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Abstract

In this paper we deal with infinite-dimensional nonlinear forward complete dynamical systems which are subject to external disturbances. We first extend the well-known Datko lemma to the framework of the considered class of systems. Thanks to this generalization, we provide characterizations of the uniform (with respect to disturbances) local, semi-global, and global exponential stability, through the existence of coercive and non-coercive Lyapunov functionals. The importance of the obtained results is underlined through some applications concerning 1) exponential stability of nonlinear retarded systems with piecewise constant delays, 2) exponential stability preservation under sampling for semilinear control switching systems, and 3) the link between input-to-state stability and exponential stability of semilinear switching systems.

Keywords: Infinite-dimensional systems, Nonlinear systems, Switching systems, Converse Lyapunov theorems, Exponential stability.

1 Introduction

Various works have been recently devoted to the characterization of the stability of infinite-dimensional systems in Banach spaces through *non-coercive* and *coercive* Lyapunov functionals (see, e.g., [9, 12, 33, 35, 36]). By non-coercive Lyapunov functional, we mean a positive definite functional decaying along the trajectories of the system which satisfies

$$0 < V(x) \leq \alpha(\|x\|), \quad \forall\, x \in X\backslash\{0\},$$

where X is the ambient Banach space and α belongs to the class $\mathcal{K}_\infty$ of continuous increasing bijections from $\mathbb{R}_+$ to $\mathbb{R}_+$. Such a function V would be coercive if there existed $\alpha_0 \in \mathcal{K}_\infty$ such that $V(x) \geq \alpha_0(\|x\|)$ for every $x \in X$. In [36] it has been proved that the existence of a coercive Lyapunov functional V represents a necessary and sufficient condition for the global asymptotic stability for a general class of infinite-dimensional forward complete dynamical systems. On the other hand, the existence of a non-coercive Lyapunov functional does not guarantee global asymptotic stability and some additional regularity assumption on the dynamics is needed (see, e.g., [12, 36]). Converse Lyapunov theorems can be helpful for many applications, such as stability analysis of interconnected systems [10] and for the characterization of input-to-state stability (see, e.g., [17, 42, 45]). Stability results based on non-coercive Lyapunov functionals may be more easily

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applied in practice, while the existence of a coercive Lyapunov functional may be exploited to infer additional information on a stable nonlinear system.

Here, we consider the same class of abstract forward complete dynamical systems, subject to a shift-invariant set of disturbances, as in [36]. The novelty of our approach is that we focus on exponential (instead of asymptotic) stability. For the rest of the paper, the word *uniform* will refer to uniformity with respect to disturbances. We provide theorems characterizing different types of uniform local, semi-global, and global exponential stability, through the existence of non-coercive and coercive Lyapunov functionals. Using a standard converse Lyapunov approach, we prove that uniform semi-global exponential stability is characterized by the existence of a 1-parameter family of Lyapunov functionals, each of them decaying uniformly on a bounded set, while the union of all such bounded sets is equal to the entire Banach space X. Concerning the non-coercive case, we first give a generalization of the Datko lemma [5, 37]. Recall that the latter characterizes the exponential behavior of a linear C_0-semigroup in a Banach space in terms of a uniform estimate of the L^p-norm of the solutions. This result has been extended in [14] to the framework of nonlinear semigroups. Here, we generalize the Datko lemma to the considered class of infinite-dimensional forward complete dynamical systems. Thanks to such a generalization, we prove that the existence of a non-coercive Lyapunov functional is sufficient, under a uniform growth estimate on the solutions of the system, for the uniform exponential stability. The importance of the obtained results is underlined through some applications as described in the sequel.

Retarded functional differential equations form an interesting class of infinite-dimensional systems that we cover by our approach. Converse Lyapunov theorems have been developed for systems described by retarded and neutral functional differential equations (see, e.g., [18, 41]). Such results have been recently extended in [9] to switching linear retarded systems through coercive and non-coercive Lyapunov characterizations. After representing a nonlinear retarded functional differential equation as an abstract forward complete dynamical system, all the characterizations of uniform exponential stability provided in the first part of the paper can be applied to this particular class of infinite-dimensional systems. In particular, we characterize the uniform global exponential stability of a retarded functional differential equation in terms of the existence of a non-coercive Lyapunov functional.

Another interesting problem when dealing with a continuous-time model is the practical implementation of a designed feedback control. Indeed, in practice, due to numerical and technological limitations (sensors, actuators, and digital interfaces), a continuous measurement of the output and a continuous implementation of a feedback control are impossible. This means that the implemented input is, for almost every time, different from the designed controller. Several methods have been developed in the literature of ordinary differential equations for sampled-data observer design under discrete-time measurements (see, e.g., [2, 19, 24, 29]), and for sampled-data control design guaranteeing a globally stable closed-loop system (see, e.g., [1, 13]). Apart from time-delays systems (see, e.g., [7, 20, 39] for sampled-data control and [29, 30] for sampled-data observer design), few results exist for infinite-dimensional systems. The difficulties come from the fact that the developed methods do not directly apply to the infinite-dimensional case, for which even the well-posedness of sampled-data control dynamics is not obvious (see, e.g., [21] for more details). Some interesting results have been obtained for infinite-dimensional linear systems [21, 23, 47]. In the nonlinear case no standard methods have been developed and the problem is treated case by case [22]. Here, we focus on the particular problem of feedback stabilization under sampled output measurements of an abstract semilinear infinite-dimensional system. In particular, we consider the dynamics

$$\dot{x}(t) = Ax(t) + f_{\sigma(t)}(x(t), u(t)), \quad t \geq 0, \tag{1}$$

where $x(t) \in X$, U is a Banach space, $u \in U$ is the input, A is the infinitesimal generator of a C_0-semigroup of bounded linear operators $(T_t)_{t\geq 0}$ on X, $\sigma : \mathbb{R}_+ \to \mathcal{Q}$ is a piecewise constant switching function, and $f_q : X \times U \to X$ is a Lipschitz continuous nonlinear operator, uniformly with respect to $q \in \mathcal{Q}$. Assume that only discrete output measurements are available

$$y(t) = x(t_k), \quad \forall\, t \in [t_k, t_{k+1}),\ \forall\, k \geq 0, \tag{2}$$

where $(t_k)_{k\geq 0}$ denotes the increasing sequence of sampling times. It is well known that, in general, no feedback of the type $u(t) = K(y(t))$ stabilizes system (1). Moreover, suppose that system (1) in closed-loop with

$$u(t) = K(x(t)), \quad \forall\, t \geq 0, \tag{3}$$

where $K : X \to U$ is a globally Lipschitz function satisfying $K(0) = 0$, is uniformly semi-globally exponentially stable. Using our converse Lyapunov theorem, we show that if the maximal sampling period is small enough then, under some additional conditions, system (1) in closed-loop with the predictor-based sampled-data control

$$u(t) = T_{t-t_k} y(t_k), \quad \forall\, t \in [t_k, t_{k+1}),\ \forall\, k \geq 0, \tag{4}$$

is uniformly locally exponentially stable in each ball around the origin. Furthermore, if the closed loop system (1)-(3) is uniformly globally exponentially stable, then the same property holds for the closed loop system (1)-(4), under sufficiently small sampling period. We give an example of a wave equation (see, e.g., [4, 26]) showing the applicability of our result.

In recent years, the problem of characterizing input-to-state stability (ISS) for infinite-dimensional systems has attracted a particular attention. Roughly speaking, the ISS property, introduced in [44] for ordinary differential equations, means that the trajectories of a perturbed system eventually approach a neighborhood of the origin whose size is proportional to the magnitude of the perturbation. This concept has been widely studied in the framework of complex systems such as switching systems (see, e.g., [25] and references therein), time-delay systems (see, e.g., [40, 46, 49] and references therein), and abstract infinite-dimensional systems (see, e.g., [32, 34]). For example, in [34] a converse Lyapunov theorem characterizing the input-to-state stability of a locally Lipschitz dynamics through the existence of a locally Lipschitz continuous coercive ISS Lyapunov functional is given. Recently in [16] it has been shown that, under regularity assumptions on the dynamics, the existence of non-coercive Lyapunov functionals implies input-to-state stability. Here, we provide a result of ISS type, proving that the input-to-state map has finite gain, under the assumption that the unforced system corresponding to (1) (i.e., with $u \equiv 0$) is uniformly globally exponentially stable.

