \int_0^\sigma M_{ij}(q^0, \lambda) dr d\sigma \\ &\leq |t| \|v_i^0 - v_j^0\| + \frac{|t|^2}{2} M_{ij}(q^0, \lambda), \end{aligned}$$

which, together with (30), implies that $q^{[n]}(t) \in \mathcal{U}_\lambda(q^0)$.

Now, in order to prove the convergence of the sequence $q^{[n]}(t)$ in the Banach space of holomorphic functions on $\mathcal{U}_\lambda(q^0)$ with norm $\|\cdot\|_*$

$$\|q\|_* = \max_{1 \leq i \leq N} \|q_i\|,$$

we write

$$q(t) - q^{[0]}(t) = \sum_{n=1}^{\infty} (q^{[n]}(t) - q^{[n-1]}(t))$$

and show that the series is absolutely convergent for all $t \in D_\lambda(q^0, v^0)$. To this aim, we write

$$\|q_i^{[1]}(t) - q_i^{[0]}(t)\| \leq \frac{|t|^2}{2} M_i(q^0, \lambda), \quad \|q_i^{[2]}(t) - q_i^{[1]}(t)\| \leq \ell_i(q^0, \lambda) \eta(\lambda) K_i(q^0) \frac{|t|^4}{4!}$$

where $\ell_i(q^0, \lambda)$ is a Lipschitz constant of g_i in $\mathcal{U}_\lambda(q^0)$. More generally, we obtain

$$\|q_i^{[n]}(t) - q_i^{[n-1]}(t)\| \leq \frac{M_i(q^0, \lambda)}{\ell_i(q^0, \lambda)} \frac{(\sqrt{\ell_i(q^0, \lambda)}|t|)^{2n}}{(2n)!}.$$

Note that we have relied on the fact that $q^{[n]}(t) \in \mathcal{U}_\lambda(q^0)$ for $t \in D_\lambda(q^0, v^0)$ in order to bound g_i by $\eta(\lambda)K_i(q^0)$ and to use the Lipschitz constant on $\mathcal{U}_\lambda(q^0)$. As a consequence, the series is absolutely convergent with

$$\sum_{n=1}^{\infty} \|q^{[n]}(t) - q^{[n-1]}(t)\| \leq \frac{M_i(q^0, \lambda)}{\ell_i(q^0, \lambda)} \left(\cosh(\sqrt{\ell_i(q^0, \lambda)}|t|) - 1 \right)$$

and this proves the existence of a solution on $D_\lambda(q^0, v^0)$, holomorphic at least on the interior of $D_\lambda(q^0, v^0)$ and continuous on the boundary. The uniqueness can be obtained through the use of the Lipschitz constant again. ■

Proof of Proposition 2. First observe that (19) implies that, for all $(q, v) \in \mathcal{U}_\lambda(q^0) \times \mathcal{V}_\lambda(q^0, v^0)$,

$$(1 - \delta) s(q^0, v^0)^{-1} \leq |s(q, v)^{-1}| \leq (1 + \delta) s(q^0, v^0)^{-1},$$

so that, in particular, $s(q, v)^{-1}$ is holomorphic in $\mathcal{U}_\lambda(q^0) \times \mathcal{V}_\lambda(q^0, v^0)$. We thus have that $s(q(t), v(t))^{-1}$ is a holomorphic function of the complex variable t , so that $\theta(t)$ (see Definition 1) is a well-defined holomorphic function in the disk

$$D_\lambda(q^0, v^0) = \{t \in \mathbb{C} : |t| \leq L(q^0, v^0, \lambda)^{-1}\}.$$

Furthermore, if $t_1, t_2 \in D_\lambda(q^0, v^0)$,

$$|\theta(t_2) - \theta(t_1)| = \left| \int_{t_1}^{t_2} s(q(t), v(t))^{-1} dt \right| \geq (1 - \delta) s(q^0, v^0)^{-1} |t_2 - t_1|.$$

Whence it follows that θ is injective in $D_\lambda(q^0, v^0)$. If $t^* \in \mathbb{C}$ is on the boundary of $D_\lambda(q^0, v^0)$, then

$$\begin{aligned} |\theta(t^*)| &= \left| \int_0^{t^*} s(q(t), v(t))^{-1} dt \right| \\ &\geq (1 - \delta) s(q^0, v^0)^{-1} |t^*| \\ &= (1 - \delta) s(q^0, v^0)^{-1} L(q^0, v^0, \lambda)^{-1}. \end{aligned}$$

Hence, the image by θ of $D_\lambda(q^0, v^0)$ contains the disk (20). We conclude that θ^{-1} is well-defined and holomorphic in the disk (20). ■

We next state without proof the following auxiliary lemma.

