Large Graphs with Given Degree and Diameter III

Jean-Claude Bermond, Charles Delorme, Guy Farhi

LARGE GRAPHS WITH GIVEN DEGREE AND DIAMETER III

J.C. BERMOND, C. DELORME and G. FARHI

Université de Paris Sud, Orsay, bât. 490
France

The following problem arises in the study of interconnection networks: find graphs of given maximum degree and diameter having the maximum number of vertices. In this article we give a construction which enables us to construct graphs of given maximum degree and diameter, having a great number of vertices from small ones; in particular we obtain a class of graphs of diameter 3, maximum degree Δ and having about $8\Delta^3/27$ vertices.

I. INTRODUCTION

We are interested in the (Δ,D) graph problem, one of the problems arising in telecommunications networks (or microprocessor networks). Let $G = (X,E)$ be an undirected graph with vertex set X and edge set E. The *distance* between two vertices x and y, denoted by $\delta(x,y)$, is the length of a shortest path between x and y. The *diameter* D of the graph G is defined as $D = \max_{(x,y)\in X^2} \delta(x,y)$. The degree $d(x)$ of a vertex x is the number of vertices adjacent to x and Δ denotes the maximum degree of G. (For a survey on diameters see [1].)

The (Δ,D) graph problem is that of finding the maximum number of vertices $n(\Delta,D)$ of a graph with given maximum degree Δ and diameter D. This problem arises quite naturally in the study of interconnection networks: the vertices represent the stations (or processors), the degree of a vertex is the number of links incident at this vertex and the diameter represents the maximum number of links to be used to transmit a message. The problem seems to have been first set in the literature by Elspas [7]. Different contributions have been made in the 70's and the known results were summarized in Storwick's article [10]. These results have been recently improved by Memmi and Raillard [8] and Quisquatter [9].

A simple bound on $n(\Delta,D)$ is given by Moore (see [3 or 4]):

$n(2,D) \le 2D+1$ and for $\Delta > 2$, $n(\Delta,D) \le \frac{\Delta(\Delta-1)^D-2}{\Delta-2}$. The graphs satisfying the equality are called *Moore graphs*. It has been proved by different authors (see Biggs [3 chap.23]) that Moore graphs can exist only if $\Delta = 2$ (the graphs being the (2D+1) cycles) or if $D = 2$ and $\Delta = 3,7,57$ (for $\Delta = 3$ and $\Delta = 7$ there exists a unique Moore graph respectively Petersen's graph on 10 vertices, and Hoffman and Singleton's graph on 50 vertices, for $\Delta = 57$ the answer is not known).

The aim of this article is to give a new construction, which gives graphs having a great number of vertices in particular for small diameters; as a corollary we obtain that $\liminf_{\Delta \to \infty} n(\Delta,3) \ge \frac{8}{27}\Delta^3$.

This paper is to be considered as a companion of [2,5,6] the construction is based on a new product of graphs which is studied in [2] and in [5,6] are given infinite families of graphs of given degree and diameter. In [5] it was proved that $n(\Delta,D) \ge (\frac{\Delta}{2})^D$, which gives only $\frac{\Delta^3}{8}$ in the case of diameter 3.

II DEFINITIONS

2.1 The * product

Let $G = (X,E)$ and $G' = (X',E')$ be two graphs. Take an arbitrary orientation of the edges of G and let U be the set of arcs. Finally, for each arc (x,y) of U, let $f_{(x,y)}$ be a one to one mapping from X' to X'.

We define the product $G*G'$ as follows:

The vertex set of $G*G'$ is the cartesian product $X \times X'$. A vertex (x,x') is joined to a vertex (y,y') in $G*G'$ if and only if

either $x = y$ and $\{x',y'\} \in E'$,

or $(x,y) \in U$ and $y' = f_{(x,y)}(x')$.

Remark: $G*G'$ can be viewed as formed by $|X|$ copies of G' where two copies generated by the vertices x and y are joined if (x,y) is an arc and in that case they are joined by a perfect matching depending on (x,y).

