Shortest coverings of graphs with cycles

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It is shown that the edges of a bridgeless graph G can be covered with cycles such that the sum of the lengths of the cycles is at most $|E(G)| + \min\{\frac{2}{3}|E(G)|, \frac{7}{3}(|V(G)|-1)\}$.

1. Introduction

1.1. *Definitions*

All graphs considered are finite, and may contain loops and multiple edges. Let G be a graph. For $S \subseteq V(G)$, we denote by $\omega(S)$ the set of edges of G with exactly one end in S. A *k-cut* of G is a set of the form $\omega(S)$ $(S \subseteq V(G))$ with $|\omega(S)| = k$. A *bridge is a* 1-cut. A *cycle* in a graph is a connected, 2-regular subgraph. The *length* of a cycle is the number of edges it contains. A *digon* is a cycle of length two. Given the graph G, a *cycle cover* of G is a set $\mathscr{C}$ of cycles of G such that each edge of G belongs to at least one cycle of $\mathscr{C}$. The *length* of $\mathscr{C}$ is the sum of the lengths of the cycles in $\mathscr{C}$ and is denoted by $l(\mathscr{C})$. It is clear that a graph admits a cycle cover if and only if it contains no bridges. Other definitions for graphs can be found in [2, 3].

1.2. *The Main Results*

Itai and Rodeh [12] have shown that every connected bridgeless graph G has a cycle cover of length at most $|E(G)| + 2\,|V(G)| \log |V(G)|$. This upper bound was improved to $\min\{3|E(G)| - 6,\ |E(G)| + 6\,|V(G)| - 7\}$ by Itai, Lipton, Papadimitriou, and Rodeh in [11]. The main result of this paper is

THEOREM 1. *Let G be a bridgeless graph. Then G has a cycle cover $\mathscr{C}$ such that $l(\mathscr{C}) \leqslant \frac{5}{3}\,|E(G)|$.*

Although Theorem 1 appears to be stronger than the previous results only if G has relatively few edges, we shall use Theorem 1 to improve these results for all graphs.

THEOREM 2. *Every bridgeless graph G has a cycle cover of length at most $|E(G)| + \frac{7}{3}\,(|V(G)| - 1)$.*

1.3. *Relationship with the Chinese Postman Problem*

Itai and Rodeh point out in [12] that one may obtain a lower bound for the length of a shortest cycle cover by considering the Chinese postman problem. That is, given a connected graph G, find a closed walk which traverses each edge of G at least once and is as short as possible. An algorithm for finding such a "postman tour" appears in [7]. The problem is equivalent to constructing a graph H_0 such that:

(i) H_0 is obtained by replacing each edge of G by one or more parallel edges,

(ii) H_0 is Eulerian, and

(iii) $|E(H_0)|$ is as small as possible.

A cycle cover $\mathscr{C}$ for a connected bridgeless graph G easily gives rise to a graph H satisfying (i) and (ii), and such that $l(\mathscr{C}) = |E(H)|$, by replacing each edge e of G by a number of parallel edges equal to the number of cycles of $\mathscr{C}$ which contain e. Thus $l(\mathscr{C}) \geqslant |E(H_0)|$. Indeed, it would seem at first sight that the two problems were equivalent, since a cycle decomposition of H_0 should give rise to a cycle cover of G. This is not necessarily the case, however, since it is possible that every cycle decomposition of H_0 contains digons which correspond to single edges in G. (We shall henceforth refer to such digons of H_0 as *forbidden digons.*) For example, if G is a cubic graph, then every H_0 satisfying (i)–(iii) is obtained by replacing each edge of some 1-factor of G by 2 parallel edges. Thus if $\mathscr{C}$ is a cycle cover of G, then $l(\mathscr{C}) \geqslant |E(H_0)| = \frac{4}{3}\,|E(G)|$. If G is the Petersen graph, however, then a shortest cycle cover of G has length 21 (see [12]), and $\frac{4}{3}\,|E(G)| = 20$. On the other hand one can prove that for planar graphs such a situation cannot occur.