Lemma 2 Consider $u \in \mathbb{R}^3$, $u \in \mathbb{C}^3$ such that $\|u - u^0\| \leq \lambda \|u^0\|$, with $\lambda \in (0, \sqrt{2} - 1)$. The following estimates hold true

- (i) $|u^T u - \|u^0\|^2| \geq (2\lambda + \lambda^2) \|u^0\|$,
- (ii) $|u^T u| \geq (1 - 2\lambda - \lambda^2) \|u^0\|$,
- (iii) $|(u^T u)^{-1} - \|u^0\|^{-2}| \leq \lambda \alpha(\lambda) \|u^0\|^{-2}$,
- (iv) $\left| (u^T u)^{1/2} - \|u^0\| \right| \leq \lambda \beta(\lambda) \|u^0\|$,
- (v) $\left| (u^T u)^{-1/2} - \|u^0\|^{-1} \right| \leq \lambda \gamma(\lambda) \|u^0\|^{-1}$,

where

$$\alpha(\lambda) = \frac{2 + \lambda}{1 - 2\lambda - \lambda^2}, \quad \beta(\lambda) = \frac{2 + \lambda}{1 + \sqrt{1 - 2\lambda - \lambda^2}}, \quad \gamma(\lambda) = \frac{\beta(\lambda)}{\sqrt{1 - 2\lambda - \lambda^2}}.$$

In what follows, we use the notations $r_{ij} = q_i - q_j$, $w_{ij} = v_i - v_j$, and likewise with zero superscript.

Proof of Proposition 3. Under the assumption that $(q, v) \in \mathcal{U}_\lambda(q^0) \times \mathcal{V}_\lambda(q^0, v^0)$, Lemma 2 implies that, for $i = 1, \dots, N$,

$$|K_i(q) - K_i(q^0)| \leq \lambda \alpha(\lambda) K_i(q^0).$$

and

$$\begin{aligned} \left| \frac{K_i(q) + K_j(q)}{(r_{ij}^T r_{ij})^{1/2}} - \frac{K_i(q^0) + K_j(q^0)}{\|r_{ij}^0\|} \right| &\leq \left| \frac{K_i(q) - K_i(q^0) + K_j(q) - K_j(q^0)}{(r_{ij}^T r_{ij})^{1/2}} \right| \\ &\quad + (K_i(q^0) + K_j(q^0)) \left| \frac{1}{(r_{ij}^T r_{ij})^{1/2}} - \frac{1}{\|r_{ij}^0\|} \right| \\ &\leq \lambda \nu(\lambda) \frac{K_i(q^0) + K_j(q^0)}{\|r_{ij}^0\|}, \end{aligned}$$

where

$$\nu(\lambda) = \frac{\alpha(\lambda)}{\sqrt{1 - 2\lambda - \lambda^2}} + \gamma(\lambda).$$

On the other hand, $\|w_{ij} - w_{ij}^0\| \leq M_{ij}(q^0, \lambda)/L(q^0, v^0, \lambda)$ implies that

$$\|w_{ij} - w_{ij}^0\| \leq \frac{\lambda \|r_{ij}^0\| M_{ij}(q^0, \lambda)}{\|w_{ij}^0\|}, \quad \|w_{ij} - w_{ij}^0\|^2 \leq \lambda \|r_{ij}^0\| M_{ij}(q^0, \lambda).$$

This, together with Lemma 1 gives

$$\begin{aligned} |w_{ij}^T w_{ij} - \|w_{ij}^0\|^2| &\leq 3\lambda \|r_{ij}^0\| M_{ij}(q^0, \lambda) \\ &= 3\lambda \eta(\lambda) \|r_{ij}^0\| (K_i(q^0) + K_j(q^0)). \end{aligned}$$

Then,

$$\begin{aligned} \left| \frac{w_{ij}^T w_{ij}}{r_{ij}^T r_{ij}} - \frac{\|w_{ij}^0\|^2}{\|r_{ij}^0\|^2} \right| &\leq \frac{|w_{ij}^T w_{ij} - \|w_{ij}^0\|^2|}{r_{ij}^T r_{ij}} + \|w_{ij}^0\|^2 \left(\frac{1}{r_{ij}^T r_{ij}} - \frac{1}{\|r_{ij}^0\|^2} \right) \\ &\leq \lambda \xi(\lambda) \frac{K_i(q^0) + K_j(q^0)}{\|r_{ij}^0\|} + \lambda \alpha(\lambda) \frac{\|w_{ij}^0\|^2}{\|r_{ij}^0\|^2}, \end{aligned}$$

where

$$\xi(\lambda) = \frac{3\eta(\lambda)}{(1 - 2\lambda - \lambda^2)^2}.$$

Hence,

$$\begin{aligned} |s(q, v)^{-2} - s(q^0, v^0)^{-2}| &\leq \lambda \alpha(\lambda) \frac{\|w_{ij}^0\|^2}{\|r_{ij}^0\|^2} + \lambda (\nu(\lambda) + \xi(\lambda)) \frac{K_i(q^0) + K_j(q^0)}{\|r_{ij}^0\|} \\ &\leq \lambda \mu(\lambda) s(q^0, v^0)^{-2}, \end{aligned}$$

where

$$\mu(\lambda) = \max(\gamma(\lambda), \nu(\lambda) + \xi(\lambda)) = \nu(\lambda) + \xi(\lambda).$$

