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# On Termination of Graph Rewriting Systems through Language Theory

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## Abstract

The termination issue we tackle is rooted in natural language processing where graph rewriting systems (GRS) may contain a large number of rules, often in the order of thousands. Decidable concepts thus become mandatory to verify the termination of such systems. The notion of graph rewriting consider does not make any assumption on the structure of graphs (they are not "term graphs", "port graphs" nor drags). The lack of algebraic structure in our setting led us to proposing two orders on graphs inspired from language theory: the matrix multiset-path order and the rational embedding order. We show that both are stable by context, which we then use to obtain the main contribution of the paper: under a suitable notion of "interpretation", a GRS is terminating if and only if it is compatible with an interpretation.

**2012 ACM Subject Classification** Theory of computation → Rewrite systems

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## 1 Introduction

Computer linguists rediscovered few years ago that graph rewriting is a good model of computation for rule-based systems. They used traditionally terms, see for instance Chomsky's Syntagmatic Structures [3]. But usual phenomena such as anaphora do not fit really well within such theories. In such situations, graphs behave much better. For examples of graph rewriting in natural language processing, we refer the reader to the parsing procedure by Guillaume and Perrier [12] or the word ordering modeling by Kahane and Lareau [14]. The first named author with Guillaume and Perrier designed a graph rewriting model called GREW [2] that is adapted to natural language processing.

The rewriting systems developed by the linguists often contain a huge number of rules, e.g., those synthesized from lexicons (e.g. some rules only apply to transitive verbs). For instance, in [12], several systems are presented, some with more than a thousand of rules. Verifying properties such as termination by hand thus becomes intractable. This fact motivates our framework for tackling the problem of GRS termination.

Following the tracks of term rewriting, for which the definition is essentially fixed by the algebraic structure of terms, many approaches to graph rewriting emerged in past years. Some definitions (here meaning semantics) are based on a categorical framework, e.g., the double pushout (DPO) and the single pushout (SPO) models, see [21]. To make use of algebraic potential, some authors make some, possibly weak, hypothesis on graph structures, see for instance the main contribution by Courcelle and Engelfriet [4] where graph decompositions, graph operations and transformations are described in terms of monadic second-order logics

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(with the underlying decidability/complexity results). In this spirit, Ogawa describes a graph algebra under a limited tree-width condition [18].

Another line of research follows from the seminal work by Lafont [15] on interaction nets. The latter are graphs where nodes have some extra structure: nodes have a label related to some arity and co-arity. Moreover, nodes have some "principal gates" (ports) and rules are actionned via them. One of the main results by Lafont is that rewriting in this setting is (strongly) confluent. This approach has been enriched by Fernandez, Kirchner and Pinaud [11], who implemented a fully operational system called PORGY with strategies and related semantics. Also, it is worth mentionning the graph rewriting as described by Dershowitz and Jouannaud [8]. Here, graphs are seen as a generalization of terms: symbols have a (fixed) arity, graphs are connected via some sprouts/variables as terms do. With such a setting, a good deal of term rewriting theory also applies to graphs.

Let us come back to the initial problem: termination of graph rewriting systems in the context of natural language processing. We already mentioned that rule sets are large, which making manual inspection impossible. Moreover, empirical studies fail to observe some of the underlying hypotheses of the previous frameworks. For instance, there is no clear bound on tree-width: even if input data such as dependency graphs are almost like trees, the property is not preserved along computations. Also, constraints on node degrees are also problematic: graphs are usually sparse, but some nodes may be highly connected. To illustrate, consider the sentence “The woman, the man, the child and the dog eat together”. The verb “eat” is related to four subjects and there is no a priori limit on this phenomenon. Typed versions (those with fixed arity) are also problematic: a verb may be transitive or not. Moreover, rewriting systems may be intrinsically nondeterministic. For instance, if one computes the semantics of a sentence out of its grammatical analysis, it is quite common there are multiple solutions. To further illustrate nondeterminism consider the well know phrasal construction “He saw a girl with a telescope” with two clear readings.

Some hypotheses are rather unusual for standard computations, e.g., fixed number of nodes. Indeed, nodes are usually related to words or concepts (which are themselves closely related to words). A paraphrase may be a little bit longer than its original version, but its length can be easily bounded by the length of the original sentence up to some linear factor. In GREW, node creations are restricted. To take into account the rare cases for which one needs extra nodes, a “reserve” is allocated at the beginning of the computation. All additional nodes are taken from the reserve. Doing so has some efficiency advantages, but that goes beyond the scope of the paper. Also, node and edge labels, despite being large, remain finite sets: they are usually related to some lexicons. These facts together have an important impact on the termination problem: since there are only finitely many graphs of a given size, rewriting only leads to finitely many outcomes. Thus, deciding termination for *a particular input graph* is decidable. However, our problem is to address termination in the class of *all* graphs. The latter problem is often referred to as *uniform termination*, whereas the former is refereed to as *non-uniform*. For word rewriting, uniform termination of non increasing systems constituted a well known problem, and was shown to be undecidable by Sénizergues in [24].

This paper proposes a novel approach for termination of graph rewriting. In a former paper [1], we proposed a solution based on label weights. Here, the focus is on the description (and the ordering) of paths within graphs. In fact, paths in a graph can be seen as regular languages. The question of path ordering thus translates into a question of regular language orderings. Accordingly, we define the *graph multi-set path ordering* that is related to that in [6]. Dershowitz and Jouannaud, in the context of drag rewriting, consider a similar notion of path ordering called GPO (see [7]). Our definitions diverge from theirs in that our graph rewriting model is quite different: here, we do not benefit (as they do) from a good algebraic structure. Our graphs have no heads, tails nor hierarchical decomposition. In fact, our ordering is not even well founded! Relating the two notions is nevertheless interesting and left for further work. Plump [20] also defines path orderings for term graphs, but those behave like sets of terms.

One of our graph orderings will involve matrices, and orderings on matrices. Nonetheless, as far as we see, there is no relationship with matrix interpretations as defined by Endrullis, Waldmann and Zantemma [10].

The paper is organised as follows. In Section 2 we recall the basic background on graphs and graph rewriting systems (GRS) that we will need throughout the paper, and introduce an example that motivated our work. In Section 3 we consider a language theory approach to the termination of GRSs. In particular, we present the language matrix, and the matrix multiset path order (Subsection 3.4) and the rational embedding order (Subsection 3.5). We also introduce the notion of stability by context (Subsection 3.6) and show that both orderings are stable under this condition (Subsection 3.7). In Section 4 we propose notion of graph interpretability and show one of our main results, namely, that a GRS is terminating if and only if it is compatible with interpretations.

**Main contributions:** The two main contributions of the paper are the following.

1. We propose two orders on graphs inspired from language theory, and we show that both are monotonic and stable by context.
2. We introduce a notion of graph interpretation, and show that terminating GRSs are exactly those compatible with such interpretations.

## 2 Notations and Graph Rewriting

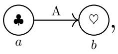
In this section we recall some general definitions and notations. Given an alphabet $\Sigma$, the set of words (finite sequences) is denoted by $\Sigma^*$. The concatenation of two words $v$ and $w$ is denoted by $v \cdot w$. The empty word, being the neutral element for concatenation, is denoted by $1_\Sigma$ or, when clear from the context, simply by 1. Note that $\langle \Sigma^*, 1, \cdot \rangle$ constitutes a monoid.

