LARGE GRAPHS WITH GIVEN DEGREE AND DIAMETER III

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The following problem arises in the study of inter-connection networks: find graphs of given maximum degree and diameter having the maximum number of vertices. In this article we give a construction which enables us to construct graphs of given maximum degree and diameter, having a great number of vertices from small ones; in particular we obtain a class of graphs of diameter 3, maximum degree Δ and having about $8\Delta^3/27$ vertices.

I. INTRODUCTION

We are interested in the (Δ, D) graph problem, one of the problems arising in telecommunications networks (or microprocessor networks). Let $G = (X, E)$ be an undirected graph with vertex set X and edge set E . The distance between two vertices x and y , denoted by $\delta(x, y)$, is the length of a shortest path between x and y . The diameter D of the graph G is defined as $D = \max_{(x, y) \in X^2} \delta(x, y)$. The degree $d(x)$ of a vertex x is the number of vertices adjacent to x and Δ denotes the maximum degree of G . (For a survey on diameters see [1].)

The (Δ, D) graph problem is that of finding the maximum number of vertices $n(\Delta, D)$ of a graph with given maximum degree Δ and diameter D . This problem arises quite naturally in the study of interconnection networks: the vertices represent the stations (or processors), the degree of a vertex is the number of links incident at this vertex and the diameter represents the maximum number of links to be used to transmit a message. The problem seems to have been first set in the literature by Elspas [7]. Different contributions have been made in the 70's and the known results were summarized in Storwick's article [10]. These results have been recently improved by Memmi and Raillard [8] and Quisquatter [9]. A simple bound on $n(\Delta, D)$ is given by Moore (see [3 or 4]):

$n(2,D) \leq 2D+1$ and for $\Delta > 2$, $n(\Delta,D) \leq \frac{\Delta(\Delta-1)^{D-2}}{\Delta-2}$. The graphs satisfying the equality are called *Moore graphs*. It has been proved by different authors (see Biggs [3 chap.23]) that Moore graphs can exist only if $\Delta = 2$ (the graphs being the $(2D+1)$ cycles) or if $D = 2$ and $\Delta = 3, 7, 57$ (for $\Delta = 3$ and $\Delta = 7$ there exists a unique Moore graph respectively Petersen's graph on 10 vertices, and Hoffman and Singleton's graph on 50 vertices, for $\Delta = 57$ the answer is not known).

The aim of this article is to give a new construction, which gives graphs having a great number of vertices in particular for small diameters; as a corollary we obtain that $\liminf_{\Delta \rightarrow \infty} n(\Delta, 3) \geq \frac{8}{27} \Delta^3$.

This paper is to be considered as a companion of [2,5,6] the construction is based on a new product of graphs which is studied in [2] and in [5,6] are given infinite families of graphs of given degree and diameter. In [5] it was proved that $n(\Delta, D) \geq (\frac{\Delta}{2})^D$, which gives only $\frac{\Delta^3}{8}$ in the case of diameter 3.

II DEFINITIONS

2.1 The * product

Let $G = (X, E)$ and $G' = (X', E')$ be two graphs. Take an arbitrary orientation of the edges of G and let U be the set of arcs. Finally, for each arc (x, y) of U , let $f_{(x,y)}$ be a one to one mapping from X' to X' .

We define the product $G * G'$ as follows:

The vertex set of $G * G'$ is the cartesian product $X \times X'$. A vertex (x, x') is joined to a vertex (y, y') in $G * G'$ if and only if

$$\begin{aligned} & \text{either } x = y \text{ and } \{x', y'\} \in E', \\ & \text{or } (x, y) \in U \text{ and } y' = f_{(x,y)}(x'). \end{aligned}$$

Remark: $G * G'$ can be viewed as formed by $|X|$ copies of G' where two copies generated by the vertices x and y are joined if (x, y) is an arc and in that case they are joined by a perfect matching depending on (x, y) .

Examples: (a) Let $G = K_2$, $(1, 2)$ being the oriented arc and let $G' = C_5$. In Fig.2.1.a, we have represented $K_2 * C_5$ where $f_{(1,2)}(x') = x'$ and in Fig.2.1.b, $K_2 * C_5$ where $f_{(1,2)}(x') = 2x' \pmod{5}$.