PROPOSITION 1. *For every connected bridgeless planar graph G, a shortest cycle cover has length equal to the length of a shortest postman tour.*

To prove this (see also [10]) we need some further definitions. Let v be a vertex of a loopless Eulerian graph H. A *transition at* v is a pair of edges incident to v. A *set of transitions for* v is a partition $T(v)$ of $\omega(\{v\})$ into transitions. If $T(v)$ is defined for every vertex v of H of degree greater than 2, the resulting family $\mathscr{T}$ of transitions is a *transition system* for H. The system is *non-separating* if the graph obtained from H by deleting any one transition of $\mathscr{T}$ is connected. Note that if H has no cut-vertices, every transition system for H is non-separating.

A cycle decomposition $\mathscr{C}$ of H is *compatible with* $\mathscr{T}$ if no cycle of $\mathscr{C}$ contains a transition of $\mathscr{T}$. It is clear that given $\mathscr{T}$, a necessary condition for the existence of a cycle decomposition which is compatible with $\mathscr{T}$ is that $\mathscr{T}$ be non-separating. Fleischner has shown

THEOREM 3 [9]. *Let H be a planar loopless Eulerian graph and $\mathscr{T}$ be a non-separating system of transitions for H. Then H has a cycle decomposition which is compatible with $\mathscr{T}$.*

Proof of Proposition 1. It is easy to see that we may assume that G has no cut-vertices. Let H_0 be a graph satisfying (i)–(iii). It follows from (iii) that H_0 is obtained by replacing each edge of G by at most 2 parallel edges. Whenever e_1 and e_2 are two parallel edges of H_0 which correspond to a single edge of G, let $\{e_1, e_2\}$ be a transition at v for each vertex v incident with both e_1 and e_2. This family of transitions can be extended to a system of transitions $\mathscr{T}$ for H_0. Since H_0 has no cut-vertices, $\mathscr{T}$ is non-separating. By Fleischner's theorem, H_0 has a cycle decomposition $\mathscr{C}$ which is compatible with $\mathscr{T}$, and hence does not contain any forbidden digons. Thus $\mathscr{C}$ gives rise to a cycle cover of G of length $|E(H_0)|$.

2. $\mathbb{Z}_2$-FLOWS AND $\mathbb{Z}_2$-CYCLES

2.1. *Definition*

Let $k \geqslant 1$ and consider the additive group $(\mathbb{Z}_2)^k$. A $(\mathbb{Z}_2)^k$-*flow* of the graph G is a mapping ϕ from $E(G)$ to $(\mathbb{Z}_2)^k$, such that: $\forall v \in V(G)$, $\sum_{e \in \omega(\{v\})} \phi(e) = 0$. (The summation and zero symbols refer to the structure of the group $(\mathbb{Z}_2)^k$.)

2.2. *Elementary properties*

(1) If ϕ is a $(\mathbb{Z}_2)^k$-flow of G, for any $S \subseteq V(G)$ we have $\sum_{e \in \omega(S)} \phi(e) = 0$.

(2) The *support* of the $(\mathbb{Z}_2)^k$-flow ϕ, denoted by $\sigma(\phi)$, is the set of edges $e \in E(G)$ such that $\phi(e) \neq 0$.

Then for $F \subseteq E(G)$ the following properties are equivalent:

(i) F is the support of some $\mathbb{Z}_2$-flow of G.

(ii) Each vertex of G is incident to an even number of edges of F.

(iii) F can be partitioned into cycles of G.

A subset F of $E(G)$ satisfying (i)–(iii) will be called a $\mathbb{Z}_2$-*cycle* of G.

(3) It easily follows from (1) (or (2)) that if e is a bridge of G, $\phi(e) = 0$ for any $(\mathbb{Z}_2)^k$-flow ϕ. A $(\mathbb{Z}_2)^k$-flow ϕ is said to be *nowhere-zero* if $\sigma(\phi) = E(G)$. Thus if a graph has a nowhere-zero $(\mathbb{Z}_2)^k$-flow, it has no bridges.