A *language* on $\Sigma$ is some subset $L \subseteq \Sigma^*$. The set of all languages on $\Sigma$ is $\mathcal{P}(\Sigma^*)$. The addition of two languages $L, L' \subseteq \Sigma^*$ is defined by $L + L' = \{w \mid w \in L \vee w \in L'\}$. The empty language is denoted by 0 and $\langle \mathcal{P}(\Sigma^*), +, 0\rangle$ is also a monoid. Given some word $w \in \Sigma^*$, we will also denote by $w$ the language made of the singleton $\{w\} \in \mathcal{P}(\Sigma^*)$. Given two languages $L, L' \subseteq \Sigma^*$, their concatenation is defined by $L \cdot L' = \{w \cdot w' \mid w \in L \wedge w' \in L'\}$. In this way, $\langle \mathcal{P}(\Sigma^*), 1, \cdot \rangle$ is also a monoid.

A *preorder* on a set $X$ is a binary relation $\preceq \subseteq X^2$ that is reflexive ($x \preceq x$, for all $x \in X$) and transitive (if $x \preceq y$ and $y \preceq z$, then $x \preceq z$, for all $x, y, z \in X$). A preorder $\preceq$ is a *partial order* if it is anti-symmetric (if $x \preceq y$ and $y \preceq x$, then $x = y$, for all $x, y \in X$). A preorder is an *equivalence relation* if it is symmetric ($x \preceq y \Rightarrow y \preceq x$) . Observe that each preorder $\preceq$ induces an equivalence relation $\sim$: $a \sim b$ if $a \preceq b$ and $b \preceq a$. The strict part of $\preceq$ is then the relation: $x \prec y$ iff $x \preceq y$ and $\neg(x \sim y)$. We also mention the "dual" preorder $\succeq$ of $\preceq$ defined by: $x \succeq y$ iff $y \preceq x$. A preorder $\preceq$ is said to be *well-founded* if there is no infinite chain $\cdots \prec x_2 \prec x_1$ or, equivalently, $x_1 \succ x_2 \succ \cdots$.

The remainder of this section may be found in [2] and we refer the reader to it for an extended presentation. We suppose given a (finite) set $\Sigma_{\mathrm{N}}$ of *node labels*, a (finite) set $\Sigma_{\mathrm{E}}$ of *edge labels* and we define graphs accordingly. A graph is a triple $G = \langle N, E, \ell \rangle$ with

$E \subseteq N \times \Sigma_\mathrm{E} \times N$ and $\ell : N \to \Sigma_\mathrm{N}$ is the labeling function of nodes. Note that there may be more than one edge between two nodes, but at most one is labeled with some $e \in \Sigma_\mathrm{E}$. In the sequel, we use the notation $m \xrightarrow{e} n$ for an edge $(m, e, n) \in E$.

Given a graph $G$, the sets $\mathcal{N}_G$, $\mathcal{E}_G$ and $\ell_G$ denote respectively the sets of nodes, edges and its labeling function. We we will also (abusively) use the notation $m \in G$ and $m \xrightarrow{e} n \in G$ instead of $m \in \mathcal{N}_G$ and $m \xrightarrow{e} n \in \mathcal{E}_G$ when the context is clear. Furthermore, in 

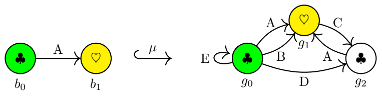
, $a, b$ are nodes, $\clubsuit, \heartsuit$ are the respective node labels and $A$ is the edge label (here between $a$ and $b$).

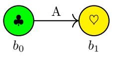
The set of graphs on node labels $\Sigma_\mathrm{N}$ and edge labels $\Sigma_\mathrm{E}$ is denoted by $\mathcal{G}_{\Sigma_\mathrm{N},\Sigma_\mathrm{E}}$ or $\mathcal{G}$ in short. Two graphs $G$ and $G'$ are said to share their nodes when $\mathcal{N}_G = \mathcal{N}_{G'}$. Given two graphs $G$ and $G'$ such that $\mathcal{N}_G \subseteq \mathcal{N}_{G'}$, set $G \blacktriangleleft G' = \langle \mathcal{N}_{G'}, \mathcal{E}_G \cup \mathcal{E}_{G'}, \ell\rangle$ with $\ell(n) = \ell_G(n)$ if $n \in \mathcal{N}_G$ and $\ell(n) = \ell_{G'}(n)$, otherwise.

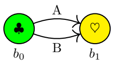
A graph morphism $\mu$ between a source graph $G$ and a target graph $H$ is a function $\mu : \mathcal{N}_G \to \mathcal{N}_H$ that preserves edges and labelings, that is, for all $m \xrightarrow{e} n \in G$, $\mu(m) \xrightarrow{e} \mu(n) \in G'$ holds, and for any node $n \in G$: $\ell_G(n) = \ell_{G'}(\mu(n))$. A *basic pattern* is a graph, and a *basic pattern matching* is an injective morphism from a basic pattern $P$ to some graph $G$. Given such a morphism $\mu : G \to G'$, we define $\mu(G)$ to be the sub-graph of $G'$ made of the nodes $\{\mu(n) \mid n \in \mathcal{N}_G\}$, of the edges $\{\mu(m) \xrightarrow{e} \mu(n) \mid m \xrightarrow{e} n \in G\}$ and node labels $\mu(n) \mapsto \ell_G(n)$.

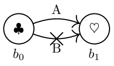
A *pattern* is a pair $P = \langle P_0, \vec{\nu}\rangle$ made of a basic pattern $P_0$ and a sequence of injective morphisms $\nu_i : P_0 \to N_i$, called *negative conditions*. The basic pattern describes what must be *present* in the target graph $G$, whereas negative conditions say what must be *absent* in the target graph. Given a pattern $P = \langle P_0, \vec{\nu}\rangle$ and a graph $G$, a *pattern morphism* is an injective morphism $\mu : P_0 \to G$ for which there is no morphism $\xi_i$ such that $\mu = \xi_i \circ \nu_i$.

▶ **Example 1.** Consider the basic pattern morphism $\mu : P_0 \to G$ (colors define the mapping):



The pattern $P = \langle P_0, [\nu]\rangle$ with $\nu$ defined by 

$\xhookrightarrow{\nu}$ 

prevents the application of the morphism above. Indeed, $\xi = b_0 \mapsto g_0, b_1 \mapsto g_1$ is such that $\xi \circ \nu = \mu$. When there is only one negative condition, we represent the pattern by crossing nodes and edges which *are not* within the basic pattern. For instance, the pattern $P$ above looks like 

that we hope is self-explanatory.

In this paper we think of graph transformations as sequences of "basic commands".