Fig. 2.1 : $K_2 * C_5$ with two different $f_{1,2}$

Note that the diameter of the graph of fig.2.1.a is 3, but the diameter of Petersen's graph (fig.2.1.b) is 2.

(b) If we choose for any arc (x,y) $f_{x,y}(x') = x'$ then $G * G'$ is nothing else than the cartesian sum (called also cartesian product) of G and G' and in this case the diameter of $G * G'$ is the sum of the diameters of G and G' .

Degrees and diameter of $G * G'$

If G is of maximum degree Δ and G' of maximum degree Δ' , then $G * G'$ has as maximum degree $\Delta + \Delta'$. If G is of diameter D and G' of diameter D' , then the diameter of $G * G'$ is less than or equal to $D + D'$ and a clever choice of the functions $f_{(x,y)}$ can give a small diameter.

2.2 The property P^*

A graph $G = (X,E)$ is said to have the property P^* if G has diameter at most 2 and if there exists an involution f of X (i.e. f^2 is the identity of X), such that for every vertex x of G we have $X = \{x\} \cup \{f(x)\} \cup \{f(\Gamma(x))\} \cup \{\Gamma(f(x))\}$.

Examples: In figure 2.2 are shown three graphs satisfying property P^* with respectively 5, 8 and 9 vertices. Other examples will be given in section 4.

Figure 2.2

2.3 The property P

A graph G will be said to satisfy property P , if any pair of vertices at distance D (where D is the diameter of G) can be joined by a path of length $D+1$.

A trivial example is given by the complete graph $K_n, n \geq 3$; other examples will be given in section 4.

III THE MAIN THEOREM

Theorem: Let G be a graph of diameter $D \geq 1$, satisfying property P and let G' be a graph satisfying property P^* (with involution f). Then $G * G'$, with $f_{(x,y)}(x') = f(x')$ for every arc (x,y) of an arbitrary orientation of G , is a graph of diameter at most $D + 1$.

Proof: Any two vertices (x,x') and (y,y') of $G * G'$ can be connected with a path of length at most $d_G(x,y) + 2$ (as G' is of diameter at most 2 by property P^*) and therefore if $d_G(x,y) < D$, then (x,x') and (y,y') are at distance at most $D + 1$. Now let x and y be at distance D and let z be the vertex preceding y on a path of length D between x and y . Let $z' = f^{D-1}(x')$. Then the vertex (z,z') is at distance $D-1$ of (x,x') ; therefore the vertices in $(z, \Gamma(z'))$ and the vertex $(y, f(z'))$ are at distance at most D from (x,x') and the vertices in $(y, f(\Gamma(z')))$ and $(y, \Gamma(f(z')))$ at distance at most $D+1$ of (x,x') . Finally by using the path of length $D+1$ between x and y (by property P) the vertex $(y, f^{D+1}(x')) = (y, z')$ is at distance at most $D+1$ from (x,x') in $G * G'$. According to the fact that by property P^* the vertex set of G' is equal to $\{z'\} \cup \{f(z')\} \cup \{f(\Gamma(z'))\} \cup \{\Gamma(f(z'))\}$ every vertex of (y, G') is at distance at most $D+1$ from (x,x') .

IV GRAPHS SATISFYING P^* AND P

4.1 Graphs satisfying P^*

We have seen three examples in 2.2 of degrees 2,3,4 and respectively 5,8,9 vertices. Note that a graph of degree Δ satisfying P^* has at most $2\Delta+2$ vertices. The next construction shows the existence of graphs of degree Δ on 2Δ vertices.

Proposition: For every integer Δ , there exists a graph G_Δ satisfying property P^* , of degree Δ and having 2Δ vertices.

Construction: Fig. 4.1.a shows such graphs with $\Delta = 1$ and $\Delta = 2$. Suppose we have constructed a graph G_Δ , satisfying P^* , of

degree Δ and such that the 2Δ vertices are A and $f(A)$ with $|A| = |f(A)| = \Delta$ and such that $\Gamma(A) \supseteq f(A)$ and $\Gamma(f(A)) \supseteq A$ (that is satisfied by G_1 and G_2). Let $G_{\Delta+2}$ be the graph obtained from G_Δ by adding 4 vertices $x, y, f(x), f(y)$ joined between themselves and to G as shown in Fig. 4.1.b. Then this graph (where the involution f extends that of G_Δ) has clearly degree $\Delta+2$, $2(\Delta+2)$ vertices and it is not difficult to show that it satisfies property P^* and the hypothesis of induction.