(4) Let ϕ be a $(\mathbb{Z}_2)^k$-flow. Let $\phi_1, \dots, \phi_k$ be $\mathbb{Z}_2$-flows such that $\forall e \in E(G): \phi(e) = (\phi_1(e), \dots, \phi_k(e))$. We shall write $\phi = (\phi_1, \dots, \phi_k)$. Then ϕ is nowhere-zero if and only if $\bigcup_{i=1}^k \sigma(\phi_i) = E(G)$. In this case, for every $i \in \{1, \dots, k\}$, there exists a partition $P_i = \{C_i^1, \dots, C_i^{r_i}\}$ of $\sigma(\phi_i)$ into cycles of G, and $\bigcup_{i=1}^k P_i$ is a cycle cover of G of length $\sum_{i=1}^k |\sigma(\phi_i)|$. This number will be denoted by $l(\phi)$. Conversely, if $\mathscr{C} = \{C_1, \dots, C_k\}$ is a cycle cover of G, let $\phi_i (i = 1, \dots, k)$ be the unique $\mathbb{Z}_2$-flow such that $\sigma(\phi_i) = C_i$. Then $\phi = (\phi_1, \dots, \phi_k)$ is a nowhere-zero $(\mathbb{Z}_2)^k$-flow and

$$l(\mathscr{C}) = \sum_{i=1}^{k} |\sigma(\phi_i)| = l(\phi).$$

We conclude that the minimum of $l(\mathscr{C})$ over the set of cycle covers $\mathscr{C}$ of G is equal to the minimum of $l(\phi)$ over the set of nowhere-sero $(\mathbb{Z}_2)^k$-flows ϕ of G $(k \geqslant 1)$.

Remark. If G is bridgeless, G has a cycle cover and hence G has a newhere-zero $(\mathbb{Z}_2)^k$-flow for some $k \geqslant 1$ (see the above discussion).

(5) Let $z = (z_1, \dots, z_k) \in (\mathbb{Z}_2)^k$.

The (Hamming) weight $w(z)$ *of* z is the number of nonzero components of z, that is, $w(z) = |\{i \in \{1, \dots, k\} : z_i = 1\}|$. Let ϕ be a nowhere-zero $(\mathbb{Z}_2)^k$-flow of G. A straightforward counting argument yields $l(\phi) = \sum_{e \in E(G)} w(\phi(e))$.

2.3. *The Double-Cover Conjecture*

A *cycle double-cover* of G is a cycle cover $\mathscr{C}$ of G such that each edge appears in exactly two cycles of $\mathscr{C}$. The double-cover conjecture asserts that every bridgeless graph has a cycle double cover [17, Conjecture 3.3]. We shall denote by D_k ($k \geqslant 2$) the subset of $(\mathbb{Z}_2)^k$ consisting of the elements of weight 2. It is easy to show (see the above discussion in 2.2(4)) that G has a cycle double-cover iff it has a $(\mathbb{Z}_2)^k$-flow with all edge-values in D_k for some $k \geqslant 2$ (such a flow will be called a *D_k-flow*).

Remark. If a graph has a D_k-flow it has a $D_{k'}$-flow for every $k' \geqslant k$. Hence the double-cover conjecture can be formulated as follows:

For every bridgeless graph G, there exists a $k \geqslant 2$ such that G has a D_k-flow. (DCC)

This conjecture is clearly related to the shortest cycle cover problem. In fact Itai and Rodeh rediscover an equivalent form of the (DCC) in [12, Problem (ii)].

We have the following result:

PROPOSITION 2. *If G has a D_k-flow ($k \geqslant 2$), it has a cycle cover $\mathscr{C}$ with $l(\mathscr{C}) \leqslant (2(k-1)/k)|E(G)|$*

Proof. Let $\phi = (\phi_1, \ldots, \phi_k)$ be a D_k-flow of G. Clearly $l(\phi) = \sum_{e \in E(G)} w(\phi(e)) = 2\,|E(G)|$.