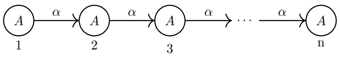
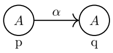
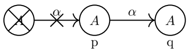
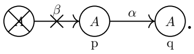
▶ **Definition 2** (The command language). *There are three basic commands:* `label`$(p, \alpha)$ *for node renaming,* `del_edge`$(p, e, q)$ *for edge deletion and* `add_edge`$(p, e, q)$ *for edge creation. In these basic commands, $p$ and $q$ are nodes, $\alpha$ is some node label and $e$ is some edge label. A pattern $\langle P_0, \vec{\nu}\rangle$ is compatible with a command whenever $p$ and $q$ are nodes in $P_0$.*

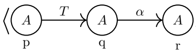
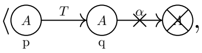
▶ **Definition 3** (Operational semantics). *Given a pattern $P = \langle P_0, \vec{\nu}\rangle$ compatible with some command $c$, and some pattern matching $\mu : P \to G$ where $G$ is the graph on which the transformation is applied, we have the following possible cases: $c =$* `label`$(p, \alpha)$ *turns the label of $\mu(p)$ into $\alpha$, $c =$* `del_edge`$(p, e, q)$ *removes $\mu(p) \xrightarrow{e} \mu(q)$ if it exists, otherwise does nothing, and $c =$* `add_edge`$(p, e, q)$ *adds the edge $\mu(p) \xrightarrow{e} \mu(q)$ if it does not exists, otherwise does nothing. The graph obtained after such an application is denoted by $G \cdot_\mu c$.*

*Given a sequence of commands* $\vec{c} = (c_1, \dots, c_n)$*, let* $G \cdot_\mu \vec{c}$ *be the resulting graph, i.e.,* $G \cdot_\mu \vec{c} = (\cdots((G \cdot_\mu c_1) \cdot_\mu c_2) \cdot_\mu \cdots c_n)$*.*

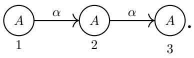
▶ **Definition 4.** *A rule is a pair* $R = \langle P, \vec{c} \rangle$ *made of a pattern and a (compatible) sequence of commands. Such a rule* $R$ *applies to a graph* $G$ *when there is a pattern morphism* $\mu : P \to G$*. Let* $G' = G \cdot_\mu \vec{c}$*, then we write* $G \to_{R,\mu} G'$*. We define* $G \to G'$ *whenever there is a rule* $R$ *and a pattern morphism* $\mu$ *such that* $G \to_{R,\mu} G'$*.*

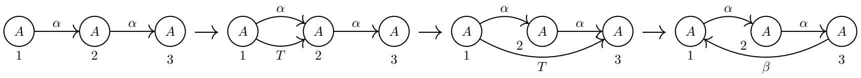
## 2.1 The main example

Let $\Sigma_{\mathrm{N}} = \{A\}$ and $\Sigma_{\mathrm{E}} = \{\alpha, \beta, T\}$. For the discussion, we suppose that $T$ is a working label, that is not present in the initial graphs. We want to add a new edge $\beta$ between node $n$ and node 1 each time we find a maximal chain:  within a graph $G$. Consider the basic pattern $P_{init}$ =  together with its two negative conditions $\nu_1$ =  and $\nu_2$ = . We consider three rules:

Init: $\langle \langle P_{init}, [\nu_1, \nu_2] \rangle, (\texttt{add\_edge}(p, T, q)) \rangle$ which fires the transitive closure.
Follow: $\langle$, $(\texttt{add\_edge}(p, T, r), \texttt{del\_edge}(p, T, q)) \rangle$ which follows the chain.
End: $\langle$, $(\texttt{del\_edge}(p, T, q), \texttt{add\_edge}(q, \beta, p)) \rangle$ which stops the processus.

To prevent all pathological cases (e.g., when the edge $\beta$ is misplaced, when two chains are crossing, and so on), we could introduce more sophisticated patterns. But, since that does not change issues around termination, we avoid obscuring rules with such technicalities.

▶ **Example 5.** Take . By applying 'Init', 'Follow' and 'End', it rewrites as:



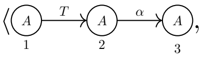
## 2.2 Three technical facts about Graph Rewriting

It is well known that the main issue with graph rewriting definitions is the way the context is related to the pattern image and its rewritten part. We shall tackle this issue with Proposition 6.

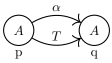
### Self-application

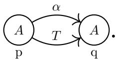
Let $R = \langle P, \vec{c} \rangle$ be the rule made of a pattern $P = \langle P_0, \vec{\nu} \rangle$ and a sequence of commands $\vec{c}$. There is the identity morphism $1_{P_0} : P_0 \to P_0$, and thus we can apply rule $R$ on $P_0$ itself, that is, $P_0 \to_{R, 1_{P_0}} P'_0 = P_0 \cdot_{1_{P_0}} \vec{c}$. We call this latter graph the *self-application of* $R$.

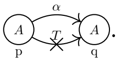
### Rule node renaming

To avoid heavy notation, we will use the following trick. Suppose that we are given a rule $R = \langle P, \vec{c} \rangle$, a graph $G$ and a pattern morphism $\mu : P \to G$. Let $P = \langle P_0, \vec{\nu} \rangle$. We define $R_\mu$ to be the rule obtained by renaming nodes $p$ in $P_0$ to $\mu(p)$ (and their references within $\vec{c}$). For instance, the rule 'Follow' can be rewritten as $Follow_\mu$ = $\langle$, $(\texttt{add\_edge}(1, T, 3), \texttt{del\_edge}(1, T, 2)) \rangle$ where $\mu$ denotes the pattern morphism used to apply 'Follow' in the derivation. Observe that: (i) the basic pattern of $R_\mu$ is actually $\mu(P_0)$, which is a subgraph of $G$, (ii) $\iota : \mu(P_0) \to G$ mapping $n \mapsto n$ is a pattern matching, and (iii) applying rule $R_\mu$ with $\iota$ is equivalent to applying rule $R$ with $\mu$. In other words, $G \to_{R,\mu} G'$ if (and only if) $G \to_{R_\mu,\iota} G'$. To sum up, we can always rewrite a rule so that its basic pattern is *actually* a subgraph of $G$.

## Uniform rules

Let us consider rule 'Init' above. It applies on: 

, and the result is the graph itself: 

. Indeed, we cannot add an already present edge (relative to a label) within a graph. Thus, depending on the graph, the rule will or will not append an edge. Such an unpredictable behavior can be easily avoided by modifying the pattern of 'Init' to: 

. The same issue may come from edge deletions. A *uniform* rule is one for which commands apply (that is, modify the graph) for each rule application. Since this is not the scope of the paper, we refer the reader to [2] for a precise definition of uniformity. We will only observe two facts.

First, any rule can be replaced by a finite set of uniform rules (using negative conditions as above) that operate identically. Thus, we can always suppose that rules are uniform.

Second, the following property holds for uniform rules (see [2]§7 for a proof).

▶ **Proposition 6.** *Suppose that* $G \to_{R,\iota} G'$ *with* $R = \langle P, \vec{c}\rangle$ *and* $P = \langle P_0, \vec{\nu}\rangle$ *(the basic pattern* $P_0$ *being a subgraph of* $G$*). Let* $C$ *be the graph obtained from* $G$ *by deleting the edges in* $P_0$*. Then* $G = P_0 \blacktriangleleft C$ *and* $G' = P_0' \blacktriangleleft C$ *with* $P_0'$ *being the self-application of the rule. Moreover,* $\mathcal{E}_C \cap \mathcal{E}_{P_0} = \emptyset$ *and* $\mathcal{E}_C \cap \mathcal{E}_{P_0'} = \emptyset$*.*

Throughout the remainder of the paper we assume that all rules are uniform.