The paper is organized as follows. Section 2 presents the problem statement with useful notations and definitions. In Section 3 we state our main results, namely three Datko-type theorems for uniform local, semi-global, and global exponential stability, together with direct and converse Lyapunov theorems. In Section 4 we compare the proposed Lyapunov theorems with the current state of art. The applications are given in Section 5. In Section 6 we consider an example of a damped wave equation. The proofs are postponed to Section 7.

1.1 Notations

By $(X, \|\cdot\|)$ we denote a Banach space with norm $\|\cdot\|$ and by $B_X(x, r)$ the closed ball in X of center $x \in X$ and radius r. By $\mathbb{R}$ we denote the set of real numbers and by $|\cdot|$ the Euclidean norm of a real vector. We use $\mathbb{R}_+$ and $\mathbb{R}_+^\star$ to denote the sets of non-negative and positive real numbers respectively. A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{K}$ if it is continuous, increasing, and satisfies $\alpha(0) = 0$; it is said to be of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. A continuous function $\kappa : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{KL}$ if $\kappa(\cdot, t)$ is of class $\mathcal{K}$ for each $t \geq 0$ and, for each $s \geq 0$, $\kappa(s, \cdot)$ is nonincreasing and converges to zero as t tends to $+\infty$.

2 Problem statement

In this paper we consider a forward complete dynamical system evolving in a Banach space X. Let us recall the following definition, proposed in [36].

Definition 1. *Let $\mathcal{Q}$ be a nonempty set. Denote by $\mathcal{S}$ a set of functions $\sigma : \mathbb{R}_+ \to \mathcal{Q}$ satisfying the following conditions:*

a) $\mathcal{S}$ is closed by time-shift, i.e., for all $\sigma \in \mathcal{S}$ and all $\tau \geq 0$, the τ-shifted function $\mathbb{T}_\tau \sigma : s \mapsto \sigma(\tau + s)$ belongs to $\mathcal{S}$;

b) $\mathcal{S}$ is closed by concatenation, i.e., for all $\sigma_1, \sigma_2 \in \mathcal{S}$ and all $\tau > 0$ the function σ defined by $\sigma \equiv \sigma_1$ over $[0, \tau]$ and by $\sigma(\tau + t) = \sigma_2(t)$ for all $t > 0$, belongs to $\mathcal{S}$.

Let $\phi : \mathbb{R}_+ \times X \times \mathcal{S} \to X$ be a map. The triple $\Sigma = (X, \mathcal{S}, \phi)$ is said to be a forward complete dynamical system *if the following properties hold:*

i) $\forall\ (x, \sigma) \in X \times \mathcal{S}$, it holds that $\phi(0, x, \sigma) = x$;

ii) $\forall\ (x, \sigma) \in X \times \mathcal{S}$, $\forall\ t \geq 0$, and $\forall\ \tilde{\sigma} \in \mathcal{S}$ such $\tilde{\sigma} = \sigma$ over $[0, t]$, it holds that $\phi(t, x, \tilde{\sigma}) = \phi(t, x, \sigma)$;

iii) $\forall\ (x, \sigma) \in X \times \mathcal{S}$, the map $t \mapsto \phi(t, x, \sigma)$ is continuous;

iv) $\forall t, \tau \geq 0$, $\forall\ (x, \sigma) \in X \times \mathcal{S}$, it holds that $\phi(\tau, \phi(t, x, \sigma), \mathbb{T}_t \sigma) = \phi(t + \tau, x, \sigma)$.

We will refer to ϕ as the transition map *of Σ.*

Observe that if Σ is a forward complete dynamical system and $\mathcal{S}$ contains a constant function σ then $(\phi(t, \cdot, \sigma))_{t \geq 0}$ is a strongly continuous nonlinear semigroup, whose definition is recalled below.

Definition 2. *Let $T_t : X \to X$, $t \geq 0$, be a family of nonlinear maps. We say that $(T_t)_{t \geq 0}$ is a strongly continuous nonlinear semigroup if the following properties hold:*

i) $\forall\ x \in X$, $T_0 x = x$;

ii) $\forall\ t_1, t_2 \geq 0$, $T_{t_1} T_{t_2} x = T_{t_1 + t_2} x$;

iii) $\forall\ x \in X$, the map $t \mapsto T_t x$ is continuous.

An example of forward complete dynamical system is given next.

Example 3 (Piecewise constant switching system)**.** *We denote by* PC *the set of piecewise constant $\sigma : \mathbb{R}_+ \to \mathcal{Q}$, and we consider here the case $\mathcal{S} = \mathrm{PC}$. Let $\sigma \in \mathrm{PC}$ be constantly equal to σ_k over $[t_k, t_{k+1})$, with $0 = t_0 < t_1 < \cdots < t_k < t < t_{k+1}$, for $k \geq 0$. With each σ_k we associate the strongly continuous nonlinear semigroup $(T_{\sigma_k}(t))_{t \geq 0} := (\phi(t, \cdot, \sigma_k))_{t \geq 0}$. By concatenating the flows $(T_{\sigma_k}(t))_{t \geq 0}$, one can associate with σ the family of nonlinear evolution operators*

$$T_\sigma(t) := T_{\sigma_k}(t - t_k) T_{\sigma_{k-1}}(t_k - t_{k-1}) \cdots T_{\sigma_1}(t_1),$$

$t \in [t_k, t_{k+1})$. By consequence, system Σ can be identified with the piecewise constant switching system

$$x(t) = T_\sigma(t) x_0,\ x_0 \in X,\ \sigma \in \mathrm{PC}. \tag{5}$$

Thanks to the representation given by (5), this paper extends to the nonlinear case some of the results obtained in [12] on the characterization of the exponential stability of switching linear systems in Banach spaces.

Various notions of uniform (with respect to the functions in $\mathcal{S}$) exponential stability of system Σ are given by the following definition.

Definition 4. *Consider the forward complete dynamical system $\Sigma = (X, \mathcal{S}, \phi)$.*

1. *We say that Σ is* uniformly globally exponentially stable at the origin *(UGES, for short) if there exist $M > 0$ and $\lambda > 0$ such that the transition map ϕ satisfies the inequality*

$$\|\phi(t, x, \sigma)\| \leq M e^{-\lambda t} \|x\|,\ \forall\ t \geq 0,\ \forall\ x \in X,\ \forall\ \sigma \in \mathcal{S}.$$

2. *We say that Σ is* uniformly locally exponentially stable at the origin *(ULES, for short) if there exist $R > 0$, $M > 0$, and $\lambda > 0$ such that the transition map ϕ satisfies the inequality, for every $t \geq 0$, $x \in B_X(0, R)$, and $\sigma \in \mathcal{S}$,*

$$\|\phi(t, x, \sigma)\| \leq M e^{-\lambda t} \|x\|. \tag{6}$$

If inequality (6) holds true for a given $R > 0$ then we say that Σ is uniformly exponentially stable at the origin in $B_X(0, R)$ *(UES in $B_X(0, R)$, for short).*

3. *We say that* Σ *is* uniformly semi-globally exponentially stable at the origin *(USGES, for short) if, for every* $r > 0$ *there exist* $M(r) > 0$ *and* $\lambda(r) > 0$ *such that the transition map* ϕ *satisfies the inequality*

$$\|\phi(t, x, \sigma)\| \leq M(r)e^{-\lambda(r)t}\|x\|, \tag{7}$$

for every $t \geq 0$, $x \in B_X(0, r)$, *and* $\sigma \in \mathcal{S}$.