Then, provided that $\lambda \mu(\lambda) < 1$ (i.e., provided that $\lambda < \lambda_0 = 0.0988424 < \sqrt{2} - 1$),

$$\begin{aligned} |s(q, v)^{-2}| &\geq (1 - \lambda \mu(\lambda)) s(q^0, v^0)^{-2}, \\ |s(q, v)^{-1}| &\geq \sqrt{1 - \lambda \mu(\lambda)} s(q^0, v^0)^{-1}, \\ |s(q, v)^{-1} - s(q^0, v^0)^{-1}| &\leq \lambda \delta(\lambda) s(q^0, v^0)^{-1}, \end{aligned}$$

where

$$\delta(\lambda) = \frac{\mu(\lambda)}{1 + \sqrt{1 - \lambda \mu(\lambda)}}.$$

■

4 Numerical experiments

We consider three examples, corresponding to the solution $(q(t), v(t))$ of (3)–(5) for two 3-body problems and one 9-body problem in certain time intervals $t \in [0, T]$. For each example, we obtain several figures:

1. A comparison of the actual radius of convergence with the lower bound given by Theorem 2. More precisely,
 - we obtain accurate approximations (q^k, v^k) of the solution $(q(t^k), v(t^k))$ for some discretization $0 = t^0 < t^1 < \dots < t^k < \dots < t^n = T$ of the time interval $[0, T]$. This is done by numerically integrating the regularized equations (16), with the time-regularization function (23), together with

$$\frac{d}{d\tau} t = s(\hat{q}, \hat{v}), \quad t(0) = 0. \quad (31)$$

To that aim, we apply a Taylor integrator of order 30 with constant step-size $\Delta\tau$. More precisely, we compute the positions $q^k \in \mathbb{R}^{3N}$, velocities $v^k \in \mathbb{R}^{3N}$, and times $t^k \in \mathbb{R}$, corresponding to $\tau^k = k \Delta\tau$ for $k = 1, 2, \dots$, in a step-by-step manner by considering the Taylor expansion of the solution $(\hat{q}(\tau), \hat{v}(\tau), t(\tau))$ at $\tau = \tau^k$, $k = 0, 1, 2, \dots$ truncated at order 30.

- the radius of convergence $\rho(q^k, v^k)$ of the Taylor expansion of the solution $q(t)$ centered at $t = t^k$ is computed numerically. This is done by computing the Taylor expansion of the solution $q(t)$ centered at $t = t^k$, ($k = 0, 1, 2, \dots$) truncated at order 30, and numerically estimating the radius of convergence.
 - the radius of convergence $\rho(q^k, v^k)$ is displayed in logarithmic scale together with the lower bound $1/L(q^k, v^k, \lambda_0)$ (with $\lambda_0 = 0.244204$) of $\rho(q^k, v^k)$.
2. A comparison of the radius of convergence of the Taylor expansion of $(\hat{q}(\tau), \hat{v}(\tau))$ in the fictitious variable τ for different time-regularization functions $s(q, v)$. We will denote the function (23) as $s_1(q, v)$, (26) as $s_2(q, v)$, (28) with $\kappa = 1$ as $s_3(q, v)$, and (29) as $s_4(q)$. For a fair comparison, each function $s(q, v)$ is scaled so that the length of the interval $[0, T]$ in physical time and the length of the corresponding interval in the fictitious time variable τ coincide. For each scaled

time-regularization function $s(q, v)$, the width of the strip around the segment $[0, T]$ of the real axis (in the complex τ -plane) where the solution $(\hat{q}(\tau), \hat{v}(\tau))$ is guaranteed to be holomorphic is computed as the minimum of the radius of convergence of the Taylor expansion of $(\hat{q}(\tau), \hat{v}(\tau))$ for $\tau \in [0, T]$.