## 3 Termination of Graph Rewriting Systems

By a *graph rewriting system* (GRS) we simply mean a set of graph rewriting rules (see Section 2). A GRS $\mathbf{R}$ is said to be *terminating* if there is no infinite sequence $G_1 \to G_2 \to \cdots$. Such sequences, whether finite or not, are called *derivations*.

Since there is no node creation (neither node deletion) in our notion of rewriting, any derivation starting from a graph $G$ will lead to graphs whose size is the size of $G$. Since there are only finitely many such graphs, we can decide the termination for this particular graph $G$. However, the question we address here is the *uniform termination problem* (see Section 1).

▶ Remark 7. Suppose that we are given a strict partial order $\succ$, not necessarily well founded. If $G \to G'$ implies $G \succ G'$ for all graphs $G$ and $G'$, then the system is terminating. Indeed, suppose it is not the case, let $G_1 \to G_2 \to \cdots$ be an infinite reduction sequence. Since there are only finitely many graphs of size of $G_1$, it means that there are two indices $i$ and $j$ such that $G_i \to \cdots \to G_j$ with $G_i = G_j$. But then, since $G_i \succ G_{i+1} \succ \cdots \succ G_j$, we have that $G_i \succ G_j = G_i$ which is a contradiction.

A similar argument was exhibited by Dershowitz in [5] in the context of term rewriting. For instance, it is possible to embed rewriting within real numbers rather than natural numbers to prove termination.

Let us try to prove the termination of our main example (see Subsection 2.1). Rules such as '*Init*' and '*End*' are "simple": we put a weight on edge labels $\omega : \Sigma_{\mathrm{E}} \to \mathbb{R}$ and we say that the weight of a graph is the sum of the weights of its edges labels. Set $\omega(\alpha) = 0, \omega(\beta) = -2$ and $\omega(T) = -1$. Then, rules 'Init' and 'End' decrease the weight by 1 and, since rule 'Follow' keeps the weight constant, it means the two former rules can be applied only finitely many times. Observe that negative weights are no problem with respect to Remark 7.

But how do we handle rule 'Follow'? No weights as above can work.

## 3.1 A language point of view

Let $G \to G'$ be a rule application. The set of nodes stays constant. Let us think of graphs as automata, and let us forget about node labeling for the time being. Let $\Sigma_{\mathrm{E}}$ be the set of edge labels. Consider a pair of states (nodes), choose one to be the initial state and one to be the final state. Thus the automaton (graph) defines some regular language on $\Sigma_{\mathrm{E}}$. In fact, the automaton describes $n^2$ languages (one for each pair of states).

Now, let us consider the effect of graph rewriting in terms of languages. Consider an application of the 'Follow' rule: $G \to G'$. Any word to state $r$ that goes through the transitions $p \xrightarrow{T} q \xrightarrow{\alpha} r$ can be mapped to a shorter one in $G'$ via the transition $p \xrightarrow{T} r$. The languages corresponding to state $r$ contain shorter words. The remainder of this section is devoted to formalizing this intuition into proper orders on graphs. For that, we will need to *count* the number of paths between any two states. Hence, we shall introduce $\mathbb{N}$-rational expressions, that is, *rational expression with multiplicity*. See, e.g., Sakarovitch's book [23] for an introduction and justifications of the upcoming constructions. We introduce here the basic ideas.

## 3.2 Formal series

A formal series on $\Sigma$ (with coefficients in $\mathbb{N}$) is a (total) function $s : \Sigma^* \to \mathbb{N}$. Given a word $w$, $s(w)$ is the multiplicity of $w$. The set of words $\underline{s} = \{w \in \Sigma^* \mid s(w) \neq 0\}$ is the *support* of $s$. Given $n \in \mathbb{N}$, let $\mathsf{n}$ be the series defined by $\mathsf{n}(w) = 0$, if $w \neq 1$, and $\mathsf{n}(1) = n$, where 1 denotes the empty word. The empty language is $\underline{0}$, the language made of the empty word is $\underline{1}$. Moreover, for $a \in \Sigma$, the series $a$ is given by $a(w) = 0$ if $w \neq a$ and $a(a) = 1$.

Given two series $s$ and $t$, their *addition* is the series $s + t$ given by $s + t(w) = s(w) + t(w)$, and their *product* is $s \cdot t$ defined by $s \cdot t(w) = \sum_{u \cdot v = w} s(u)t(v)$. The *star operation* is defined by $s^* = 1 + s + s^2 + \cdots$. The monoïd $\Sigma^*$ being graded, the operation is correctly defined whenever $s(1) = 0$.

Given a series $s$, let $s^{\leq k}$ be its restriction to words of length less or equal to $k$, i.e., $s^{\leq k}(w) = 0$ whenever $|w| > k$ and $s^{\leq k}(w) = s(w)$, otherwise.

An $\mathbb{N}$-rational expression on an alphabet $\Sigma$ is built upon the grammar [22]:

$$\mathsf{E} ::= a \in \Sigma \mid n \in \mathbb{N} \mid (\mathsf{E} + \mathsf{E}) \mid (\mathsf{E} \cdot \mathsf{E}) \mid (\mathsf{E}^*).$$

Thus, given the constructions mentioned in the previous paragraph, any $\mathbb{N}$-rational expression $E \in \mathsf{E}$ denotes some formal series. To each $\mathbb{N}$-rational expression corresponds an $\mathbb{N}$-automaton, which is standard automaton with transitions labeled by a non empty linear combination $\sum_{i \leq k} n_i a_i$ with $n_i \in \mathbb{N}$ and $a_i \in \Sigma$ for all $i \leq k$.

## 3.3 The language matrix

Let us suppose given an edge label set $\Sigma_{\mathrm{E}}$. Let $\mathsf{E}$ denote the $\mathbb{N}$-expressions over $\Sigma_{\mathrm{E}}$. A matrix $M$ of dimension $P \times P$ for some (finite) set $P$ is an array $(M_{i,j})_{i \in P, j \in P}$ whose components are in $\mathsf{E}$. Let $\mathfrak{M}_E$ be the set of such matrices. Given a graph $G$, we define the matrix $M_G$ of dimension $\mathcal{N}_G \times \mathcal{N}_G$ as follows: $M_{G\,i,j} = T_1 + \cdots + T_\ell$ with $T_1, \ldots, T_\ell$ the set of labels on the transitions between state $i$ and $j$ if such transitions exist, otherwise 0.

Let $1_P$ be the unit matrix of dimension $P \times P$, that is $(1_P)_{i,j} = 0$ if $i \neq j$ else 1. From now on, we abbreviate the notation from $1_P$ to 1 if the context is clear. Then, let $M_G^* = 1 + M_G + M_G^2 + \cdots$. Each component of $M_G^*$ is actually an $\mathbb{N}$-regular expression (see Sakarovitch Ch. III, §4 for instance). The (infinite) sum is correctly defined since for all $i, j$, $(M_G)_{i,j} = T_1 + \cdots + T_\ell$. Thus, $1 \notin \underline{(M_G)_{i,j}}$.