On the other hand, $l(\phi) = \sum_{i=1}^{k} |\sigma(\phi_i)|$. We may assume without loss of generality that $\forall i \in \{1, \ldots, k-1\}, |\sigma(\phi_k)| \geqslant |\sigma(\phi_i)|$. Consider now the $(\mathbb{Z}_2)^{k-1}$-flow $\phi' = (\phi_1, \ldots, \phi_{k-1})$. It is clearly nowhere-zero. Moreover

$$l(\phi') = l(\phi) - |\sigma(\phi_k)| \leqslant \frac{k-1}{k}\, l(\phi) = \frac{2(k-1)}{k}\,|E(G)|.$$

This completes the proof.

2.4. *Some Consequences of Proposition* 2

(1) For $k = 2$ the situation is quite simple. If G has a D_2-flow, it has a cycle cover $\mathscr{C}$ with $l(\mathscr{C}) = |E(G)|$, i.e., $E(G)$ can be partitioned into cycles, and conversely.

(2) For $k = 3$, we obtain that if G has a D_3-flow it has a cycle cover $\mathscr{C}$ with $l(\mathscr{C}) \leqslant \frac{4}{3}\,|E(G)|$.

We may now use the following easy result:

PROPOSITION 3. *A graph has a D_3-flow* iff *it has a nowhere-zero $(\mathbb{Z}_2)^2$-flow.*

Proof. If $\phi = (\phi_1, \phi_2, \phi_3)$ is a D_3-flow, then $\phi' = (\phi_1, \phi_2)$ is a nowhere-zero $(\mathbb{Z}_2)^2$-flow. Conversely, if $\phi' = (\phi_1, \phi_2)$ is a nowhere-zero $(\mathbb{Z}_2)^2$-flow, $\phi = (\phi_1, \phi_2, \phi_1 + \phi_2)$ is a D_3-flow.

Now, applying Propositions 2 and 3 together with some known results on the existence of nowhere-zero $(\mathbb{Z}_2)^2$-flows, we obtain

COROLLARY 1. *Every bridgeless planar graph G has a cycle cover* $\mathscr{C}$ *with* $l(\mathscr{C}) \leqslant \frac{4}{3}|E(G)|$.

Proof. Use the four color theorem [1; 13, Proposition 3].

COROLLARY 2. *Let G be a cubic 3-edge-colorable graph. The length of a shortest cycle cover of G is equal to* $\frac{4}{3}|E(G)|$.

Proof. As already seen in Subsection 1.3, the length of a shortest cycle cover of G is at least $\frac{4}{3}|E(G)|$. The equality follows from Propositions 2, 3 and [13, Proposition 2].

COROLLARY 3. *Every bridgeless graph G without 3-cuts has a cycle cover* $\mathscr{C}$ *with* $l(\mathscr{C}) \leqslant \frac{4}{3}|E(G)|$.

Proof. Use [13, Proposition 10].

(3) For $k = 4$, we shall use the following observation:

PROPOSITION 4. *If a graph has a* D_4*-flow, it has a* D_3*-flow.*

Proof. Let $\phi = (\phi_1, \phi_2, \phi_3, \phi_4,)$ be a D_4-flow. Then it is easy to check that $\phi' = (\phi_1 + \phi_2, \phi_1 + \phi_3, \phi_1 + \phi_4)$ is a D_3-flow. Hence this case reduces to the previous one.

(4) For $k = 5$, we have nothing but a conjecture which has been proposed by several authors [4, 16], and which is stronger than the double-cover conjecture.

Conjecture. Every bridgeless graph has a D_5-flow.
By Proposition 1, this implies the following:

Conjecture. Every bridgeless graph G has a cycle cover $\mathscr{C}$ with $l(\mathscr{C}) \leqslant \frac{8}{5}|E(G)|$.

(5) For $k = 6$, the existence of a D_6-flow in G implies that G has a cycle cover $\mathscr{C}$ with $l(\mathscr{C}) \leqslant \frac{5}{3}|E(G)|$.
We shall show (by different methods) that this last property holds for every bridgeless graph G.

3. The Main Result

3.1. *Introduction*

The following result is proved in [13].