Fig. 4.1.a

Fig. 4.1.b

4.2 Graphs satisfying P

As already noticed the complete graphs K_n , $n \geq 3$, satisfy P . One can show also that Petersen's graph, Hoffmann-Singleton's graph, the products $K_n * X$, $n \geq 3$, where X is one of the graphs of Fig. 2.2, satisfy property P .

An important class of graphs (which will be used in section 5) is formed by the graphs P_q of diameter 2 associated to projective planes. Let us recall, that if $q = p^r$ where p is a prime number, there exists a projective plane of order q with a polarity, that is a numbering of the points M_i , $1 \leq i \leq q^2 + q + 1$ and of the lines, D_j , $1 \leq j \leq q^2 + q + 1$ such that if $M_i \in D_j$ then $M_j \in D_i$. Thus $\bigcup_{j \in D_i} D_j$ contains all the points of the plane.

Let P_q be the graph whose vertices are the points of the projective plane, the vertices M_i and M_j are joined if and only if $M_j \in D_i$. The properties of the numbering shows that P_q is of degree $q+1$; it has q^2+q+1 vertices and its diameter is 2. Now, let M_i and M_j be two vertices at distance 2, then $M_j \notin D_i$. As $|D_i \cap D_j| = 1$, there exists a vertex M_k , such that $M_k \in D_i$ and $M_k \in D_j$; as M_k is at distance 1 from M_i and 2 from M_j , it is clear that M_i and M_j are joined by a path of length 3. Similarly one can prove that the quotient of Levi graphs of

generalized quadrangles and hexagons, having a polarity, by that polarity have property P . (See [5].)

V APPLICATIONS

Let $G = K_n$ and $G' = X_8$ the graph on 8 vertices of degree 3 (see Fig. 2.2) then $K_n * X_8$ has $8n$ vertices, degree $n+2$ and diameter 2 . In particular with $n = 3$ (resp. 4) we obtain graphs of diameter 2 , degree 5 (resp. 6) on 24 (resp. 32) vertices. The better graph of diameter 2 degree 6, known before had only 31 vertices (namely P_5) . By taking $K_3 * X_8$ as graph G and again X_8 as G' we obtain a graph of diameter 3, degree 8, on 192 vertices (the better graph known before has 114 vertices). By taking for G the graph on 585 vertices of degree 9 and diameter 3 and for G' the graphs on 5(8,9) vertices of Fig. 2.1 we give examples of graphs of diameter 4 of degree 11 (resp. 12, 13) on 2925 (resp. 4680, 5265) vertices which are the best known now. Another application is the following result.

Theorem: $\liminf_{\Delta \rightarrow \infty} n(\Delta, 3) \geq \frac{8}{27} \Delta^3$

Proof: Let G be the graph P_q defined in section 4, with q odd, q a prime power, of diameter 2, degree $q+1$ and on q^2+q+1 vertices which satisfies property P . Let $\Delta = \frac{3(q+1)}{2}$ and let $G' = G_{\Delta/3}$ be the graph of degree $\frac{\Delta}{3}$ on $\frac{2\Delta}{3}$ vertices satisfying property P^* constructed in 4.1. By the main theorem $P_q * G_{\Delta/3}$ is a graph of diameter 3 degree $q+1+\frac{\Delta}{3} = \Delta$, having $(q^2+q+1) \frac{2\Delta}{3} = (4\frac{\Delta^2}{9} - \frac{2\Delta}{3} + 1) \frac{2\Delta}{3} = \frac{8\Delta^3}{27} - 4\frac{\Delta^2}{9} + 2\frac{\Delta}{3}$ vertices proving therefore the theorem.

VI GENERALIZATIONS

6.1 The property P_i

A graph G is said to have property P_i , if there exists a permutation f of its vertices, such that f^2 is an automorphism of G and that the graph G^* , which has the same vertices as G and whose edges are the edges of G or their images by f , has diameter i .