The question about termination can be reformulated in terms of matrices whose components are languages (with words counted with their multiplicity). To prove the termination of the rewriting system, it is then sufficient to prove that for any two graphs $G \to G'$, $M_G^* > M_{G'}^*$. To prove such a property in the infinite class of finite graphs, we will use the notion of "stable orders".

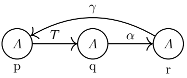
Recall the 'Follow' rule and consider the basic pattern $L$ and the self-application $R$. Then,

$$M_L = \begin{pmatrix} 0 & T & 0 \\ 0 & 0 & \alpha \\ 0 & 0 & 0 \end{pmatrix} \quad M_R = \begin{pmatrix} 0 & 0 & T \\ 0 & 0 & \alpha \\ 0 & 0 & 0 \end{pmatrix}.$$

Observe that $(M_R)_{13} > (M_L)_{13}$. This matrix deals with edges/transitions. In order to consider paths, we need to compute $M_L^*$ and $M_R^*$ that are given by:

$$M_L^* = \begin{pmatrix} 1 & T & T \cdot \alpha \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix} \quad M_R^* = \begin{pmatrix} 1 & 0 & T \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix}.$$

Any word within $M_R^*$'s components is a sub-word of the corresponding component in $M_L^*$.

▶ **Example 8.** Consider now a variation of 'Follow': ⟨

$$(\texttt{add\_edge}(p, T, r), \texttt{del\_edge}(p, T, q))\rangle.$$

By setting $L'$ as the pattern and $R'$ as the self-application, we get the following matrices:

$$M_{L'}^* = \begin{pmatrix} (T\alpha\gamma)^* & T(\alpha\gamma T)^* & T\alpha(\gamma T\alpha)^* \\ \alpha\gamma(T\alpha\gamma)^* & (\alpha\gamma T)^* & \alpha(\gamma T\alpha)^* \\ \gamma(T\alpha\gamma)^* & \gamma T(\alpha\gamma T)^* & (\gamma T\alpha)^* \end{pmatrix} \quad M_{R'}^* = \begin{pmatrix} (T\gamma)^* & 0 & T(\gamma T)^* \\ \alpha\gamma(T\gamma)^* & 1 & \alpha(\gamma T)^* \\ \gamma(T\gamma)^* & 0 & (\gamma T)^* \end{pmatrix}.$$

Again, words within $M_{R'}^*$ are sub-words of the corresponding ones in $M_{L'}^*$.

## 3.4 The matrix multiset path order

The order we shall introduce in this section is inspired by the notion of *multiset path ordering* within the context of term rewriting (see for instance [6]). However, in the present context of graph rewriting (to be compared with Dershowitz and Jouannaud's [7] or with Plump's [20]), the definition is a bit more complicated. Here, we do not consider an order on letters as it is done for terms.

Let $\trianglelefteq$ be the word embedding on $\Sigma^*$, that is, the smallest partial order such that $1 \trianglelefteq w$, and if $u \trianglelefteq v$, then $(u \cdot w \trianglelefteq v \cdot w$ and $w \cdot u \trianglelefteq w \cdot v$, for all $u, v, w \in \Sigma^*$. This order $\trianglelefteq$ can be extended to formal series, that is, the multiset-path ordering, see Dershowitz and Manna [9] or Huet and Oppen [13].

▶ **Definition 9** (Multiset path order). *The multiset path order is the smallest partial order on finite series such that*

- *if there is $w \in \underline{t}$ such that for all $v \in \underline{s}$, $v \lhd w$, then $s \unlhd t$, and*
- *if $r \unlhd s$ and $t \unlhd u$, then $r + t \unlhd s + u$.*

*We write $s \lhd t$ when $s \unlhd t$ and $s \neq t$.*

▶ **Proposition 10.** *Addition and product are monotonic with respect to the multiset-path order. Moreover, addition is strictly monotonic with respect to $\unlhd$, and if $r \lhd s$, then $r \cdot t \lhd s \cdot t$ and $t \cdot r \lhd t \cdot s$, whenever $t \neq 0$ (otherwise, we have equality).*

**Proof.** It is not difficult to see that addition is monotonic. So suppose that $r \lhd s$. We prove that $r + t \lhd s + t$, by induction (see Definition 9). Suppose that there is $w \in \underline{s}$ such that for all $v \in \underline{r}$ we have $v \lhd w$, then $r \lhd s$. Since $r(w) = 0$, then $(r + t)(w) = t(w) < s(w) + t(w) = (s + t)(w)$, and we are done. Otherwise, $r = r_0 + r_1$ and $s = s_0 + s_1$ with $r_0 \unlhd s_0$ and $r_1 \unlhd s_1$. One of the two inequalities must be strict (otherwise $r = s$). Suppose $r_0 \lhd s_0$. By definition, observe that $r_1 + t \unlhd s_1 + t$. But then, $r + t = r_0 + (r_1 + t)$ and $s = s_0 + (s_1 + t)$ and we apply induction on $(r_0, s_0)$. As addition is commutative, the result holds.

For the product, suppose that $r \unlhd s$ and let $t$ be some series. We prove $r \cdot t \unlhd s \cdot t$; the other inequality $t \cdot r \unlhd t \cdot s$ is similar. Again, we proceed by induction on Definition 9:

- Suppose there is $w \in \underline{s}$ such that for all $v \in \underline{r}, v \lhd w$. By induction on $t$, if $t = 0$, $r \cdot t = 0 \unlhd 0 = s \cdot t$. Otherwise, $t = t_0 + v_0$ for a word $v_0$. Observe that $r \cdot v_0 = \sum_{v \in \underline{r}} r(v) v \cdot v_0$. Since for all $v \in \underline{r}$, $v \cdot v_0 \lhd w \cdot v_0$, we have $r \cdot v_0 \lhd w \cdot v_0 \unlhd s \cdot v_0$. Now, $r \cdot t = r \cdot (t_0 + v_0) = r \cdot t_0 + r \cdot v_0$ and $s \cdot t = s \cdot t_0 + s \cdot v_0$. By induction, $r \cdot t_0 \unlhd s \cdot t_0$ and since $r \cdot v_0 \unlhd s \cdot v_0$, the result holds.
- Otherwise, $r = r_0 + r_1$. In this case, $s \cdot r = s \cdot r_0 + s \cdot r_1$ and $t \cdot r = t \cdot r_0 + t \cdot r_1$. The result then follows by induction.

To show strict monotonicity, suppose $r \lhd s$ and again proceed by case analysis. Suppose that there is some $w \in \underline{s}$ such that for all $v \in \underline{r}, v \lhd w$. Since $t \neq 0$, it contains at least one word $v_0$ such that $t = t_0 + v_0$. By $r \lhd s$ , $r \cdot v_0 = \sum_{v \in \underline{r}} r(v) v \cdot v_0 \lhd \sum_{v \in \underline{s}} s(v) v \cdot v_0 = s \cdot v_0$. The result then follows by induction on the expansion of $t$ and using the strict monotonicity of addition. ◀

▶ **Definition 11** (Matrix multiset-path order)**.** *Let $M$ and $M'$ be two matrices with dimension $P \times P$. Write $M \unlhd M'$ if for all $k \geq |P|$ and for all $(i, j) \in P \times P$, we have $M_{i,j}^{\leq k} \unlhd {M'}_{i,j}^{\leq k}$.*

▶ **Corollary 12.** *The addition and the multiplication are monotonic with respect to the matrix multiset-path order.*

**Proof.** It follows from Proposition 10 since addition and product of matrices are defined as addition and product of their components. ◀

## 3.5 The Rational Embedding Order

Let $\Sigma$ be some fixed alphabet. For a transducer $\tau$, we denote the function it computes by $[\tau]$.