Examples: (a) Let $G = K_2$, $(1,2)$ being the oriented arc and let $G' = C_5$. In Fig.2.1.a, we have represented K_2*C_5 where $f_{(1,2)}(x') = x'$ and in Fig.2.1.b, K_2*C_5 where $f_{(1,2)}(x') = 2x'$ (mod.5).

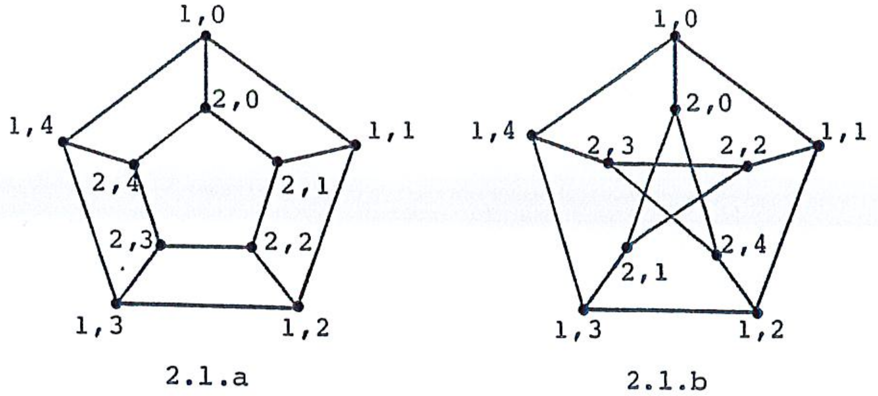

Fig. 2.1 : K_2*C_5 with two different $f_{1,2}$

Note that the diameter of the graph of fig.2.1.a is 3, but the diameter of Petersen's graph (fig.2.1.b) is 2.

(b) If we choose for any arc (x,y) $f_{x,y}(x') = x'$ then G*G' is nothing else than the cartesian sum (called also cartesian product) of G and G' and in this case the diameter of G+G' is the sum of the diameters of G and G'.

Degrees and diameter of G*G'

If G is of maximum degree Δ and G' of maximum degree Δ', then G*G' has as maximum degree $\Delta + \Delta'$. If G is of diameter D and G' of diameter D', then the diameter of G*G' is less than or equal to D + D' and a clever choice of the functions $f_{(x,y)}$ can give a small diameter.

2.2 The property P*

A graph G = (X,E) is said to have the property P* if G has diameter at most 2 and if there exists an involution f of X (i.e. f^2 is the identity of X), such that for every vertex x of G we have $X = \{x\} \cup \{f(x)\} \cup \{f(\Gamma(x))\} \cup \{\Gamma(f(x))\}$.

Examples: In figure 2.2 are shown three graphs satisfying property P* with respectively 5,8 and 9 vertices. Other examples will be given in section 4.

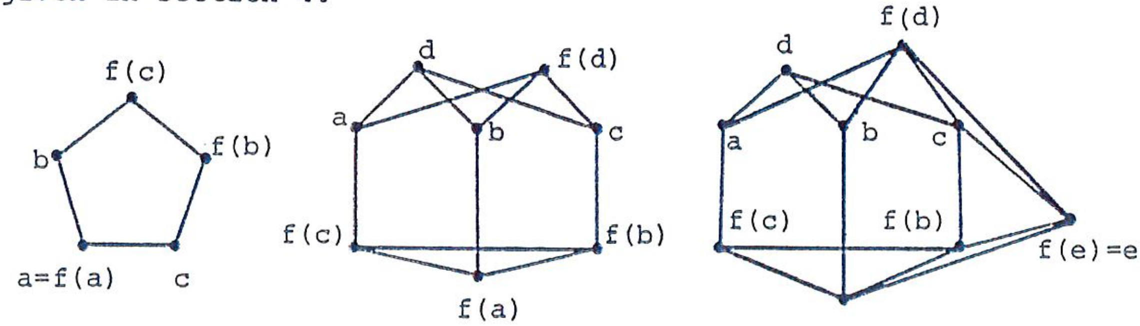

Figure 2.2

2.3 The property P

A graph G will be said to satisfy property P, if any pair of vertices at distance D (where D is the diameter of G) can be joined by a path of length $D+1$.
A trivial example is given by the complete graph $K_n, n \geq 3$; other examples will be given in section 4.