8-Flow Theorem. *Every bridgeless graph has a nowhere-zero* $(\mathbb{Z}_2)^3$-*flow.*

Using this result only, one can prove that every bridgeless graph G has a cycle cover $\mathscr{C}$ with $l(\mathscr{C}) \leqslant \frac{12}{7}|E(G)|$. We now present the proof of Theorem 1. It uses a refinement of an alternative proof for the 8-flow Theorem (the idea is indicated in [13, Sect. V].

3.2. Lemma. *Every bridgeless graph G has a $\mathbb{Z}_2$-cycle C such that $|C| \geqslant \frac{2}{3}|E(G)|$ and C intersects every 3-cut of G.*

Proof. It is clear that to prove the lemma, it is enough to prove it for loopless 2-edge-connected graphs. Let G be such a graph, and let $v \in V(G)$. *A splitting of G at v* is the graph G' obtained by replacing v by two distinct vertices v' and v'', each edge of G with end vertices v, x ($x \in V(G) - \{v\}$) being replaced by an edge with end vertices v', x or v'', x in such a way that v' has degree 2 in G'. *A splitting of G* is any graph obtained from G by a succession of vertex-splittings. It follows from a result of Fleischner [8] (see also Mader [14]) that G has a 2-edge-connected splitting G' which has no vertices of degree greater than three. Note that identifying edges of G' with edges of G in the obvious way, every $\mathbb{Z}_2$-cycle of G' is a $\mathbb{Z}_2$-cycle of G, and every 3-cut of G is a 3-cut of G'. We conclude that:

(1) *To prove the lemma it is enough to prove it for loopless 2-edge-connected graphs with no vertex of degree greater than three.*

Let G be such a graph. If G has no vertices of degree 3, the result is clear. Otherwise there exists a cubic 2-edge-connected graph H such that G can be obtained from H by replacing each edge e of H by a simple path P_e of length $f(e) \geqslant 1$. For $F \subseteq E(H)$ we shall denote by $f(F)$ the sum $\sum_{e \in F} f(e)$. It follows from a result of Edmonds [6] that there exists an integer $k \geqslant 1$ and a family $(M_1, ..., M_{3k})$ of $3k$ perfect matchings of H (not necessarily distinct) such that every edge of H appears in exactly k of the M_i's.

Let K be a 3-cut of H. For every perfect matching M of H, $E(H) - M$ is a 2-factor of H and hence a $\mathbb{Z}_2$-cycle. Hence $|K \cap (E(H) - M)|$ is even, so that $|K \cap M|$ equals 1 or 3. Now each one of the 3 edges of K appears in exactly k of the M_i's ($i = 1, ..., 3k$), so that $\sum_{i=1}^{3k} |K \cap M_i| = 3k$. It follows that $\forall i \in \{1, ..., 3k\}$, $|K \cap M_i| = 1$.

Finally we note that $\sum_{i=1}^{3k} f(M_i) = kf(E(H))$. Hence there exists

$i \in \{1,..., 3k\}$ with $f(M_i) \leqslant \frac{1}{3} f(E(H))$. Then $F = E(H) - M_i$ is a $\mathbb{Z}_2$-cycle of H which intersects every 3-cut of H and such that $f(F) \geqslant \frac{2}{3} f(E(H))$. Let C be the subset of edges of G equal to $\bigcup_{e \in F} P_e$. Clearly C is a $\mathbb{Z}_2$-cycle of G and $|C| = f(F) \geqslant \frac{2}{3} f(E(H)) = \frac{2}{3} |E(G)|$. Moreover, no 3-cut of G contains two edges of a single path P_e, $e \in E(H)$ (the remaining edge of the 3-cut would be a bridge). Hence every 3-cut of G is obtained by considering some 3-cut $\{e_1, e_2, e_3\}$ of H and choosing exactly one edge from each of $P_{e_1}, P_{e_2}, P_{e_3}$. It follows that C intersects every 3-cut of G. This completes the proof.

3.3. *A Consequence of the Lemma*

PROPOSITION 5. *Every connected bridgeless graph G has a postman tour of length at most $\frac{4}{3} |E(G)|$.*

Proof. Let C be a $\mathbb{Z}_2$-cycle of G with $|C| \geqslant \frac{2}{3} |E(G)|$. Replace every edge of $E(G) - C$ by two parallel edges. This yields an Eulerian graph H with $|E(H)| \leqslant \frac{4}{3} |E(G)|$.