▶ **Definition 13** (Rational Embedding Order)**.** *Given two regular languages $L$ and $L'$ on $\Sigma$, write $L \lesssim L'$ if:*

- *there is an injective function $[\tau] : L' \to L$ and*
- *$[\tau]$ can be computed by a transducer $\tau$ such that $|\tau(w)| \leq |w|$, for every $w \in L'$. Such transducers are said to be decreasing (in [16]).*

The transducer $\tau$ is said to be a *witness* of $L \lesssim L'$.

We say that a transition of a transducer is *deleting* when it is of the form $a \mid 1$ for some $a \in \Sigma$. A transducer whose transitions are of the form $X \mid Y$, with $|Y| \leq |X|$, is itself decreasing. If a path corresponding to an input $w$ passes through a deleting transition, then $|\tau(w)| < |w|$.

In the sequel we will make use of the following results that are direct consequences of Nivat's Theorem [17].

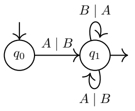
▶ **Proposition 14.** *Let $[\tau] : L \to L'$ be computed by a transducer $\tau$, and let $L''$ be a regular language. Then the following assertions hold.*

1. *The restriction $\tau_{|L''} : L'' \cap L \to L'$ mappping $w \mapsto \tau(w)$ is computable by a transducer.*
2. *The co-restriction $\tau^{|L''} : L \to L' \cap L''$ mapping $w \mapsto \tau(w)$ if $\tau(w) \in L''$ and otherwise undefined, is computable by a transducer.*
3. *The function $\tau' : L \to L'$ defined by $\tau'(w) = \tau(w)$ if $w \in L''$ and otherwise undefined, is computable by a transducer.*

Observe that the identity on $\Sigma^*$ is computed by a transducer (made of a unique initial/final state with transitions $a \mid a$ for all $a \in \Sigma$). Then, the identity on $L$ is obtained by Proposition 14(1,2). Thus we have that $\lesssim$ is reflexive. Also, it is well known that both transducers and injective functions can be composed. Hence, we also have that $\lesssim$ is transitive. Thus, $\lesssim$ is a preorder.

However, we do not have anti-reflexivity in general.

▶ **Example 15.** $L_1 = A \cdot (A + B)^* \lesssim L_2 = B \cdot (A + B)^* \lesssim L_1$. Consider the transducer (in the drawing, the initial state is shown with an in-arrow and the final one by an out-arrow):



. That shows $L_1 \lesssim L_2$. Swap 'A' and 'B', the converse holds.

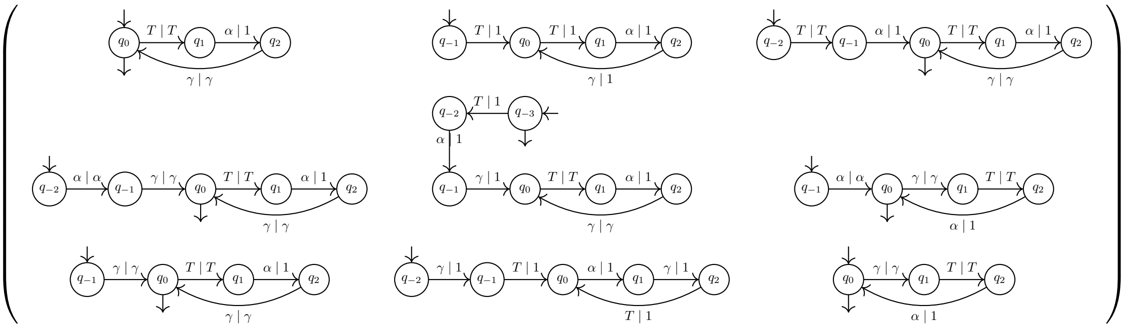
However, there is a simple criterion to ensure a strict inequality. Suppose $L_1 \lesssim L_2$ has a witness $\tau : L_2 \to L_1$. If $\tau$ contains one (accessible and co-accessible) deleting transition, then, the relation is strict.

As before, set $L_1 < L_2$ whenever $L_1 \lesssim L_2$ but not $L_2 \lesssim L_1$. Suppose $L_1 \lesssim L_2 \lesssim L_1$ with a transducer $\theta : L_1 \to L_2$ and $\tau$ as above. Let $w$ be the smallest input word from the initial state to a final state through the transition $a \mid 1$. Then $\theta \circ \tau$ (the composition of the two transducers) defines an injective function. Define the set $M^{<|w|} = \{u \in L_2 \mid |u| < |w|\}$. Generally speaking, for any word $u$, $|\theta \circ \tau(u)| \leq |u|$. Thus $\theta \circ \tau(M^{<w}) \subseteq M^{<w}$. Since $M^{<w}$ is a finite set and $\theta \circ \tau$ is injective, it is actually bijective when restricted to $M^{<w}$. However, $|\theta \circ \tau(w)| \leq |\tau(w)| < w$ implies $\theta \circ \tau(w) \in M^{<w}$. By the Pigeon-hole Principle, there is one word in $M^{<w}$ that has two pre-images via $\theta \circ \tau$. Thus, $\theta \circ \tau$ cannot be injective, which yields a contradiction.

▶ Remark 16. Observe that, when two regular languages verify $L \subseteq L'$, it follows from Proposition 14 that $L \lesssim L'$.

▶ **Definition 17.** *The rational embedding order extends to matrices by pointwise ordering: Let $M$ and $N$ with dimension $P \times P$, and write $M \lesssim N$ if for every $i, j \in P \ \times P$, we have $M_{i,j} \lesssim N_{i,j}$.*

Recall the modified version of 'Follow' (Example 8). The following transducers show that all components strictly decrease.



In the following, to compare two graphs by means of the rational embedding order, we transform graphs into matrices as follows. Given a graph $G$, let $M'_G$ be the matrix of dimension $\mathcal{N}_G \times \mathcal{N}_G$ such that $(M'_G)_{i,j} = T_1^{i,j} + T_2^{i,j} + \cdots T_\ell^{i,j}$ with $T_1, \ldots, T_\ell$ the labels of the edges from $i$ to $j$. In other words, we "decorate" the labels with the source and target nodes. Then, $G \lesssim G'$ whenever $M'_G \lesssim M'_{G'}$.

## 3.6 Stable orders on matrices

A matrix on $\mathsf{E}$ is said to be *finite* whenever all its component are finite. Two matrices $M$ and $M'$ (of same dimension) on $\mathsf{E}$ are said to be disjoint if for every $i, j$, $M_{i,j} \cdot M'_{i,j} = \mathsf{0}$.