III THE MAIN THEOREM

Theorem: Let G be a graph of diameter $D \geq 1$, satisfying property P and let G' be a graph satisfying property P^* (with involution f). Then $G*G'$, with $f_{(x,y)}(x') = f(x')$ for every arc (x,y) of an arbitrary orientation of G, is a graph of diameter at most $D+1$.

Proof: Any two vertices (x,x') and (y,y') of $G*G'$ can be connected with a path of length at most $d_G(x,y) + 2$ (as G' is of diameter at most 2 by property P^*) and therefore if $d_G(x,y) < D$, then (x,x') and (y,y') are at distance at most $D+1$.
Now let x and y be at distance D and let z be the vertex preceding y on a path of length D between x and y. Let $z' = f^{D-1}(x')$. Then the vertex (z,z') is at distance $D-1$ of (x,x'); therefore the vertices in $(z,\Gamma(z'))$ and the vertex $(y,f(z'))$ are at distance at most D from (x,x') and the vertices in $(y,f(\Gamma(z')))$ and $(y,\Gamma(f(z')))$ at distance at most $D+1$ of (x,x'). Finally by using the path of length $D+1$ between x and y (by property P) the vertex $(y,f^{D+1}(x')) = (y,z')$ is at distance at most $D+1$ from (x,x') in $G*G'$. According to the fact that by property P^* the vertex set of G' is equal to $\{z'\} \cup \{f(z')\} \cup \{f(\Gamma(z'))\} \cup \{\Gamma(f(z'))\}$ every vertex of (y,G') is at distance at most $D+1$ from (x,x').

IV GRAPHS SATISFYING P* AND P

4.1 Graphs satisfying P*

We have seen three examples in 2.2 of degrees 2,3,4 and respectively 5,8,9 vertices. Note that a graph of degree Δ satisfying P^* has at most $2\Delta+2$ vertices. The next construction shows the existence of graphs of degree Δ on 2Δ vertices.
Proposition: For every integer Δ, there exists a graph G_Δ satisfying property P^*, of degree Δ and having 2Δ vertices.

Construction: Fig. 4.1.a shows such graphs with $\Delta = 1$ and $\Delta = 2$. Suppose we have constructed a graph G_Δ, satisfying P^*, of degree Δ and such that the 2Δ vertices are A and $f(A)$ with $|A| = |f(A)| = \Delta$ and such that $\Gamma(A) \supseteq f(A)$ and $\Gamma(f(A)) \supseteq A$ (that is satisfied by G_1 and G_2). Let $G_{\Delta+2}$ be the graph obtained from G_Δ by adding 4 vertices $x, y, f(x), f(y)$ joined between themselves and to G as shown in Fig. 4.1.b. Then this graph (where the involution f extends that of G_Δ) has clearly degree $\Delta+2$, $2(\Delta+2)$ vertices and it is not difficult to show that it satisfies property P^* and the hypothesis of induction.

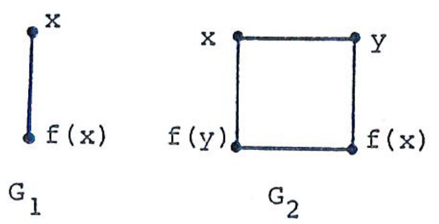

Fig. 4.1.a

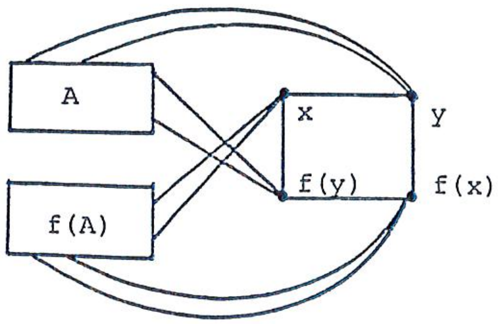

Fig. 4.1.b

4.2 Graphs satisfying P

As already noticed the complete graphs K_n, $n \geq 3$, satisfy P. One can show also that Petersen's graph, Hoffmann-Singleton's graph, the products $K_n * X$, $n \geq 3$, where X is one of the graphs of Fig.2.2, satisfy property P.