Remark. Propositions 5 and 1 together give another proof of Corollary 1 which does not rely on the four color theorem.

3.4. *Proof of Theorem 1*

Let G be a bridgeless graph with $|E(G)| = m$. By the lemma, there exists a $\mathbb{Z}_2$-flow ϕ_1 of G such that $|\sigma(\phi_1)| \geqslant 2m/3$ and $\sigma(\phi_1)$ intersects every 3-cut of G. For each edge e of $\sigma(\phi_1)$, add to G an edge e' parallel to e (i.e., with the same pair of ends). We obtain a new bridgeless graph G' which contains G as a subgraph. Moreover it is clear that G' has no 3-cuts. By Proposition 10 of [13], G' has a nowhere-zero $(\mathbb{Z}_2)^2$-flow $\phi' = (\phi_2', \phi_3')$.

For $e \in E(G)$ and $i \in \{2, 3\}$ let $\phi_i(e) = \phi_i'(e)$ if $e \notin \sigma(\phi_1)$ and $\phi_i(e) = \phi_i'(e) + \phi_i'(e')$ if $e \in \sigma(\phi_1)$. This defines two $\mathbb{Z}_2$-flows ϕ_2, ϕ_3 of G. Since $\phi' = (\phi_2', \phi_3')$ is nowhere-zero, the $(\mathbb{Z}_2)^2$-flow (ϕ_2, ϕ_3) of G takes nonzero values on $E(G) - \sigma(\phi_1)$. It follows that (ϕ_1, ϕ_2, ϕ_3) is a nowhere-zero $(\mathbb{Z}_2)^3$-flow of G.

Consider the vector space $[GF(2)]^3$ (over $GF(2)$) of the 3-tuples $x = (\alpha_1, \alpha_2, \alpha_3)$ ($\alpha_i \in GF(2)$, $i = 1, 2, 3$). To every element $x = (\alpha_1, \alpha_2, \alpha_3)$ of this space we associate the flow $\phi_x = \Sigma_{\alpha_i = 1} \phi_i$. In particular,

$$\phi_{(1,0,0)} = \phi_1, \qquad \phi_{(0,1,0)} = \phi_2, \qquad \text{and} \qquad \phi_{(0,0,1)} = \phi_3.$$

It is easy to show that for every basis $\{x_1, x_2, x_3\}$ of $[GF(2)]^3$, $(\phi_{x_1}, \phi_{x_2}, \phi_{x_3})$ is a nowhere-zero $(\mathbb{Z}_2)^3$-flow of G. Denote by X the set $[GF(2)]^3 - \{(0, 0, 0)\}$ and by X' the set $X - \{(1, 0, 0)\}$. One can easily check that each edge appears in exactly 4 of the $\sigma(\phi_x)$ ($x \in X$), and hence $\Sigma_{x \in X} |\sigma(\phi_x)| = 4m$. Then $\sum_{x \in X'} \sigma(\phi_x)| = 4m - |\sigma(\phi_1)| \leqslant 4m - \frac{2}{3} m = \frac{10}{3} m$. Let $\mathscr{B}$ be the set of bases of

$[GF(2)]^3$ which do not contain the vector $(1,0,0)$. Every vector of X' appears in exactly 8 elements of $\mathscr{B}$. Hence $\Sigma_{B\in\mathscr{B}}(\Sigma_{x\in B}|\sigma(\phi_x)|) = 8\,\Sigma_{x\in X'}|\sigma(\phi_x)| \leqslant 80m/3$. Since $|\mathscr{B}|=16$, there exists $B\in\mathscr{B}$ with

$$\sum_{x\in B}|\sigma(\phi_x)| \leqslant \frac{1}{16}\frac{80m}{3} = \frac{5m}{3}.$$

Then the supports of the $\mathbb{Z}_2$-flows ϕ_x for $x\in B$ will give a cycle cover $\mathscr{C}$ with $l(\mathscr{C})\leqslant 5m/3$. This completes the proof.