3. A comparison of the efficiency of numerical integration of the regularized equations (16) with constant step-size with the numerical integration of the original equations (3)–(4) with standard adaptive step-size strategy. For that purpose, the embedded pair of explicit Runge-Kutta schemes of order 9(8) constructed by Verner [16] is chosen, implemented (and recommended for high precision computations) in the julia package DifferentialEquations.jl [11]. We first integrate the problem (without time-regularization) with adaptive implementation of Verner’s 9(8) embedded RK pair with relative and absolute tolerance of 10^{-14} , which results in the application of, say, n successful steps of the 9th order RK scheme. We then apply, for each time-regularization function $s_i(q, v)$, $i = 1, 2, 3, 4$, the same 9th order RK scheme to the equations (16) (together with (31)) with constant time-step $\Delta\tau$ chosen so that also n steps are required to integrate from $t = 0$ to $t = T$. In all examples, we use high precision floating point arithmetic (Julia language’s Bigfloat numbers with default precision, that is, floating point numbers with 256-bit mantissa from GNU MPFR Library) in order to minimize the influence of roundoff errors in the numerical results.

4.1 The Pythagorean 3-body problem

This is a planar 3-body problem first investigated numerically by Burrau (1913). It consists in three masses in the ratio 3:4:5 placed at rest at the vertices of a 3:4:5 right triangle. Victor Szebehely and C. Frederick Peters (1967) found, by numerically integrating the problem (using Levi-Civita regularization near close encounters), that after several binary close encounters a stable binary is formed between the two heaviest bodies while the third one is ejected from the system.

As in [13], we consider the problem in adimensional units, with $Gm_i = 5, 4, 3$, initial positions $q_i^0 = (1, -1), (-2, -1), (1, 3)$, and zero initial velocities. We compute the solution in the time interval $t \in [0, 63]$. The trajectories of the three bodies are displayed in Figure 1.

In Figure 1, the radius of convergence $\rho(q^k, v^k)$ is displayed in logarithmic scale together with the lower bound $1/L(q^k, v^k, \lambda_0)$ given by Theorem 2. The minimum and maximum of $\rho(q^k, v^k)L(q^k, v^k, \lambda_0)$ are 3.41111 and 7.9689 respectively.

The widths of the strips around $[0, T]$ for the scaled time-regularization functions $s_i(q, v)$, $i = 1, 2, 3, 4$, together with the width of the strip corresponding to $s(q, v) = 1$. are displayed in Table 1. The corresponding maximum errors in energy are also displayed in the same table.

The position errors $\|q_i(t^k) - q_i^k\|$ ($i = 1, 2$) for the different numerical integrations are plotted in Figure 2. A very similar plot (not shown) is obtained for $\|q_3(t^k) - q_3^k\|$. A clear superiority of the constant step-size integrations of the time-regularized equations over the adaptive numerical integration of the equations in physical time can be observed for all the considered choices of the time-regularization functions.