6.2 Examples

The graph G of Fig. 6, on 4 vertices, where $f(x) = x+1$ (modulo 4)

has property P_1 , G^* being the complete graph K_4 .

Fig. 6

The graph G , on 29 vertices, where the vertex x and y are joined if $y - x \equiv 1, -1, 7, -7 \pmod{29}$ and where $f(x) \equiv 12x \pmod{29}$ has property P_2 .

If $4e+1$ is a prime power, consider the graph whose $4e+1$ vertices are the elements of the field GF_{4e+1} ; two vertices x and y are adjacent if $x-y$ is a square in GF_{4e+1} . This graph has property P_1 (diameter 2 and degree $2e$). (These graphs are known as Paley's graphs and are examples of strongly regular graphs.)

6.3 Theorem

Let G be a graph of diameter $D \geq i+1$ and let G' be a graph satisfying property P_i . Then $G \circ G'$, with $f_{(x,y)}(x') = f(x')$ for every arc (x,y) of an arbitrary orientation of G , has diameter $D+i$.

Proof: Let x and y be two vertices of G joined by a path P (not necessarily elementary) of length j . If this path contains a arcs of G and therefore $j - a$ arcs of G' , let $\varepsilon(P) = 2a - j$. We will prove that if there exists a path of length j' from $f^{\varepsilon(P)}(x')$ to y' in G'^* , with $j' \leq j$, then there exists a path of length $j+j'$ in $G \circ G'$ between (x, x') and (y, y') . The proof goes by induction on j (that is true for $j = 0$) and for j fixed by induction on j' ($j' \leq j$); that is true for $j' = 0$ ($y, f^{\varepsilon(P)}(x')$) being joined to (x, x') by the obvious path of length j . Suppose $j' > 0$ and let (z', y') be the last edge of the path between $f^{\varepsilon(P)}(x')$ and y' in G'^* . If $(z', y') \in G'$, we obtain a path of length $(j+j')$ from (x, x') to (y, y') by adding the arc $((y, z'), (y, y'))$ to the path of length $j+j'-1$ from (x, x') to (y, z') (which exists by induction hypothesis). If $(z', y') \notin G'$ let z be the vertex preceding y on the path P of G ; then $f^{\varepsilon(y,z)}(z')$ and $f^{\varepsilon(y,z)}(y')$ are joined by an edge in G' (recall that $\varepsilon(y,z)$ equals 1 or -1 according (y,z) belongs or not to G).

By induction hypothesis with $j-1$ and $j'-1$ ($j'-1 \leq j-1$) there exists in $G \ast G'$ a path of length $j+j'-2$ from (x, x') to $(z, f^\varepsilon(y, z)(z'))$ and then by adding the path of length 2 of $G \ast G'$ $(z, f^\varepsilon(y, z)(z')) (z, f^\varepsilon(y, z)(y')) (y, y')$ we obtain a path of length $j+j'$ from (x, x') to (y, y') .

Now let (x, x') and (y, y') be any two vertices of $G \ast G'$. There is a (not necessarily elementary) path P from x to y of length $j = D$ or $D-1$. Since $D \geq i+1$, $j \geq i$ holds. As G'^* is of diameter i , there exists a path of length $j' \leq i$ between $f^\varepsilon(P)(x')$ and y' in G'^* .

We have $j' \leq j$ (as $j' \leq i$ and $i \leq j$) and by the property proved above there exists a path of length $j+j' \leq D+i$ from (x, x') to (y, y') .

6.4 Applications

By using the example of 6.2, we can build new good graphs. For example with the Paley's graphs we obtain infinitely many graphs of diameter 3 which have actually the highest known number of vertices, although the asymptotic value obtained in theorem of section 5 is not improved. By taking for G the graph of diameter 3, degree 9 on 585 vertices and for G' the Paley's graph on 13 vertices we obtain a graph of diameter 4, degree 15 on 7605 vertices. (This is in fact the best known value).

Remark: The condition on G , $D \geq i+1$ can be deleted if one adds extra conditions on G' ; for example if G is of diameter 1 and G' satisfies P_1 and is of diameter 2, then $G \ast G'$ has diameter 2.

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