3.5. 4-*Covers*

We observe that using the seven $\mathbb{Z}_2$-cycles $\sigma(\phi_x)$ $(x\in[GF(2)]^3 - \{(0,0,0)\})$ defined in the above proof it is possible to obtain a cycle cover $\mathscr{C}$ such that every edge appears in exactly 4 cycles of $\mathscr{C}$. Calling such a cycle cover a *cycle* 4-*cover*, we have

PROPOSITION 6. *Every bridgeless graph has a cycle* 4-*cover.*

4. PROOF OF THEOREM 2

Let G be a bridgeless graph. We may assume G is connected . Let H be a subset of $E(G)$ such that the graph $(V(G),H)$ is 2-edge-connected and minimal with this property. It is easy to show, using [5, 15], that $|H|\leqslant 2|V(G)|-2$. By Theorem 1, $(V(G),H)$ has a cycle cover $\mathscr{C}_1$ with $l(\mathscr{C}_1)\leqslant\frac{5}{3}|H|$. Let $F=E(G)-H$, and consider a spanning tree T contained in H. For every e in F, there is a unique $\mathbb{Z}_2$-flow ϕ_e such that $e\in\sigma(\phi_e)\subseteq T\cup\{e\}$. Let $\phi=\Sigma_{e\in F}\phi_e$. Then clearly $F\subseteq\sigma(\phi)\subseteq T\cup F$. Let $\mathscr{C}_2$ be a cycle decomposition of $\sigma(\phi)$. Now $\mathscr{C}_1\cup\mathscr{C}_2$ is a cycle cover $\mathscr{C}$ of G, with

$$l(\mathscr{C}) = l(\mathscr{C}_1)+l(\mathscr{C}_2) = l(\mathscr{C}_1)+|\sigma(\phi)| \leqslant \tfrac{5}{3}|H| + |T\cup F|.$$

Since $|T\cup F|=|T|+|F|=|V(G)|-1+|E(G)|-|H|$ we have $l(\mathscr{C})\leqslant |E(G)|+|V(G)|-1+\frac{2}{3}|H|\leqslant|E(G)|+\frac{7}{3}(|V(G)|-1)$. This completes the proof.

5. VERTEX CYCLE COVERS

Given a graph G, a *vertex cycle cover* of G is a set of cycles $\mathscr{C}$ of G such that each vertex of G belongs to at least one cycle of $\mathscr{C}$.

PROPOSITION 7. *Let G be a graph such that each vertex of G lies in a cycle. Then G has a vertex cycle cover $\mathscr{C}$ such that* $l(\mathscr{C})\leqslant\frac{10}{3}(|V(G)|-1)$.

Proof. We may assume that G is 2-edge-connected. Let H be a critically 2-edge-connected spanning subgraph of G, so that $|E(H)| \leqslant 2\,|V(G)| - 2$. By Theorem 1, H has a cycle cover $\mathscr{C}$ such that $l(\mathscr{C}) \leqslant \frac{5}{3}\,|E(H)|$. Clearly $\mathscr{C}$ is a vertex cycle cover of G and $l(\mathscr{C}) \leqslant \frac{10}{3}\,(|V(G)| - 1)$.

6. Covering of the Vertices of a Strong Digraph with Circuits

In this section, circuit means "directed circuit." Let $f(2p) = p^2 + p$ and $f(2p+1) = (p+1)^2$.

Proposition 8. *For any strong digraph D with n vertices, there exists a vertex circuit cover $\mathscr{C}$ such that $l(\mathscr{C}) \leqslant f(n)$.*

Proof. Let k be the length of the longest circuit of D and let C_0 be such a longest circuit. We can cover the vertices of D with C_0 and for each vertex not in C_0 with a circuit of length at most k. Therefore we can cover with $(n-k+1)$ circuits of length at most k. This yields a vertex circuit cover $\mathscr{C}$ with $l(\mathscr{C}) \leqslant k(n-k+1)$. But it is known that $\max_k k(n-k+1) = f(n)$.