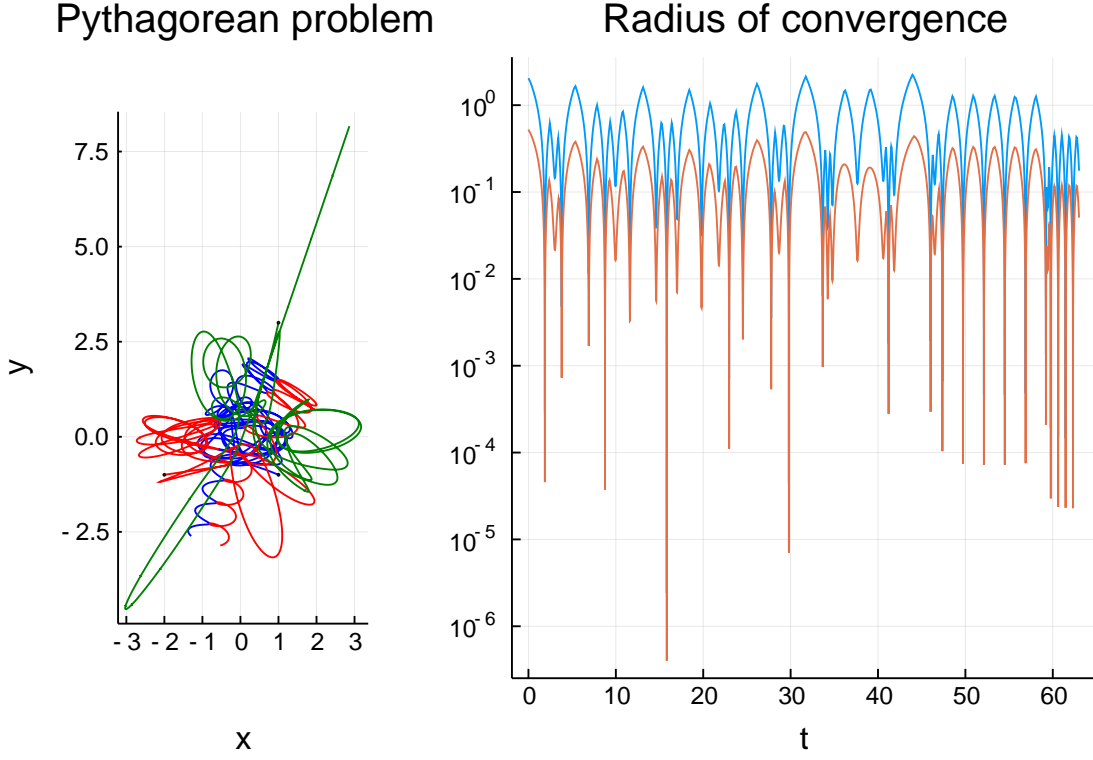


Figure 1: Trajectories of the three bodies in the Pythagorean 3-body problem (left). Values of radius of convergence $\rho(q^k, v^k)$ together with its lower bound $1/L(q^k, v^k, \lambda_0)$ for the Pythagorean 3-body problem (right)

Table 1: Pythagorean problem

	$s(q, v) = 1$	$s_1(q, v)$	$s_2(q, v)$	$s_3(q, v)$	$s_4(q)$
Width of strip	4.8×10^{-6}	0.2532	0.2716	0.2359	0.2891
Energy error	6×10^{-12}	9×10^{-15}	1.3×10^{-14}	6×10^{-15}	2.9×10^{-14}

4.2 A binary-visitor problem

A binary with a mass ratio of 2 : 1, eccentricity $e = 0.9$, and semi-major axis $a = 10$ is visited by a smaller third body with a 1 : 100 mass ratio (with respect to the most massive body of the binary), which crosses their orbital plane through their center of masses perpendicularly at velocity $\|v_3\| = 100$ when the components of the binary are in their closest position. The third body has enough energy to avoid being captured by the binary.

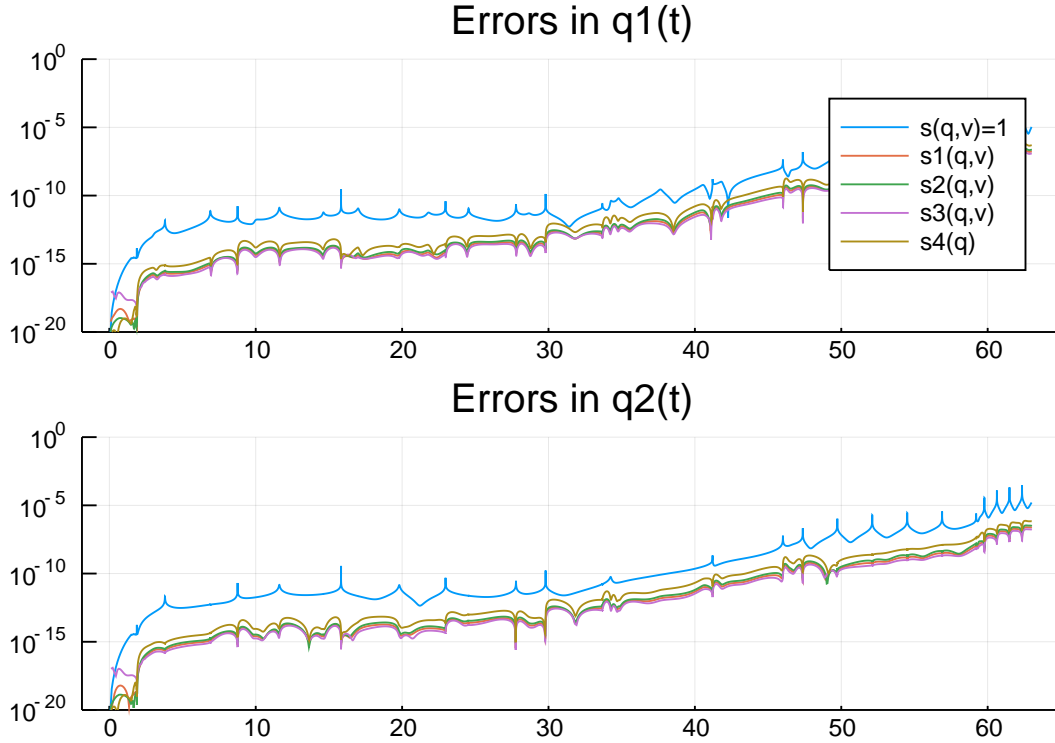


Figure 2: Evolution of position errors for the Pythagorean 3-body problem for different numerical integrations (see main text)

In Figure 3, the radius of convergence $\rho(q^k, v^k)$ is displayed in logarithmic scale together with the lower bound $1/L(q^k, v^k, \lambda_0)$ given by Theorem 2. The minimum and maximum of $\rho(q^k, v^k)L(q^k, v^k, \lambda_0)$ are 4.56306 and 6.223617 respectively.