▶ **Definition 18.** *Let $M$ be a matrix of dimension $P \times P$ and $P \subseteq G$. The extension of $M$ to dimension $G \times G$ is the matrix $M^{\uparrow G}$ defined by:*

$$(M^{\uparrow G})_{i,j} = \begin{cases} M_{i,j} & \text{if } i, j \in P \\ \mathsf{0} & \text{otherwise} \end{cases}$$

*The notation $M^{\uparrow G}$ is shortened to $M^{\uparrow}$ when $G$ is clear from the context.*

▶ **Fact 1.** *Let $M$ be a matrix of dimension $P \times P$, with $P \subseteq G$. Then $(M^{\uparrow G})^* = (M^*)^{\uparrow G}$.*

▶ **Definition 19.** *We say that a partial order $\preceq$ on $\mathsf{E}$ is* stable by context *if for every $P \subseteq G$, all matrices $L$ and $R$ of dimension $P \times P$, and every $C$ of dimension $G \times G$, the following assertions hold.*

1. *If $L, R, C$ are finite, $L$ being disjoint from $C$, $R$ being disjoint from $C$ and $R^* \prec L^*$, then $(R + C)^* \prec (L + C)^*$;*
2. *If $R \prec L$, then $R^{\uparrow G} \prec L^{\uparrow G}$.*

▶ **Lemma 20.** *Let $\preceq$ be partial order stable by context and consider finite matrices $L, R$ of dimension $P \times P$ and let $C$ be a finite matrix of dimension $G \times G$ with $P \subseteq G$. Then, $R^* \prec L^*$ implies $(R^{\uparrow} + C)^* \prec (L^{\uparrow} + C)^*$.*

**Proof.** If $R^* \prec L^*$, then, $(R^*)^{\uparrow} \prec (R^*)^{\uparrow}$ by Definition 19.2. By Lemma 1, it follows that $(R^{\uparrow})^* \prec (L^{\uparrow})^*$. Clearly, $R^{\uparrow}$ and $L^{\uparrow}$ are finite, and from Definition 19.1, we have $(R^{\uparrow} + C)^* \prec (L^{\uparrow} + C)^*$ ◀

▶ **Theorem 21.** *Let $\preceq$ be a partial order stable by context. Suppose that for every rule $R = \langle P, \vec{c} \rangle$ with $P = \langle P_0, \vec{\nu} \rangle$ and $P'_0$ the self-application of $R$, we have $(P'_0)^* \prec (P_0)^*$. Then the corresponding GRS is terminating.*

**Proof.** Let $\preceq$ be a partial order on graphs and consider the corresponding order on matrices: $G \prec G'$ iff $M^*_G \prec M^*_{G'}$. We show that for every rule, we have $G \to G'$ implies $G' \prec G$. So let $R$ be a graph rewriting rule and let $\mu$ be a morphism such that $G \to_{R,\mu} G'$. By the discussion in the beginning of Section 3, without loss of generality, we can suppose that $\mu$ is actually the inclusion of pattern $P_0$ in $G$. Now, let $P_0andP'_0$ be respectively the basic pattern and the self-application of $R$. Define $C$ to be the graph made of the nodes of $G$ without edges in $P_0$. By Proposition 6, $M_G = M^{\uparrow}_{P_0} + M_C$ and $M_{G'} = M^{\uparrow}_{P'_0} + M_C$. Moreover, $M_{P_0}, M_{P'_0}$ and $M_C$ are finite and $M_{P_0}$ is disjoint from $M_C$ and $M_{P'_0}$ is disjoint from $M_C$. Thus, we can apply Lemma 20, and we get $M^*_{G'} = (M^{\uparrow}_{P'_0} + M_C)^* \prec (M^{\uparrow}_{P_0} + M_C)^* = M^*_G$. ◀

## 3.7 Stability of the orderings

We can now prove the two announced stability results.

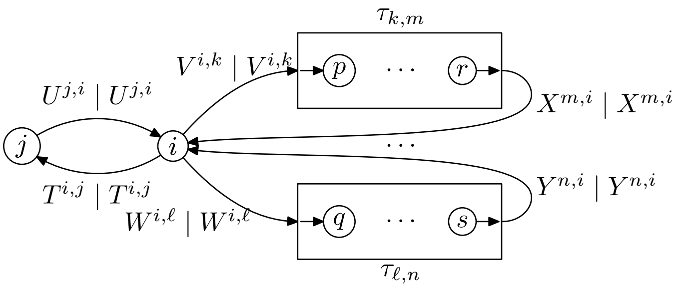
▶ **Proposition 22.** *The multiset path ordering is stable by context.*

**Proof.** We first verify that condition 2 of Definition 19 holds. Suppose that $R \triangleleft L$ with $R, L$ of dimension $P \times P$. Then, for all $(i,j) \notin P \times P$, $R^{\uparrow G}_{i,j} = 0 \trianglelefteq 0 = L^{\uparrow G}_{i,j}$. Now, for all $k \geq |G| \geq |P|$ and for all $(i,j) \in P \times P$, we have $(R^{\uparrow G})^{\leq k}_{i,j} = R^{\leq k}_{i,j} \trianglelefteq L^{\leq k}_{i,j} = (L^{\uparrow G})^{\leq k}_{i,j}$.

To verify that condition 1 also holds, let $G \times G$ be the dimension of $L, R$ and $C$. Take $k \geq |G|$. We have on one side $(R + C)^{*^{\leq k}} = \sum_{(A_1,\ldots,A_\ell)\in\{R,C\}^*,\, \ell\leq k} \prod_{i\leq \ell} A_i$, and on the other side $(L + C)^{*^{\leq k}} = \sum_{(A_1,\ldots,A_\ell)\in\{R,C\}^* | \ell\leq k} \prod_{i\leq \ell} A_i\{R \leftarrow L\}$ where $A_i\{R \leftarrow L\} = L$ if $A_i = R$, and $C$ otherwise. As the product and the addition are (strictly) monotonic, the result follows. ◀

▶ **Proposition 23.** *The rational embedding order is stable by context.*

**Proof.** Since we use a component-wise ordering, it is easy to verify that condition 2 of Definition 19 holds. To verify that condition 1 also holds, let $G \times G$ be the shared dimension of $L, R$ and $C$. Since $R < L$, there are decreasing transducers $\tau_{i,j} : L_{i,j} \to R_{i,j}$ with at least one of them deleting. Let $P$ be the set of nodes corresponding to the pattern $L$. We build the family of transducers $(\theta_{p,q})_{p,q\in G\times G}$ as follows. The family of transducers will share the major part of the construction. First, we make a copy of all transducers $(\tau_{i,j})_{i,j}$. Then, we add as states all the nodes of $C$. Given an edge $i \xrightarrow{T} j \in C$, we set a transition $i \xrightarrow{T^{i,j}|T^{i,j}} j$. That is the transducer copies the paths within $C$. For a transition $i \xrightarrow{T} j$ with $i \notin P, j \in P$, we set the transitions: $i \xrightarrow{T^{i,j}|T^{i,j}} q_n$ for all $q_n$ initial state of the transducer $\tau_{j,n}, n \in P$. Similarly, for any transition $i \xrightarrow{T} j$ with $i \in P, j \notin P$, we set the transitions: $r_n \xrightarrow{T^{i,j}|T^{i,j}} j$ for each terminal state $r_n$ of the transducer $\tau_{n,i}, n \in P$. This construction can be represented as follows:



where $U, T, W, X, Y$ range over the edge labels. Take $k, \ell \not\in P$. Any path from state $k$ to state $\ell$ describes a path in $C + L$ on the input side and a path in $C + R$ on the output side. Indeed, transitions within $C$ are simply copied and the transducers $\tau_{i,j}$ transform paths in $L$ into paths in $R$.