Remark 5. *Up to modifying* $r \mapsto M(r)$, *one can assume without loss of generality in the definition of USGES that* $r \mapsto \lambda(r)$ *can be taken constant and* $r \mapsto M(r)$ *nondecreasing. Indeed, let us fix* $M := M(1)$ *and* $\lambda := \lambda(1)$. *One has, by definition of* (M, λ),

$$\|\phi(t, x, \sigma)\| \leq Me^{-\lambda t}\|x\|,$$

for every $t \geq 0$, $x \in B_X(0, 1)$, *and* $\sigma \in \mathcal{S}$. *For* $R > 1$, *by using* (7) *there exists* t_R *such that, for every* $x \in B_X(0, R)$ *with* $\|x\| \geq 1$, *one has for* $t \geq t_R$,

$$M(R)e^{-\lambda(R)t_R}R = 1, \ \ \|\phi(t, x, \sigma)\| \leq \frac{\|x\|}{R} \leq \min\{1, \|x\|\}.$$

This implies that, for every $t \geq t_R$, $x \in B_X(0, R)$, *and* $\sigma \in \mathcal{S}$,

$$\|\phi(t, x, \sigma)\| \leq Me^{-\lambda(t-t_R)}\|\phi(t_R, x, \sigma)\| \leq Me^{-\lambda(t-t_R)}\|x\|.$$

By setting

$$\widehat{M}(R) = \begin{cases} M & R \leq 1, \\ \max\{M(R), M(R)e^{|\lambda-\lambda(R)|t_R}, Me^{\lambda t_R}\} & R > 1, \end{cases}$$

one has that $\|\phi(t, x, \sigma)\| \leq \widehat{M}(r)e^{-\lambda t}\|x\|$, *for every* $t \geq 0$, $x \in B_X(0, r)$, *and* $\sigma \in \mathcal{S}$. *Finally, we may replace* $r \mapsto \widehat{M}(r)$ *with the nondecreasing function* $r \mapsto \inf_{\rho \geq r} \widehat{M}(\rho)$.

The property of semi-global exponential stability introduced in Definition 4 turns out to be satisfied by some interesting class of infinite-dimensional systems, as described in the following two examples.

Example 6. *For* $L > 0$, *let* $\Omega = (0, L)$ *and consider the controlled Korteweg–de Vries (KdV) equation*

$$\begin{cases} \eta_t + \eta_x + \eta_{xxx} + \eta\eta_x + \rho(t, x, \eta) = 0 & (x, t) \in \Omega \times \mathbb{R}_+, \\ \eta(t, 0) = \eta(t, L) = \eta_x(t, L) = 0 & t \in \mathbb{R}_+, \\ \eta(0, x) = \eta_0(x) & x \in \Omega, \end{cases} \tag{8}$$

where $\rho : \mathbb{R}_+ \times \Omega \times \mathbb{R} \to \mathbb{R}$ *is a sufficiently regular nonlinear function. The case* $\rho \equiv 0$ *is a well known model describing waves on shallow water surfaces [15]. The controllability and stabilizability properties of* (8) *have been extensively studied in the literature (see, e.g., [31, 43]). In the case where the feedback control is of the form* $\rho(t, x, \eta) = a(x)\eta$, *for some non-negative function* $a(\cdot)$ *having nonempty support in* Ω, *system* (8) *is globally exponentially stable in* $X = L^2(0, L)$. *In [27] the authors prove that, when a saturation is introduced in the feedback control* ρ, *the system is only semi-globally exponentially stable in* X.

Example 7. *Consider the 1D wave equation with boundary damping*

$$\begin{cases} \psi_{tt} - \Delta\psi = 0 & (x, t) \in (0, 1) \times \mathbb{R}_+, \\ \psi(0, t) = 0 & t \in \mathbb{R}_+, \\ \psi_x(1, t) = -\sigma(t, \psi_t(1, t)) & t \in \mathbb{R}_+, \\ \psi(0) = \psi_0, \psi_t(0) = \psi_1 & x \in (0, 1), \end{cases} \tag{9}$$

where $\sigma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ *is continuous. This system is of special interest when the damping term* $\sigma(t, \psi_t(1, t))$ *represents a nonlinear feedback control. Once again, different types of stability can be established (for global and semi-global exponential stability, see e.g., [28, 3]). In particular, if* σ *is a nonlinearity of saturation type, only semi-global exponential stability holds true in*

$$X = \{(\psi_0, \psi_1) \mid \psi_0(0) = 0, \ \psi_0' \text{ and } \psi_1 \in L^\infty(0, 1)\}.$$

The systems considered in Examples 6 and 7 do not depend on $\sigma \in \mathcal{S}$ as in Definition 1. In Section 6 we introduce and study variants of such systems in a switching framework.

3 Main results

3.1 Datko-type theorems

In this section we give Datko-type theorems [5] for an abstract forward complete dynamical system Σ. The uniform (local, semi-global, and global) exponential stability is characterized in terms of the L^p-norm of the trajectories of the system. This provides a generalization of the results obtained in [14] for nonlinear semigroups.

The following theorem characterizes the local exponential stability of system Σ.

Theorem 8. *Consider a forward complete dynamical system* $\Sigma = (X, \mathcal{S}, \phi)$. *Let* $t_1, G_0 > 0$, *and* β *be a function of class* $\mathcal{K}_\infty$ *such that* $\limsup_{r \downarrow 0} \frac{\beta(r)}{r}$ *is finite and*

$$\|\phi(t, x, \sigma)\| \leq G_0 \beta(\|x\|), \quad \forall\, t \in [0, t_1],\ \forall\, x \in X,\ \forall\, \sigma \in \mathcal{S}. \tag{10}$$

The following statements are equivalent

i) *System* Σ *is ULES;*

ii) *for every* $p > 0$ *there exist a nondecreasing function* $k : \mathbb{R}_+ \to \mathbb{R}_+$ *and* $R > 0$ *such that*

$$\int_0^{+\infty} \|\phi(t, x, \sigma)\|^p dt \leq k(\|x\|)^p \|x\|^p, \tag{11}$$

for every $x \in B_X(0, R)$ *and* $\sigma \in \mathcal{S}$;

iii) *there exist* $p > 0$, $k : \mathbb{R}_+ \to \mathbb{R}_+$ *nondecreasing, and* $R > 0$ *such that* (11) *holds true.*

Remark 9. *Observe that hypothesis* (10) *in Theorem 8 is global over* X. *Indeed, we do not know if the stability at 0 may be deduced from inequality* (11) *if one restricts* (10) *to a ball* $B_X(0, r)$.

The following theorem characterizes the semi-global exponential stability of system Σ.

Theorem 10. *Consider a forward complete dynamical system* $\Sigma = (X, \mathcal{S}, \phi)$. *Let* $t_1, G_0 > 0$, *and* β *be a function of class* $\mathcal{K}_\infty$ *such that* $\limsup_{r \downarrow 0} \frac{\beta(r)}{r}$ *is finite and*

$$\|\phi(t, x, \sigma)\| \leq G_0 \beta(\|x\|), \quad \forall\, t \in [0, t_1],\ \forall\, x \in X,\ \forall\, \sigma \in \mathcal{S}. \tag{12}$$

The following statements are equivalent

i) *System* Σ *is USGES;*

ii) *for every* $p > 0$ *there exists a nondecreasing function* $k : \mathbb{R}_+ \to \mathbb{R}_+$ *such that, for every* $x \in X$ *and* $\sigma \in \mathcal{S}$

$$\int_0^{+\infty} \|\phi(t, x, \sigma)\|^p dt \leq k(\|x\|)^p \|x\|^p; \tag{13}$$

iii) *there exist* $p > 0$ *and* $k : \mathbb{R}_+ \to \mathbb{R}_+$ *nondecreasing such that* (13) *holds true.*

The particular case of uniformly globally exponentially stable systems is considered in the following theorem.

Theorem 11. *Consider a forward complete dynamical system $\Sigma = (X, \mathcal{S}, \phi)$. Let $t_1 > 0$ and $G_0 > 0$ be such that*

$$\|\phi(t,x,\sigma)\| \leq G_0\|x\|, \ \forall\ t \in [0,t_1],\ \forall\ x \in X,\ \forall\ \sigma \in \mathcal{S}. \tag{14}$$

The following statements are equivalent

i) *System Σ is UGES;*

ii) *for every $p > 0$ there exists $k > 0$ such that*

$$\int_0^{+\infty} \|\phi(t,x,\sigma)\|^p dt \leq k^p\|x\|^p, \tag{15}$$

for every $x \in X$ and $\sigma \in \mathcal{S}$;

iii) *there exist $p, k > 0$ such that* (15) *holds true.*

Remark 12. *By the shift-invariance properties given by items a) and iv) of Definition 1, it is easy to see that* (14) *implies*

$$\|\phi(t,x,\sigma)\| \leq Me^{\lambda t}\|x\|, \ \forall\ t \geq 0,\ \forall\ x \in X,\ \forall\ \sigma \in \mathcal{S}, \tag{16}$$

where $M = G_0$ and $\lambda = \max\{0, \log(\frac{G_0}{t_1})\}$. Notice that inequality (16) *is a nontrivial requirement on system Σ. Even in the linear case, and even if* (16) *is satisfied for each constant $\sigma \equiv \sigma_c$, uniformly with respect to σ_c, it does not follow that a similar exponential bound holds for the corresponding system Σ (see [12, Example 1]).*

3.2 Lyapunov characterization of exponential stability

In this section we characterize the exponential stability of a forward complete dynamical system through the existence of a Lyapunov functional. First, let us recall the definition of Dini derivative of a functional $V : X \to \mathbb{R}_+$.