An important class of graphs (which will be used in section 5) is formed by the graphs P_q of diameter 2 associated to projective planes. Let us recall, that if $q = p^r$ where p is a prime number, there exists a projective plane of order q with a polarity, that is a numbering of the points M_i, $1 \leq i \leq q^2+q+1$ and of the lines, D_j, $1 \leq j \leq q^2+q+1$ such that if $M_i \in D_j$ then $M_j \in D_i$. Thus $\bigcup_{j \in D_i} D_j$ contains all the points of the plane.

Let P_q be the graph whose vertices are the points of the projective plane, the vertices M_i and M_j are joined if and only if $M_j \in D_i$. The properties of the numbering shows that P_q is of degree $q+1$; it has q^2+q+1 vertices and its diameter is 2. Now, let M_i and M_j be two vertices at distance 2, then $M_j \notin D_i$. As $|D_i \cap D_j| = 1$, there exists a vertex M_k, such that $M_k \in D_i$ and $M_k \notin D_j$; as M_k is at distance 1 from M_i and 2 from M_j, it is clear that M_i and M_j are joined by a path of length 3. Similarly one can prove that the quotient of Levi graphs of generalized quadrangles and hexagons, having a polarity, by that polarity have property P. (See [5].)

V APPLICATIONS

Let $G = K_n$ and $G' = X_8$ the graph on 8 vertices of degree 3 (see Fig. 2.2) then $K_n * X_8$ has $8n$ vertices, degree $n+2$ and diameter 2. In particular with $n = 3$ (resp.4) we obtain graphs of diameter 2, degree 5 (resp.6) on 24 (resp.32) vertices. The better graph of diameter 2 degree 6, known before had only 31 vertices (namely P_5). By taking $K_3 * X_8$ as graph G and again X_8 as G' we obtain a graph of diameter 3, degree 8, on 192 vertices (the better graph known before has 114 vertices). By taking for G the graph on 585 vertices of degree 9 and diameter 3 and for G' the graphs on $5(8,9)$ vertices of Fig.2.1 we give examples of graphs of diameter 4 of degree 11 (resp. 12,13) on 2925 (resp. 4680, 5265) vertices which are the best known now. Another application is the following result.

Theorem: $\liminf_{\Delta\to\infty} n(\Delta,3) \geq \frac{8}{27}\Delta^3$

Proof: Let G be the graph P_q defined in section 4, with q odd, q a prime power, of diameter 2, degree $q+1$ and on q^2+q+1 vertices which satisfies property P. Let $\Delta = \frac{3(q+1)}{2}$ and let $G' = G_{\Delta/3}$ be the graph of degree $\frac{\Delta}{3}$ on $\frac{2\Delta}{3}$ vertices satisfying property P^* constructed in 4.1. By the main theorem $P_q * G_{\Delta/3}$ is a graph of diameter 3 degree $q+1+\frac{\Delta}{3} = \Delta$, having $(q^2+q+1)\frac{2\Delta}{3} = (4\frac{\Delta^2}{9} - \frac{2\Delta}{3} + 1)\frac{2\Delta}{3} = \frac{8\Delta^3}{27} - 4\frac{\Delta^2}{9} + 2\frac{\Delta}{3}$ vertices proving therefore the theorem.

VI GENERALIZATIONS

6.1 The property P_i

A graph G is said to have property P_i, if there exists a permutation f of its vertices, such that f^2 is an automorphism of G and that the graph G^*, which has the same vertices as G and whose edges are the edges of G or their images by f, has diameter i.

6.2 Examples

The graph G of Fig.6, on 4 vertices, where $f(x) = x+1$ (modulo 4) has property P_1, G^* being the complete graph K_4.