It remains to specify initial and final states. Given some component $p, q \in G$, if $i \not\in P$, we set the initial state to be $p$. Otherwise, we introduce a new state $\iota$ which is set to be initial, and we add a transition $\iota \xrightarrow{1|1} i$ for any state $i$ initial in $\tau_{p,r}$ for some $r$. If $q \not\in P$, then, $q$ is the final state. Otherwise, any state $j$ within some $\tau_{r,q}$, $r \in P$, is final.

Consider some pair $p, q \in G$. We prove that the transducer $\theta_{p,q}$ is injective. Consider a path $w$ in $C + L$. It can be decomposed as follows: $w = w_1\ell_1 \cdots w_k\ell_k$ where the $\ell_i$'s are the sub-words within $L$ (that is the $w_i$'s have the shape $v_i a_i$ where $a_i$ is a transition from $C$ to $L$). Consider a second word $w' = w'_1\ell'_1 \cdots w'_{k'}\ell'_{k'}$ such that the transducer $\theta_{p,q}(w) = \theta_{p,q}(w') = u$. Given the construction of $\theta_{p,q}$, consider the word $u = u_1 r_1 \cdots u_k r_k$ with $r_1, \ldots, r_k$ some path within $R$. Indeed, only a letter within $L$ can produce a letter within $R$. Consider the case where $r_k$ is non empty. When the transducer reaches the first letter in $\ell_k$, it is in a state $\tau_{k,m}$ for some $m$. Actually, $m = q$ since only $\tau_{k,q}$ contains a final state. Thus, the path is fixed within $\tau_{k,p}$ and then, the injection of $\tau_{k,p}$ applies. So, $\ell'_{k'} = \ell_k$. We can go back within $w_k$. On this part of the word, the transitions have the shape $T^{i,j} \mid T^{i,j}$. Thus, $w_k = w'_{k'}$. We can continue this process up to the beginning of $w$ and $w'$. ◀

## 4 Interpretations for Graph Rewriting Termination

Interpretations methods are well known in the context of term rewriting, see for instance Dershowitz and Jouannaud's survey on rewriting [6]. Their usefulness comes from the fact that they belong to the class of simplification orderings, i.e., orderings for which if $t \trianglelefteq u$, then $t \preceq u$. In the context of graphs, we introduce a specific notion of "interpretation", that we will still call interpretation.

▶ **Definition 24.** *A graph* interpretation *is a triple $\langle X, \prec, \phi \rangle$ where $\langle X, \prec \rangle$ is a partially ordered set and $\phi : \mathcal{G} \to X$ is such that given two graphs $P$ and $P'$ having the same set of nodes and $C$ disjoint of $P$ and $P'$, if $\phi(P) \prec \phi(P')$, then $\phi(P + C) \prec \phi(P' + C)$.*

An interpretation $\Omega = \langle X, \prec, \phi \rangle$ is *compatible* with a rule $R$ if $\phi(P'_0) \prec \phi(P_0)$ where $P_0$ is the basic pattern of $R$ and $P'_0$ its self-application. Similarly, an interpretation is compatible with a GRS if it is compatible with all of its rules.

▶ **Theorem 25.** *Every GRS compatible with an interpretation $\Omega$ is terminating.*

The theorem being a more abstract form of Theorem 21, its proof follows exactly the same steps.

**Proof.** Suppose that $G \prec G'$ iff $\phi(G) \prec \phi(G')$. We prove that for each rule $R$ of the GRS, $G \to G'$ implies $G' \prec G$. Indeed, suppose that $G \to_{R,\mu} G'$. Let $P_0$ and $P'_0$ be respectively the basic pattern and the self-application of $R$. Then, there is a graph $C$ such that $G = P_0 + C$, $G' = P'_0 + C$, such that $P_0$ and $P'_0$ are disjoint from $C$. Since $\phi(P'_0) \prec \phi(P)$, we then have $\phi(G') \prec \phi(G)$. ◀

▶ **Example 26.** The triple $\langle \mathfrak{M}, \trianglelefteq, (M_{(-)})^* \rangle$ is an interpretation for 'Follow'.

▶ **Example 27.** Let us come back to the weight analysis. Define $\overline{\omega}(G) = \sum_{p \xrightarrow{e} q \in G} \omega(e)$ with $\omega(\alpha) = 0, \omega(T) = -1, \omega(\beta) = -1$. Then, $\langle \mathbb{R}, <, \overline{\omega}(-) \rangle$ is an interpretation for 'Init' and 'End'.

▶ **Example 28.** Let $\langle X_1, \prec_1, \phi_1\rangle$ be an interpretation for a set of rules $\mathcal{R}_1$, and let $\langle X_2, \prec_2, \phi_2\rangle$ be an interpretation for a set of rules $\mathcal{R}_2$. Suppose that for every rule $R$ in $\mathcal{R}_2$, $G \rightarrow_{R,\mu} G'$ implies $G' \preceq_1 G$ (that is without strict inequality). Then the lexicographic ordering on $X_1 \times X_2$ defined by $(x_1, x_2) \prec_{1,2} (y_1, y_2)$ iff $x_1 \prec_1 y_1$, or $x_1 \preceq_1 y_1$ and $x_2 \prec_2 y_2$, constitutes an interpretation $\langle X_1 \times X_2, \prec_{1,2}, \phi_1 \times \phi_2\rangle$ for $\mathcal{R}_1 \cup \mathcal{R}_2$.

Thus, combining Example 26 and Example 27, we have a proof of the termination of our main Example.

▶ **Corollary 29.** *The GRS given in Subsection 2.1 is terminating.*

▶ **Example 30.** Let $\mathcal{R}$ be a terminating GRS. Then there is an interpretation that "justifies" this fact. Indeed, take $\langle \mathcal{G}, \prec, 1_{\mathcal{G}}\rangle$ with $\prec$ defined to be the transitive closure of the rewriting relation $\rightarrow$. The termination property ensures that the closure leads to an irreflexive relation. The compatibility of $\prec$ with respect to $1_{\mathcal{G}}$ is immediate.

Thus the following corollary.

▶ **Corollary 31.** *A GRS is terminating iff it is compatible with some interpretation.*

## 5 Conclusion

We proposed a new approach based on the theory of regular languages to decide the termination of graph rewriting systems, which does not account for node additions but settles the uniform termination problem for these GRS. We think that there is room to reconsider some old results of this theory under the new light. In particular, we think of profinite topology [19], is a powerful tool that could give us some insight on underlying structure of the orders. In the two cases, we can extend the orders to take into account orders on the edge labels.

As the next natural step, we intend to consider graph rewriting with node creations and that take into account node labels. Moreover, in the experiments mentioned in the introduction about natural language processing, in principle, these two orders should still be sufficient to ensure termination. However, we need to implement these new results for an extensive and complete evaluation.

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