Definition 13. *Consider a forward complete dynamical system $\Sigma = (X, \mathcal{S}, \phi)$. The upper and lower Dini derivatives $\overline{D}_\sigma V : X \to \mathbb{R} \cup \{\pm\infty\}$ and $\underline{D}_\sigma V : X \to \mathbb{R} \cup \{\pm\infty\}$ of a functional $V : X \to \mathbb{R}_+$ are defined, respectively, as*

$$\overline{D}_\sigma V(x) = \limsup_{h \downarrow 0} \frac{1}{h}\left(V(\phi(h,x,\sigma)) - V(x)\right),$$

and

$$\underline{D}_\sigma V(x) = \liminf_{h \downarrow 0} \frac{1}{h}\left(V(\phi(h,x,\sigma)) - V(x)\right),$$

where $x \in X$ and $\sigma \in \mathcal{S}$.

Remark 14. *When $\mathcal{S}$ contains* PC*, we can associate with every $q \in \mathcal{Q}$ the upper and lower Dini derivatives $\overline{D}_q V$ and $\underline{D}_q V$ corresponding to $\sigma \equiv q$. Notice that for every $\sigma \in \mathrm{PC}$ and sufficiently small $h > 0$, we have $\sigma_{|(0,h)} \equiv q$, for some $q \in \mathcal{Q}$. By consequence, we have $\overline{D}_\sigma V(\varphi) = \overline{D}_q V(\varphi)$ and $\underline{D}_\sigma V(\varphi) = \underline{D}_q V(\varphi)$.*

The regularity of a Lyapunov functional associated with an exponentially stable forward complete dynamical system $\Sigma = (X, \mathcal{S}, \phi)$ is recovered, in our results, from the regularity of the transition map ϕ. The $\mathcal{S}$-uniform continuity of the transition map ϕ with respect to the initial condition is defined as follows.

Definition 15. *We say that the transition map ϕ of $\Sigma = (X, \mathcal{S}, \phi)$ is $\mathcal{S}$-uniformly continuous if, for any $\bar{t} > 0$, $x \in X$, and $\varepsilon > 0$, there exists $\eta > 0$ such that*

$$\|\phi(t,x,\sigma) - \phi(t,y,\sigma)\| \leq \varepsilon,$$

for every $t \in [0,\bar{t}]$, $y \in B_X(x,\eta)$, and $\sigma \in \mathcal{S}$.

Similarly, the notion of $\mathcal{S}$-uniform Lipschitz continuity of the transition map is given by the following definition.

Definition 16. *We say that the transition map ϕ of $\Sigma = (X, \mathcal{S}, \phi)$ is $\mathcal{S}$-uniformly Lipschitz continuous (respectively, $\mathcal{S}$-uniformly Lipschitz continuous on bounded sets) if, for any $\bar{t} > 0$ (respectively, $\bar{t} > 0$ and $R > 0$), there exists $l(\bar{t}) > 0$ (respectively, $l(\bar{t}, R) > 0$) such that*

$$\|\phi(t, x, \sigma) - \phi(t, y, \sigma)\| \leq l(\bar{t})\|x - y\|,$$

for every $t \in [0, \bar{t}]$, $x, y \in X$, and $\sigma \in \mathcal{S}$ (respectively,

$$\|\phi(t, x, \sigma) - \phi(t, y, \sigma)\| \leq l(\bar{t}, R)\|x - y\|,$$

for every $t \in [0, \bar{t}]$, $x, y \in B_X(0, R)$, and $\sigma \in \mathcal{S}$).

The following theorem shows that the existence of a non-coercive Lyapunov functional is sufficient for proving the uniform exponential stability of the forward complete dynamical system Σ, provided that inequality (10) holds true.

Theorem 17. *Consider a forward complete dynamical system $\Sigma = (X, \mathcal{S}, \phi)$. Let $t_1, G_0 > 0$, and β be a function of class $\mathcal{K}_\infty$ such that $\limsup_{r\downarrow 0} \frac{\beta(r)}{r}$ is finite and*

$$\|\phi(t, x, \sigma)\| \leq G_0\beta(\|x\|), \quad \forall\ t \in [0, t_1],\ \forall\ x \in X,\ \forall\ \sigma \in \mathcal{S}. \tag{17}$$

Then,

i) if there exist $R > 0$, $V : B_X(0, R) \to \mathbb{R}_+$, and $p, c > 0$ such that, for every $x \in B_X(0, R)$ and $\sigma \in \mathcal{S}$,

$$V(x) \leq c\|x\|^p, \tag{18}$$
$$\underline{D}_\sigma V(x) \leq -\|x\|^p, \tag{19}$$

and $V(\phi(\cdot, x, \sigma))$ is continuous from the left at every $t > 0$ such that $\phi(t, x, \sigma) \in B_X(0, R)$, then system Σ is ULES;

ii) if i) holds true for every $R > 0$ with $V = V_R$, $c = c_R$ and $p = p_R$ and

$$\limsup_{R\to+\infty} \beta^{-1}\left(\frac{R}{G_0}\right) \min\left\{1, \left(\frac{t_1}{c_R}\right)^{\frac{1}{p_R}}\right\} = +\infty, \tag{20}$$

then system Σ is USGES;

iii) if β in (17) is equal to the identity function and there exist $p, c > 0$ and a functional $V : X \to \mathbb{R}_+$ such that, for every $x \in X$ and $\sigma \in \mathcal{S}$, the map $t \mapsto V(\phi(t, x, \sigma))$ is continuous from the left, and V satisfies inequalities (18)-(19) in X, then system Σ is UGES.

The following theorem states that the existence of a coercive Lyapunov functional is necessary for the uniform exponential stability of a forward complete dynamical system.

Theorem 18. *Consider a forward complete dynamical system $\Sigma = (X, \mathcal{S}, \phi)$ and assume that the transition map ϕ is $\mathcal{S}$-uniformly continuous. If Σ is USGES then, for every $r > 0$ there exist $\underline{c}_r, \overline{c}_r > 0$ and a continuous functional $V_r : X \to \mathbb{R}_+$, such that*

$$\underline{c}_r\|x\| \leq V_r(x) \leq \overline{c}_r\|x\|, \quad \forall\ x \in B_X(0, r), \tag{21}$$
$$\overline{D}_\sigma V_r(x) \leq -\|x\|, \quad \forall\ x \in B_X(0, r),\ \forall\ \sigma \in \mathcal{S}, \tag{22}$$
$$V_r = V_R \text{ on } X, \quad \forall R > 0 \text{ such that } \lambda(r) = \lambda(R) \text{ and } M(r) = M(R), \tag{23}$$

where $\lambda(\cdot), M(\cdot)$ are as in (7). Moreover, in the case where the transition map ϕ is $\mathcal{S}$-uniformly Lipschitz continuous (respectively, $\mathcal{S}$-uniformly Lipschitz continuous on bounded sets), V_r can be taken Lipschitz continuous (respectively, Lipschitz continuous on bounded sets).

To conclude this section, we state the following corollary which characterizes the uniform global exponential stability of a forward complete dynamical system, completing Item iii) of Theorem 17.