The result is best possible in the sense that there exists a strong digraph D of order n such that for any covering family $\mathscr{C}$, $l(\mathscr{C}) \geqslant f(n)$. Consider the digraph D consisting of a directed circuit of length $k = \lceil n/2 \rceil$ in which we have replaced one vertex by a stable set of $(n-k+1)$ vertices (see Fig. 1). Each vertex y_i belongs to the unique circuit $C_i = (x_1, y_i, x_2, \ldots, x_{k-1})$. Therefore to cover all the y_i we need to use all the circuits C_i. But $\Sigma\, l(C_i) = k(n-k+1) = \lceil n/2 \rceil\,(\lfloor n/2 \rfloor + 1) = f(n)$.

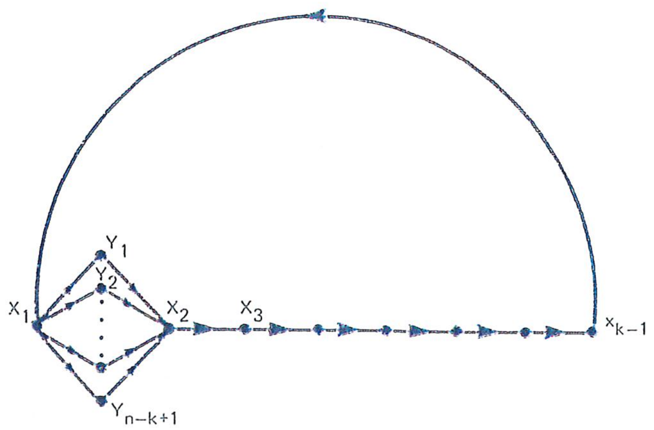

FIGURE 1.

7. Open Problems

7.1.

In view of Theorem 1, the main problem is to find the infimum ρ of the set of numbers r with the property that every bridgeless graph G has a cycle cover $\mathscr{C}$ with $l(\mathscr{C}) \leqslant r\,|E(G)|$. All we know is that $\frac{7}{5} \leqslant \rho \leqslant \frac{5}{3}$. The lower bound $\frac{7}{5}$ is given by the Petersen graph (see subsection 1.3). In fact, by combining several Petersen graphs together as in Fig. 2, we obtain an infinite family of graphs G whose shortest cycle cover $\mathscr{C}$ satisfies $l(\mathscr{C}) = \frac{7}{5}\,|E(G)|$. We note further that both the Blanuša snarks on 18 vertices, the flower snark on 20 vertices, and both the Loupekhine snarks on 22 vertices, have cycle covers of length $\frac{4}{3}\,|E(G)|$.

7.2.

A problem related to Theorem 2 is proposed by Itai and Rodeh [12, Open Problem (i)]. Does every bridgeless graph G have a cycle cover $\mathscr{C}$ with $l(\mathscr{C}) \leqslant |E(G)| + |V(G)| - 1$? They prove this for graphs with two edge disjoint spanning trees. By Theorem 1, the result is true for graphs G with $|E(G)| \leqslant \frac{3}{2}\,(|V(G)| - 1)$ (e. g., subdivisions of cubic graphs containing at least three vertices of degree 2).

By Proposition 1 and using the obvious property that a shortest postman tour of G has length at most $|E(G)| + |V(G)| - 1$ it follows that the result is also true for planar graphs. On the other hand, it can be checked that if G is the complete bipartite graph $K_{n,3}$, the length of a shortest cycle cover is $|E(G)| + |V(G)| - 3$.

7.3.

Finally we propose the following conjecture

> Every 2-connected graph G has a vertex cycle cover of length at most $2\,|V(G)| - 2$.

Note that this conjecture would be best possible because of the complete bipartite graph $K_{n,2}$ (n odd).

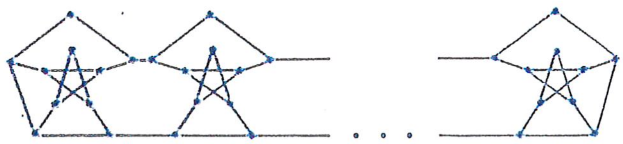

Figure 2.

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