Fig. 6

The graph G, on 29 vertices, where the vertex x and y are joined if $y - x \equiv 1, -1, 7, -7$ (mod.29) and where $f(x) \equiv 12x$ (mod.29) has property P_2.

If $4e+1$ is a prime power, consider the graph whose $4e+1$ vertices are the elements of the field GF_{4e+1}; two vertices x and y are adjacent if $x-y$ is a square in GF_{4e+1}. This graph has property P_1 (diameter 2 and degree 2e). (These graphs are known as Paley's graphs and are examples of strongly regular graphs.)

6.3 Theorem

Let G be a graph of diameter $D \geq i+1$ and let G' be a graph satisfying property P_i. Then $G*G'$, with $f_{(x,y)}(x') = f(x')$ for every arc (x,y) of an arbitrary orientation of G, has diameter $D+i$.

Proof: Let x and y be two vertices of G joined by a path P (not necessarily elementary) of length j. If this path contains a arcs of G and therefore $j - a$ arcs of $-G$, let $\varepsilon(P) = 2a-j$. We will prove that if there exists a path of length j' from $f^{\varepsilon(P)}(x')$ to y' in G'^*, with $j' \leq j$, then there exists a path of length $j+j'$ in $G*G'$ between (x,x') and (y,y'). The proof goes by induction on j (that is true for $j = 0$) and for j fixed by induction on $j'(j' \leq j)$; that is true for $j' = 0$ $(y, f^{\varepsilon(P)}(x'))$ being joined to (x,x') by the obvious path of length j. Suppose $j' > 0$ and let (z',y') be the last edge of the path between $f^{\varepsilon(P)}(x')$ and y' in G'^*. If $(z',y') \in G'$, we obtain a path of length $(j+j')$ from (x,x') to (y,y') by adding the arc $((y,z')\ (y,y'))$ to the path of length $j+j'-1$ from (x,x') to (y,z') (which exists by induction hypothesis). If $(z',y') \notin G'$ let z be the vertex preceding y on the path P of G; then $f^{\varepsilon(y,z)}(z')$ and $f^{\varepsilon(y,z)}(y')$ are joined by an edge in G' (recall that $\varepsilon(y,z)$ equals 1 or -1 according (y,z) belongs or not to G).

By induction hypothesis with $j-1$ and $j'-1$ $(j'-1 \le j-1)$ there exists in $G*G'$ a path of length $j+j'-2$ from (x,x') to $(z,f^{\varepsilon(y,z)}(z'))$ and then by adding the path of length 2 of $G*G'$ $(z,f^{\varepsilon(y,z)}(z'))$ $(z,f^{\varepsilon(y,z)}(y'))$ (y,y') we obtain a path of length $j+j'$ from (x,x') to (y,y').

Now let (x,x') and (y,y') be any two vertices of $G*G'$. There is a (not necessarily elementary) path P from x to y of length $j = D$ or $D-1$. Since $D \ge i+1$, $j \ge i$ holds. As G'^* is of diameter i, there exists a path of length $j' \le i$ between $f^{\varepsilon(P)}(x')$ and y' in G'^*.

We have $j' \le j$ (as $j' \le i$ and $i \le j$) and by the property proved above there exists a path of length $j+j' \le D+i$ from (x,x') to (y,y').

6.4 Applications

By using the example of 6.2, we can build new good graphs. For example with the Paley's graphs we obtain infinitely many graphs of diameter 3 which have actually the highest known number of vertices, although the asymptotic value obtained in theorem of section 5 is not improved. By taking for G the graph of diameter 3, degree 9 on 585 vertices and for G' the Paley's graph on 13 vertices we obtain a graph of diameter 4, degree 15 on 7605 vertices. (This is in fact the best known value).

Remark: The condition on G, $D \ge i+1$ can be deleted if one adds extra conditions on G'; for example if G is of diameter 1 and G' satisfies P_1 and is of diameter 2, then $G*G'$ has diameter 2.

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