Corollary 19. *Consider a forward complete dynamical system $\Sigma = (X, \mathcal{S}, \phi)$. Assume that the transition map ϕ is $\mathcal{S}$-uniformly continuous. If there exist $t_1 > 0$ and $G_0 > 0$ such that*

$$\|\phi(t, x, \sigma)\| \leq G_0 \|x\|, \quad \forall\ t \in [0, t_1],\ \forall\ x \in X,\ \forall\ \sigma \in \mathcal{S}, \tag{24}$$

then the following statements are equivalent:

i) System Σ is UGES;

ii) there exists a continuous functional $V : X \to \mathbb{R}_+$ and positive reals p, $\underline{c}$, and $\overline{c}$ such that

$$\underline{c}\|x\|^p \leq V(x) \leq \overline{c}\|x\|^p, \quad \forall\ x \in X,$$

and

$$\overline{D}_\sigma V(x) \leq -\|x\|^p, \quad \forall\ x \in X,\ \forall\ \sigma \in \mathcal{S}; \tag{25}$$

iii) there exist a functional $V : X \to \mathbb{R}_+$ and positive reals p and c such that, for every $x \in X$ and $\sigma \in \mathcal{S}$, the map $t \mapsto V(\phi(t, x, \sigma))$ is continuous from the left, inequality (25) is satisfied and the following inequality holds

$$V(x) \leq c\|x\|^p, \quad \forall\ x \in X.$$

Proof. The fact that item i) implies ii) is a straightforward consequence of Theorem 18, using, in particular, (23). Moreover ii) clearly implies iii). Finally, iii) implies i), as follows from Theorem 17. □

4 Discussion: comparison with the current state of the art

We compare here the results stated in Section 3.2 with some interesting similar results, obtained recently in [35], concerning the Lyapunov characterization of the uniform global asymptotic stability of a forward complete dynamical system. In order to make this comparison, we briefly recall some definitions and assumptions from [35].

Definition 20. *We say that a forward complete dynamical system $\Sigma = (X, \mathcal{S}, \phi)$ is* uniformly globally asymptotically stable at the origin *(UGAS, for short) if there exists a function κ of class $\mathcal{KL}$ such that the transition map ϕ satisfies the inequality*

$$\|\phi(t, x, \sigma)\| \leq \kappa(\|x\|, t), \quad \forall\ t \geq 0,\ \forall\ x \in X,\ \forall\ \sigma \in \mathcal{S}.$$

The notion of *robust forward completeness* of system Σ is given by the following.

Definition 21. *The forward complete dynamical system $\Sigma = (X, \mathcal{S}, \phi)$ is said to be* robustly forward complete *(RFC, for short) if for any $C > 0$ and any $\tau > 0$ it holds that*

$$\sup_{\|x\| \leq C, t \in [0,\tau], \sigma \in \mathcal{S}} \|\phi(t, x, \sigma)\| < \infty.$$

Notice that RFC property of Σ is equivalent to inequality (17), although it does not necessarily imply that $\limsup_{r \downarrow 0} \frac{\beta(r)}{r}$ is finite.

The notion of *robust equilibrium point*, which may be seen as a form of weak stability at the origin, is given as follows.

Definition 22. *We say that $0 \in X$ is a* robust equilibrium point *(REP, for short) of the forward complete dynamical system $\Sigma = (X, \mathcal{S}, \phi)$ if for every $\varepsilon, h > 0$, there exists $\delta = \delta(\varepsilon, h) > 0$, so that*

$$\|x\| \leq \delta \implies \|\phi(t, x, \sigma)\| \leq \varepsilon, \quad \forall t \in [0, h], \forall \sigma \in \mathcal{S}.$$

One of the main results obtained in [35] relates the UGAS property with the existence of a non-coercive Lyapunov functional, i.e., a continuous function $V : X \to \mathbb{R}_+$ satisfying $V(0) = 0$ and the two inequalities

$$0 < V(x) \leq \alpha_1(\|x\|), \quad \forall\, x \in X\backslash\{0\},$$

and

$$\overline{D}_\sigma V(x) \leq -\alpha_2(\|x\|), \quad \forall\, x \in X,\ \forall\, \sigma \in \mathcal{S},$$

where $\alpha_1 \in \mathcal{K}_\infty$ and $\alpha_2 \in \mathcal{K}$. This is formulated by the following theorem.

Theorem 23 ([35])**.** *Consider a forward complete dynamical system $\Sigma = (X, \mathcal{S}, \phi)$ and assume that Σ is robustly forward complete and that 0 is a robust equilibrium point of Σ. If Σ admits a non-coercive Lyapunov functional, then it is UGAS.*

The existence of a non-coercive Lyapunov functional V is not sufficient in order to get the uniform global asymptotic stability of system Σ. Indeed, as shown in [35, Example 6.1], the existence of such a V does not imply either the RFC or REP property, hence the necessity of both assumptions. Even in the linear case, an RFC like condition is required (see [12, Remark 4]).

Let us compare Theorem 23 with our corresponding result for UGES. As we already noticed, RFC is equivalent to (17), and in particular it is ensured by the stronger condition (24). Moreover, it is easy to check that (24) also implies the REP property. Thus, the hypotheses of Theorem 17 imply those of Theorem 23. As a counterpart, a stronger stability property is obtained (namely, UGES). Also notice that our theorem relaxes the requirements on the functional V, since the lower Dini derivatives, instead of the upper one, is used and discontinuities of V along the trajectories of Σ are allowed.

5 Applications

5.1 Nonlinear retarded systems with piecewise constant delays

Let $r \geq 0$ and set $\mathcal{C} = \mathcal{C}([-r, 0], \mathbb{R}^n)$, the set of continuous functions from $[-r, 0]$ to $\mathbb{R}^n$. Consider the nonlinear retarded system

$$\begin{array}{ll} \dot{x}(t) = f_{\sigma(t)}(x_t), & t \geq 0, \\ x(\theta) = \varphi(\theta), & \theta \in [-r, 0], \end{array} \tag{26}$$

where $x(t) \in \mathbb{R}^n$, $\varphi \in \mathcal{C}$, $x_t : [-r, 0] \to \mathbb{R}^n$ is the standard notation for the history function defined by

$$x_t(\theta) = x(t + \theta), \quad -r \leq \theta \leq 0,$$

$\sigma : \mathbb{R}_+ \to \mathcal{Q}$ is a piecewise constant function, and $f_q : \mathcal{C} \to \mathbb{R}^n$ is a continuous functional such that $f_q(0) = 0$ for all $q \in \mathcal{Q}$.

For every $q \in \mathcal{Q}$ and $\varphi \in \mathcal{C}$, we assume that there exists a unique solution x over $[-r, +\infty)$ of (26) with $\sigma(t) = q$ for every $t \geq 0$. This defines a family $(T_q(t))_{t \geq 0}$ of nonlinear maps from $\mathcal{C}$ into itself by setting

$$T_q(t)\varphi = x_t,$$

for $t \geq 0$. According to [11, 48], $(T_q(t))_{t \geq 0}$ is a strongly continuous semigroup of nonlinear operators on $\mathcal{C}$. We denote by $\Sigma_r = (\mathcal{C}, \mathrm{PC}, \phi)$ the corresponding forward complete dynamical system constructed as in Example 3.

As a consequence of the switching representation of the nonlinear time-varying delay system (26), the results of the previous section (in particular Theorem 17 and Corollary 19) apply to system Σ_r. Let us explicitly provide an application of Corollary 19.

Theorem 24. *Let $L > 0$ be such that*

$$|f_q(\psi_1) - f_q(\psi_2)| \leq L\|\psi_1 - \psi_2\|,\ \forall\, \psi_1, \psi_2 \in \mathcal{C},\ \forall\, q \in \mathcal{Q}. \tag{27}$$

The following statements are equivalent:

i) System Σ_r *is UGES;*

ii) there exists a continuous functional $V : X \to \mathbb{R}_+$ *and positive reals* p, $\underline{c}$, *and* $\overline{c}$ *such that*

$$\underline{c}\|\psi\|^p \leq V(\psi) \leq \overline{c}\|\psi\|^p, \quad \forall\ \psi \in \mathcal{C},$$

and

$$\overline{D}_q V(\psi) \leq -\|\psi\|^p, \quad \forall\ \psi \in \mathcal{C},\ \forall\ q \in \mathcal{Q}; \tag{28}$$

iii) there exist a functional $V : \mathcal{C} \to \mathbb{R}_+$ *and positive reals* p *and* c *such that, for every* $\psi \in \mathcal{C}$ *and* $q \in \mathcal{Q}$, *the map* $t \mapsto V(T_q(\cdot)\psi)$ *is continuous from the left, inequality* (28) *is satisfied, and*

$$V(\psi) \leq c\|\psi\|^p, \quad \forall\ \psi \in \mathcal{C}.$$

Proof. The proof is based on Corollary 19. In order to apply it, we have to prove that the transition map is PC-uniformly continuous and that (24) holds true. Using (27), we easily get

$$\|\phi(t,\varphi_1,\sigma)-\phi(t,\varphi_2,\sigma)\| \leq L\int_0^t \|\phi(s,\varphi_1,\sigma)-\phi(s,\varphi_2,\sigma)\|ds, \quad \forall\ t \geq 0,$$

which implies

$$\|\phi(t,\varphi_1,\sigma)-\phi(t,\varphi_2,\sigma)\| \leq e^{Lt}\|\varphi_1-\varphi_2\|, \quad \forall\ t \geq 0.$$