Table 2: A binary-visitor problem

	$s(q, v) = 1$	$s_1(q, v)$	$s_2(q, v)$	$s_3(q, v)$	$s_4(q)$
Width of strip	0.008	10.354	10.354	10.354	0.4941
Energy error	5×10^{-15}	6×10^{-19}	6×10^{-19}	6×10^{-19}	4×10^{-6}

The widths of the strips around $[0, T]$ for the scaled time-regularization functions $s_i(q, v)$, $i = 1, 2, 3, 4$ and $s(q, v) = 1$, together with the corresponding maximum errors in energy are displayed in Table 2. In this example, one observes that the time-reparametrization function $s_4(q)$ does not perform well. That ill-behavior is exacerbated if the velocity when the binary is crossed by the visitor at a higher velocity: the

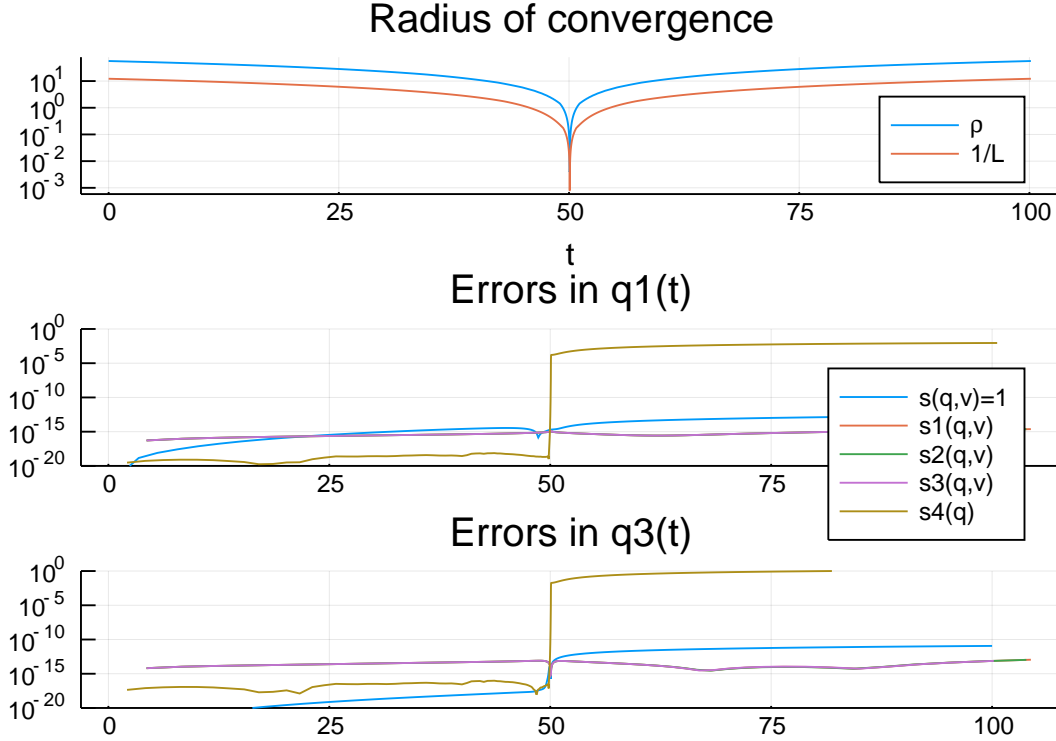


Figure 3: Top: Values of radius of convergence $\rho(q^k, v^k)$ and its lower bound $1/L(q^k, v^k, \lambda_0)$ for the binary-visitor 3-body problem. Middle (resp. bottom): Evolution of position errors of m_1 (resp. m_3) for the binary-visitor 3-body problem for different numerical integrations

scaled width of the corresponding strip is approximately $49.8/\|v_3\|$ as $\|v_3\| \uparrow \infty$. (For the equations in physical time, it is approximately $0.86/\|v_3\|$ as $\|v_3\| \uparrow \infty$.)

The position errors $\|q_i(t^k) - q_i^k\|$ ($i = 1, 3$) for the different numerical integrations are plotted in Figure 3. A clear superiority of the constant step-size integrations of the time-regularized equations over the adaptive numerical integration of the equations in physical time can be observed for $s_i(q, v)$, $i = 1, 2, 3$ (their error curves are virtually identical). The very poor performance of $s_4(q)$ is in accordance with the narrowness of the strip noted in previous item.

4.3 Solar system

We consider a 9-body model of the Solar System, with the Sun, Mercury, Venus, the Earth (the Earth-Moon barycenter), Mars, Jupiter, Saturn, Neptune, and Uranus. The initial values (and the product of the gravitational constant with their corresponding masses) are taken from DE430, Julian day (TDB) 2440400.5 (June 28, 1969). We consider a time interval of 2000 days, that is, $t \in [0, T]$ with $T = 2000$.

In Figure 4, the radius of convergence $\rho(q^k, v^k)$ is displayed together with the lower

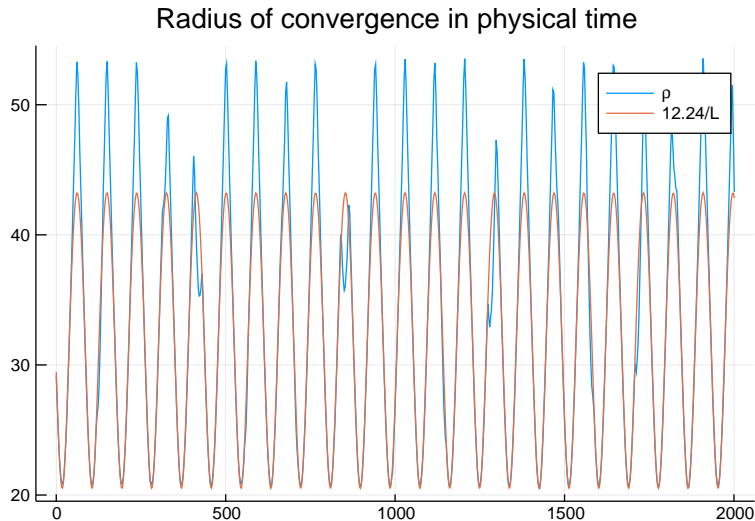


Figure 4: Values of radius of convergence $\rho(q^k, v^k)$ together with its lower bound $1/L(q^k, v^k, \lambda_0)$ (scaled so that their minimum values coincide) for the Solar-System 9-body problem

bound $1/L(q^k, v^k, \lambda_0)$ given by Theorem 2 scaled by a factor of 12.24. The minimum and maximum of $\rho(q^k, v^k)L(q^k, v^k, \lambda_0)$ are 10.1610416 and 15.1623627 respectively. One can observe that the radius of convergence is dominated by Mercury's influence, as expected.

The strip around $[0, T]$ for the original equations has a width of 41.02. In this example, the scaled time-regularization functions $s_i(q, v)$, $i = 1, 2, 3$, are not able to get wider strips (their corresponding widths are 40.07114, 38.007644, and 28.8 respectively). The widest strip is obtained in the case of $s_4(q)$: 47.4334.

We have included the integration with constant step-size of the original equations, in order to stress the fact that neither adaptive step-size strategy nor time-regularization could be safely avoided in that example. The constant step-size integration of the original equations gives the largest maximum relative error in energy (7.8976×10^{-16}), followed by the adaptive integration of the original equations (2.8864×10^{-16}). The maximum energy errors of the integration with constant step-size with time-regularization functions $s_i(q, v)$, $i = 1, 2, 3, 4$ are in all cases smaller. The smallest maximum error in energy is obtained with $s_3(q, v)$ (7.12216×10^{-18}). As for the position errors $\|q_i(t^k) - q_i^k\|$ ($i = 1, 2, \dots, 9$), the time-regularization function $s_3(q, v)$ gives the most precise results among the four time-regularization functions $s_i(q, v)$, $i = 1, 2, 3, 4$. In Figure 5, the position errors for each of the 9 bodies are displayed for the adaptive numerical integration in physical time (top), the constant step-size integration in physical time (middle), and for the constant step-size implementation with time-regularization function $s_3(q, v)$. One can observe that the position error of Mercury is smaller in the later case, but this is achieved at the expense of having larger position errors for the rest of the bodies.