Hence, the transition map ϕ is PC-uniformly Lipschitz continuous and, using the fact that $f_q(0) = 0$ for every $q \in \mathcal{Q}$, we have that (24) holds true with $G_0 = e^{Lt_1}$. □

5.2 Predictor-based sampled data exponential stabilization

Let X, U be two Banach spaces. Consider the semilinear control system

$$\begin{cases} \dot{x}(t) = Ax(t) + f_{\sigma(t)}(x(t),u(t)), & t \geq 0, \\ x(0) = x_0 \in X, \end{cases} \tag{29}$$

where $u \in \mathcal{C}(\mathbb{R}_+, U)$ is the control input, A is the infinitesimal generator of a C_0-group $(T_t)_{t\in\mathbb{R}}$ of bounded linear operators on X, $\sigma : \mathbb{R}_+ \to \mathcal{Q}$ is a piecewise constant function, and $f_q : X \times U \to X$, for $q \in \mathcal{Q}$, is a Lipschitz continuous nonlinear operator, with Lipschitz constant $L_f > 0$ independent of q, such that $f_q(0,0) = 0$. Let $K : X \to U$ be a globally Lipschitz function with Lipschitz constant $L_K > 0$, satisfying $K(0) = 0$, and consider system (29) in closed-loop with

$$u(t) = K(x(t)), \quad \forall\ t \geq 0. \tag{30}$$

Observe that, since A is the infinitesimal generator of a C_0-group, the following inequality holds for the corresponding induced norm: there exist $\Gamma, \omega > 0$ such that

$$\|T_t\| \leq \Gamma e^{\omega|t|}, \quad \forall\ t \in \mathbb{R}. \tag{31}$$

The aim of this section is to show the usefulness of our converse theorems in the study of exponential stability preservation under sampling for the semilinear control switching system (29). For convenience of the reader we give the following definition.

Definition 25. *Let* $0 < s_1 < \cdots < s_k < \cdots$ *be an increasing sequence of times such that* $\lim_{k\to+\infty} s_k = +\infty$. *The instants* s_k *are called* sampling instants *and the quantity*

$$\delta = \sup_{k\geq 0}(s_{k+1} - s_k)$$

is called the maximal sampling time. *By* predictor-based sampled data controller *we mean a feedback* $u(\cdot)$ *of the type*

$$u(t) = K(T_{t-s_k}x(s_k)), \quad \forall\ t \in [s_k, s_{k+1}),\ \forall\ k \geq 0. \tag{32}$$

5.2.1 Converse Lyapunov result for the closed-loop system (29)–(30)

For every $\sigma \in \mathrm{PC}$, there exist an increasing sequence of times $(t_k)_{k\geq 0}$ and a sequence $(q_k)_{k\geq 0}$ taking values in $\mathcal{Q}$ such that $t_0 = 0$, $\lim_{k\to\infty} t_k = \infty$ and σ is constantly equal to q_k on $[t_k, t_{k+1})$ for every $k \geq 0$. For every $q \in \mathcal{Q}$ and $x_0 \in X$, letting $\sigma(t) \equiv q$, there exists a unique mild solution of (29) over $[0, +\infty)$, i.e., a continuous function $x(\cdot, x_0, q)$ satisfying

$$x(t, x_0, q) = T_t x_0 + \int_0^t T_{t-s} f_q\left(x(s, x_0, q), K(x(s, x_0, q))\right) ds,$$

for every $t \geq 0$. This defines a family $(T_q(t))_{t\geq 0}$ of nonlinear maps by setting $T_q(t)x_0 = x(t, x_0, q)$, for $t \geq 0$. We denote by $\Sigma_0 = (X, \mathrm{PC}, \phi)$ the corresponding forward complete dynamical system constructed as in Example 3. By [6, Corollary 1.6], system Σ_0 is PC-uniformly Lipschitz continuous. As a consequence of Theorem 18, we have the following lemma.

Lemma 26. *Suppose that Σ_0 is USGES. Then, for every $r > 0$, there exist $\underline{c}_r, \overline{c}_r > 0$ and a Lipschitz continuous Lyapunov functional $V_r : X \to \mathbb{R}_+$ such that*

$$\underline{c}_r \|x\| \leq V_r(x) \leq \overline{c}_r \|x\|, \quad \forall\, x \in B_X(0, r),$$

and

$$\overline{D}_q V_r(x) \leq -\|x\|, \quad \forall\, x \in B_X(0, r),\ \forall\, q \in \mathcal{Q}.$$

5.2.2 Predictor-based sampled data feedback

Consider the predictor-based sampled data switching control system

$$\begin{cases} \dot{x}(t) = Ax(t) + f_{\sigma(t)}(x(t), K(T_{t-s_k} x(s_k))),\ s_k \leq t < s_{k+1}, \\ x(0) = x_0, \end{cases} \tag{33}$$

where $(s_k)_{k\geq 0}$ is the increasing sequence of sampling instants. We will say that $x^\Sigma : \mathbb{R}_+ \to X$ is a solution of system (33) if $t \mapsto x^\Sigma(t)$ is continuous and for every $k \geq 0$ and $s_k \leq t < s_{k+1}$ one has

$$x^\Sigma(t) = T_{t-s_k} x^\Sigma(s_k) + \int_0^{t-s_k} T_{t-s_k-s} f_{\sigma(s+s_k)}\left(x^\Sigma(s+s_k), K\left(T_s x^\Sigma(s_k)\right)\right) ds, \tag{34}$$

that is, the restriction of x^Σ on $[s_k, s_{k+1}]$ is a mild solution.

By applying [38, Theorem 6.1.2] on every interval $[s_k, s_{k+1}]$, we deduce the following.

Lemma 27. *For every $x_0 \in X$ and $\sigma \in \mathrm{PC}$, system (33) admits a unique solution $x^\Sigma : \mathbb{R}_+ \to X$.*

For $x_0 \in X$ and $\sigma \in \mathrm{PC}$, following the same reasoning as before, we identify the dynamics of system (33) with the transition map

$$\phi(t, x_0, \sigma) = x^\Sigma(t), \quad t \geq 0,$$

where x^Σ is given by (34), and we denote by $\Sigma = (X, \mathrm{PC}, \phi)$ the corresponding forward complete dynamical system. Notice that Σ depends on the sequence of sampling instants $(s_k)_{k\in\mathbb{N}}$.

Theorem 28. *If system Σ_0 is USGES and*

$$\lim_{r\to\infty} \frac{r}{M(r)} = +\infty, \tag{35}$$

where $M(r)$ is as in (7), then for every $r > 0$ there exists $\delta^\star(r) > 0$ such that Σ is UES in $B_X(0, r)$, provided that the maximal sampling time of $(s_k)_{k\in\mathbb{N}}$ is smaller than $\delta^\star(r)$.

When Σ_0 is uniformly globally exponentially stable, the associated Lyapunov functional is globally uniformly Lipschitz, and by consequence (see the proof of Theorem 28), the following corollary holds true.

Corollary 29. *Suppose that system Σ_0 is UGES. Then there exists $\delta^\star > 0$ such that system Σ is UGES provided that the maximal sampling time of $(s_k)_{k\in\mathbb{N}}$ is smaller than $\delta^\star$.*

5.3 Link between uniform global exponential stability and uniform input-to-state stability

Let X and U be two Banach spaces and consider the control system (29), where A is the infinitesimal generator of a C_0-semigroup of bounded linear operators $(T_t)_{t\geq 0}$ on X, $\sigma \in \mathrm{PC}$, and $f_q : X \times U \to X$, for $q \in \mathcal{Q}$, is a Lipschitz continuous nonlinear operator, with Lipschitz constant $L_f > 0$ independent of q, such that $f_q(0,0) = 0$. We assume here that the set of admissible controls is $L^p(U) := L^p(\mathbb{R}_+, U)$ with $1 \leq p \leq +\infty$. Following the reasoning in Section 5.2 we can define (see, e.g., [6]), for every $x_0 \in X$, $\sigma \in \mathrm{PC}$, and $u \in L^p(U)$, the corresponding trajectory $\phi_u(t, x_0, \sigma)$ on $\mathbb{R}_+$, which is absolutely continuous with respect to t and continuous with respect to $(x_0, u) \in X \times L^p(U)$.

We next provide a result of ISS type in the same spirit as those obtained in [16, 34]. In our particular context (UGES and global Lipschitz assumption) we are able to prove that the input-to-state map $u \mapsto \phi_u(\cdot, 0, \sigma)$ has finite gain.

Theorem 30. *Assume that the forward complete dynamical system (X, PC, ϕ_0) is UGES. Then for every $1 \leq p \leq +\infty$ and $\sigma \in \mathrm{PC}$, the input-to-state map $u \mapsto \phi_u(\cdot, 0, \sigma)$ is well defined as a map from $L^p(U)$ to $L^p(X)$ and has a finite L^p-gain independent of σ, i.e., there exists $c_p > 0$ such that*

$$\|\phi_u(\cdot, 0, \sigma)\|_{L^p(X)} \leq c_p \|u\|_{L^p(U)}, \quad \forall\, u \in L^p(U),\ \forall\, \sigma \in \mathrm{PC}. \tag{36}$$

6 Example: Sample-data exponential stabilization of a switching wave equation

Let Ω be a bounded open domain of class $\mathcal{C}^2$ in $\mathbb{R}^n$, and consider the switching damped wave equation

$$\begin{cases} \frac{\partial^2 \psi}{\partial t^2} - \Delta x + \rho_{\sigma(t)}(\frac{\partial \psi}{\partial t}) = 0 & \text{in } \Omega \times \mathbb{R}_+, \\ \psi = 0 & \text{on } \partial\Omega \times \mathbb{R}_+, \\ \psi(0) = \psi_0, \psi'(0) = \psi_1 & \text{on } \Omega, \end{cases} \tag{37}$$

where $\sigma : \mathbb{R}_+ \to \mathcal{Q}$ is a piecewise constant function and $\rho_q : \mathbb{R} \to \mathbb{R}$, for $q \in \mathcal{Q}$, is a uniformly Lipschitz continuous function satisfying

$$\rho_q(0) = 0, \quad \alpha|v| \leq |\rho_q(v)| \leq \frac{|v|}{\alpha}, \quad \forall\, v \in \mathbb{R},\ \forall\, q \in \mathcal{Q},$$

for some $\alpha > 0$. In the case where $\tilde{\rho}(t, v) := \rho_{\sigma(t)}(v)$ is sufficiently regular, namely a continuous function differentiable on $\mathbb{R}_+ \times (-\infty, 0)$ and $\mathbb{R}_+ \times (0, \infty)$, and $v \mapsto \tilde{\rho}(t, v)$ is nondecreasing, for each initial condition (ψ_0, ψ_1) taken in $H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$ there exists a unique strong solution for (37) in the class $\mathbf{H} = W^{2,\infty}_{\mathrm{loc}}(\mathbb{R}_+, L^2(\Omega)) \cap W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}_+, H_0^1(\Omega)) \cap L^{\infty}_{\mathrm{loc}}(\mathbb{R}_+, H^2(\Omega) \cap H_0^1(\Omega))$ (see [26] for more details). For the switching damped wave equation (37) the existence and uniqueness of a strong solution (in $\mathbf{H}$) is given by concatenation. Defining the energy of the solution of (37) by

$$E(t) = \frac{1}{2} \int_{\Omega} \left(\frac{\partial \psi}{\partial t}^2 + |\nabla \psi|^2 \right) dx,$$

we can prove, following the same lines of the proof of [26, Theorem 1], that the energy of the solutions in $\mathbf{H}$ decays uniformly (with respect to the initial condition) exponentially to zero as

$$E(t) \leq E(0) \exp\left(1 - \mu t\right), \quad \forall\, t \geq 0,$$

for some $\mu > 0$ that depends only on α. Let X be the Banach space $H_0^1(\Omega) \times L^2(\Omega)$ endowed with the norm

$$\|x\| = \|\nabla x_1\|^2_{L^2(\Omega)} + \|x_2\|^2_{L^2(\Omega)},$$

and let A be the linear operator defined on X by

$$D(A) = \Big\{ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X \mid x_1 \in H^2(\Omega) \cap H_0^1(\Omega), \\ x_2 \in H_0^1(\Omega) \Big\},$$

$$A = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix},$$

where I is the identity operator and Δ denotes the Laplace operator. It is well known that $D(A)$ is dense in X and that A is the infinitesimal generator of a C_0-group of bounded linear operators $(T_t)_{t\in\mathbb{R}}$ on X satisfying

$$\|T_t\| \leq \Gamma e^{|t|}, \quad \forall\ t \in \mathbb{R}.$$

With this formulation, equation (37) can be rewritten as the initial value problem

$$\begin{cases} \dot{x}(t) = Ax(t) + f_{\sigma(t)}(x(t), u(t)), \quad t \geq 0, \\ x(0) = x_0, \end{cases} \tag{38}$$

with feedback

$$u(t) = x_2(t) \text{ and } f_s(x, u) = \begin{pmatrix} 0 \\ \rho_s(u) \end{pmatrix}.$$

The associated transition map satisfies the inequality

$$\|\phi(t, x, \sigma)\| \leq e^{1-\mu t}\|x\|, \quad \forall\ t \geq 0. \tag{39}$$

Note that the constant μ does not depend on the solution. Using the density of $D(A)$ in X, inequality (39) holds true for weak solutions. Thus system (38) is UGES. The assumptions of Corollary 29 being satisfied, we deduce that for a sufficiently small maximal sampling time the predictor-based sampled data feedback (32) preserves the exponential decay to zero of the energy of the solutions of (38).

7 Proof of the main results

As a preliminary step, let us state the following useful and straightforward lemma.

Lemma 31. *Given a continuous function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\limsup_{r\downarrow 0} \frac{\beta(r)}{r} < +\infty$, the function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ defined by*

$$\alpha(0) = 1 + \limsup_{r\downarrow 0} \frac{\beta(r)}{r}, \ \alpha(r) = 1 + \sup_{s\in(0,r]} \frac{\beta(s)}{s}, \ r > 0, \tag{40}$$

is a continuous nondecreasing function satisfying $\beta(r) \leq r\alpha(r)$ for every $r \geq 0$.

7.1 Proof of Theorem 8

By using inequality (6), one deduces that i) implies ii) with $r \mapsto k(r) \equiv \frac{M}{(p\lambda)^{1/p}}$. Moreover, ii) clearly implies iii). It then remains to prove that iii) implies i). Without loss of generality, we assume that $G_0 \geq 1$. Let $C : \mathbb{R}_+ \to \mathbb{R}_+$ be the nondecreasing function defined by

$$C(r) = G_0 \max\left\{\alpha(r), \frac{k(r)}{t_1^{1/p}} \alpha\left(\frac{k(r)r}{t_1^{1/p}}\right)\right\}, \tag{41}$$

where $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is given by (40).