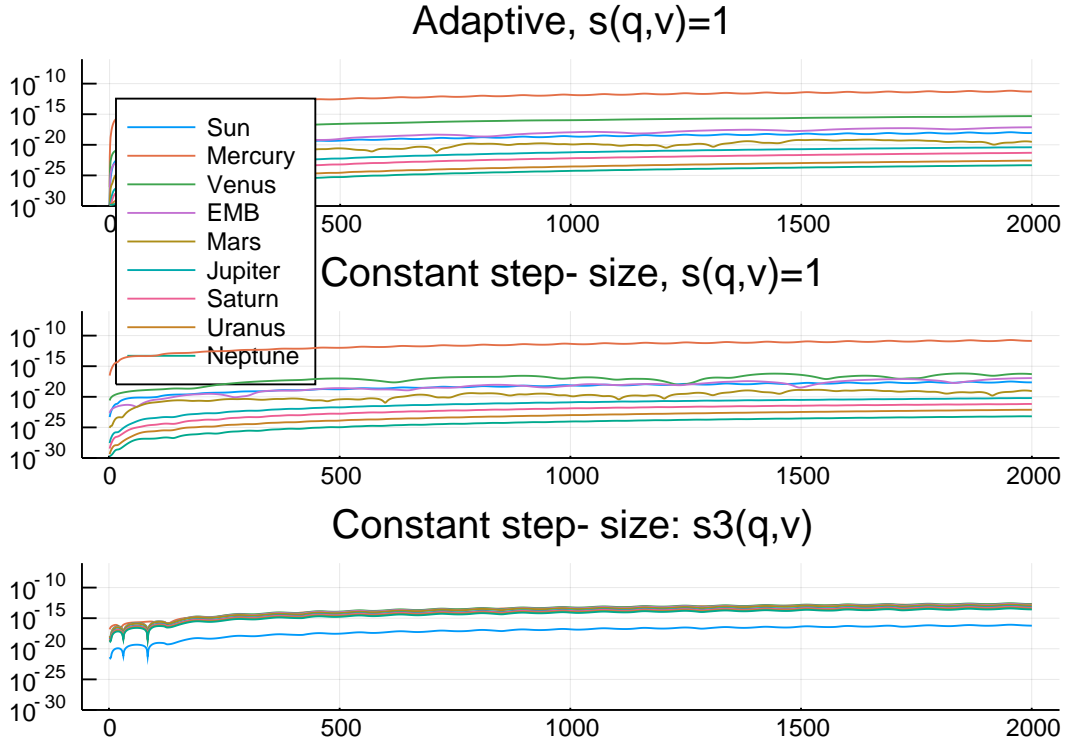


Figure 5: Evolution of position errors for the Solar System 9-body problem for adaptive numerical integration in physical time (top), constant step-size in physical time (middle), and constant step-size for the time-regularized equations with $s_3(q, v)$ (bottom)

4.4 Summary of numerical experiments

In the three considered examples, the estimate of the radius of convergence obtained in Theorem 2 gives useful information on how the radius of convergence evolve along the solution.

In the Pythagorean problem, a classical three-body problem with extreme close encounters, all the considered time-regularization functions ($s_i(q, v)$, $i = 1, 2, 3$, and $s_4(q)$) show similar behavior: Constant step-size numerical integration of the time-regularized equations is clearly more efficient than the adaptive integration of the original equations. There seem to be a correlation between the width of the strip around the integration interval in τ for each scaled time-regularization function, with the accuracy of the numerical integration with constant step-size.

The second example has been cooked up to illustrate the fact that $s_4(q)$ is not a global time-regularization function. That is, there exists no $\beta > 0$ such that all the solutions of (16) with $s(q, v) := s_4(q)$ can be extended analytically in the strip $\{\tau \in \mathbb{C} : |\text{Im}(\tau)| \leq \beta\}$. Whence, the function $s(q, v) := s_4(q)$ may be appropriate for the numerical integration with constant step-size of many solution trajectories of (16) but not for all of them.

In the third example, the 9-body model of the Solar System, there is actually no need for time-regularization, as it can be integrated numerically with constant step-size. In fact, similar position errors have been obtained by integrating the original formulation of the considered initial value problem in constant step-size mode and with the adaptive implementation. The only close approaches correspond to the perihelion of Mercury, and are very mild compared to those occurring in previous two examples. This example has been chosen to illustrate the performance of time-regularization in unfavorable conditions: The estimates of the radius of convergence given by Theorem 2 are able to detect the perihelion passages of Mercury. Furthermore, the errors in energy of the numerical integrations of the time-regularized equations are smaller than in the case of the integration with the physical time. The position errors of Mercury are also smaller for the time-regularized formulations, but larger for the rest of the bodies. This is due to the hierarchical nature of the Solar System, and in particular of our 9-body model: the comparatively high-frequency oscillations due to Mercury's orbit have in practice limited effect on the smoothness of the positions of the rest of the bodies in the original formulation. However, the high-frequency oscillations of Mercury's orbit are inherited by the time-regularization function $s(q, v)$. This implies that the positions of all the bodies as functions of the fictitious time τ become as oscillatory as Mercury's position.

5 Concluding remarks

In this work, we have introduced new time-regularizations defined as real analytic functions $s(q, v)$ depending on both velocities and positions. Our main contribution has been to exhibit *global* time-regularizations such that any solution of the reformulated N -body system is defined (as a function of the complex variable τ) on a strip of width $2\beta = 0.0888886$ along the complete real line \mathbb{R} . Noteworthy, our results improve over previous ones by being insensitive to the possible occurrence of vanishing mass-ratios in the system. Such global time-regularizations allow for the use of constant time-steps (in the new fictitious time) without degrading the accuracy of the computed trajectories with close approaches.

We have performed some preliminary numerical experiments on three different examples: two three-body problems with close encounters, and a 9-body problem corresponding to the Solar System. The performance of time-regularization compared to a highly efficient adaptive general purpose integrator has been checked for short integration intervals. We plan to address more exhaustive numerical tests, and in particular, to check the long term behavior of the proposed time-regularization functions for different symplectic numerical integrators.

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