For $t > t_1$, $x \in X$, and $\sigma \in \mathcal{S}$, we have

$$\|\phi(t,x,\sigma)\| = \|\phi\left(\tau, \phi(t-\tau,x,\sigma), \mathbb{T}_{t-\tau}\sigma\right)\| \leq G_0 \beta\left(\|\phi(t-\tau,x,\sigma)\|\right), \quad \forall\, \tau \in [0,t_1]. \tag{42}$$

One deduces from equations (11) and (42) that, for $t > t_1$, $x \in B_X(0,R)$, and $\sigma \in \mathcal{S}$,

$$t_1 \left(\beta^{-1}\left(\frac{\|\phi(t,x,\sigma)\|}{G_0}\right)\right)^p \leq \int_{t-t_1}^{t} \|\phi(\tau,x,\sigma)\|^p d\tau \leq \int_0^{+\infty} \|\phi(\tau,x,\sigma)\|^p d\tau \leq k(\|x\|)^p \|x\|^p.$$

Therefore, for $t > t_1$, $x \in B_X(0,R)$, and $\sigma \in \mathcal{S}$, we have

$$\|\phi(t,x,\sigma)\| \leq G_0 \beta\left(\frac{k(\|x\|)\|x\|}{t_1^{1/p}}\right). \tag{43}$$

By consequence, bundling together (10) and (43), we get for $t \geq 0$ and $x \in B_X(0,R)$ that

$$\|\phi(t,x,\sigma)\| \leq G_0 \max\left\{\beta(\|x\|), \beta\left(\frac{k(\|x\|)\|x\|}{t_1^{1/p}}\right)\right\} \leq C(\|x\|)\|x\|, \tag{44}$$

where we used that $\beta(\|x\|) \leq \|x\|\alpha(\|x\|)$.

Let $r > 0$ be such that $rC(r) < R$. In particular, for every $t \geq 0$ and $x \in B_X(0,r)$ we have $\|\phi(t,x,\sigma)\| < R$. It follows from (44) that, for every $t \geq 0$ and $x \in B_X(0,r)$,

$$t\|\phi(t,x,\sigma)\|^p = \int_0^t \|\phi(t-\tau, \phi(\tau,x,\sigma), \mathbb{T}_\tau \sigma)\|^p d\tau \leq \int_0^t C(r)^p \|\phi(\tau,x,\sigma)\|^p d\tau \leq \left(k(r)C(r)\right)^p \|x\|^p,$$

where the last inequality follows from (11). By consequence, one has, for every $t \geq 0$, $x \in B_X(0,r)$, and $\sigma \in \mathcal{S}$

$$\|\phi(t,x,\sigma)\| \leq \frac{k(r)C\,(r)}{t^{1/p}}\|x\|.$$

So, for each $0 < c < 1$, there exists a positive real number $t_0 = t_0(c,r)$ such that

$$\|\phi(t,x,\sigma)\| \leq c\|x\|, \quad \forall\, t \geq t_0,\ \forall\, x \in B_X(0,r),\ \forall\, \sigma \in \mathcal{S}.$$

Now, let $t \geq 0$, $x \in B_X(0,r)$, and $\sigma \in \mathcal{S}$ be fixed. There exists an integer $n \geq 0$ such that $t = nt_0 + s$, with $0 \leq s < t_0$. Notice that $\phi(jt_0,x,\sigma) \in B_X(0,r)$ for every $j \in \mathbb{N}$. We have

$$\|\phi(t,x,\sigma)\| = \|\phi(s, \phi(nt_0,x,\sigma), \mathbb{T}_{nt_0}\sigma)\| \leq C(r)\|\phi(nt_0,x,\sigma)\| \leq C(r)c^n\|x\| \leq M(r)e^{-\lambda(r)t}\|x\|,$$

with $M(r) = \frac{C(r)}{c}$ and $\lambda(r) = -\frac{\log(c)}{t_0(c,r)} > 0$. The uniform local exponential stability of system Σ is established.

7.2 Proof of Theorem 10

If Σ is USGES, then by Remark 5 we may assume without loss of generality that $r \mapsto \lambda(r)$ is constant and $r \mapsto M(r)$ is nondecreasing. Hence, i) implies ii) with $k(r) := \frac{M(r)}{(p\lambda(r))^{1/p}}$. Moreover, ii) clearly implies iii). It then remains to prove that iii) implies i). For this, let $r > 0$. Let $R > rC(r)$, where C is defined by (41), and observe that (13) is satisfied in $B_X(0,R)$. Following the same proof as in Theorem 8, we get the existence of $M(r) > 0$ and $\lambda(r) > 0$ such that

$$\|\phi(t,x,\sigma)\| \leq M(r)e^{-\lambda(r)t}\|x\|,$$

for all $t \geq 0$, $x \in B_X(0,r)$, and $\sigma \in \mathcal{S}$, whence the uniform semi-global exponential stability of system Σ.

7.3 Proof of Theorem 11

The proof follows the same steps as that of Theorem 10 with β equal to the identity function and k equal to a constant function. In such a case, the functions $r \mapsto M(r)$ and $r \mapsto \lambda(r)$ are constant.

7.4 Proof of Theorem 17

We start by proving item i). Let $x \in B_X(0, R)$ and $\sigma \in \mathcal{S}$. Assume that there exists a first time $0 < t^\star < +\infty$ such that $\|\phi(t^\star, x, \sigma)\| = R$. If $t^\star \leq t_1$ then we have $R \leq G_0\beta(\|x\|)$ which implies that $\|x\| \geq \beta^{-1}(R/G_0)$. Then let us take $x \in B_X(0, \beta^{-1}(R/G_0))$, so that $t^\star > t_1$. For every $t_1 \leq t \leq t^\star$, by repeating the same reasoning as in (42), we have

$$t_1\beta^{-1}\left(\frac{\|\phi(t, x, \sigma)\|}{G_0}\right)^p \leq \int_{t-t_1}^{t} \|\phi(\tau, x, \sigma)\|^p d\tau. \tag{45}$$

In addition, since $t \mapsto V(\phi(t, x, \sigma))$ is continuous from the left, it follows from (19) (see [8, Theorem 9]) that

$$-V(x) \leq V(\phi(t, x, \sigma)) - V(x) \leq -\int_0^t \|\phi(\tau, x, \sigma)\|^p d\tau \leq -\int_{t-t_1}^{t} \|\phi(\tau, x, \sigma)\|^p d\tau. \tag{46}$$

By consequence, from inequalities (45) and (46) together with (18), one gets

$$t_1\beta^{-1}\left(\frac{\|\phi(t, x, \sigma)\|}{G_0}\right)^p \leq c\|x\|^p. \tag{47}$$

Thus, evaluating (47) at $t = t^\star$, one gets

$$\|x\| \geq \left(\frac{t_1}{c}\right)^{1/p} \beta^{-1}\left(\frac{R}{G_0}\right).$$

If we take $x \in B_X(0, \gamma)$ with

$$\gamma := \min\left\{\beta^{-1}\left(\frac{R}{G_0}\right), \left(\frac{t_1}{c}\right)^{1/p} \beta^{-1}\left(\frac{R}{G_0}\right)\right\}$$

we get a contradiction, that is, we then have

$$\|\phi(t, x, \sigma)\| \leq R, \quad \forall\ t \geq 0,\ \forall\ x \in B_X(0, \gamma),\ \forall\ \sigma \in \mathcal{S}.$$

From (19), we have, for every $t \geq 0$, $x \in B_X(0, \gamma)$, and $\sigma \in \mathcal{S}$,

$$\underline{D}_\sigma V(\phi(t, x, \sigma)) \leq -\|\phi(t, x, \sigma)\|^p,$$

from which, by repeating the same reasoning as in (46), we deduce the inequality

$$\int_0^\infty \|\phi(\tau, x, \sigma)\|^p d\tau \leq c\|x\|^p,\ \forall\ x \in B_X(0, \gamma),\ \forall\ \sigma \in \mathcal{S}. \tag{48}$$

Thanks to Theorem 8, the uniform local exponential stability of system Σ follows from (48) together with (17).

The statement ii) of the theorem follows from (20), (48), and the definition of γ. The last statement follows from the fact that (48) holds true for every $x \in X$ with c independent of $\|x\|$, using Theorem 11.

7.5 Proof of Theorem 18

Fix $r > 0$ and let $M = M(r)$ and $\lambda = \lambda(r)$ be as in Definition 4. Choose $\gamma = \gamma(r) > 0$ such that $\gamma - \lambda < 0$. Let $\bar{t} = \frac{\log(M)}{\lambda - \gamma}$. Let $V_r : X \to \mathbb{R}_+$ be the functional defined by

$$V_r(x) = \frac{1}{\gamma} \sup_{\sigma \in \mathcal{S}, t \in [0,\bar{t}]} \|e^{\gamma t} \phi(t, x, \sigma)\|, \quad x \in X. \tag{49}$$

The functional V_r is well defined, since Σ is USGES, which implies that, for every $R > 0$, $t \geq 0$, and $x \in B_X(0, R)$,

$$\|e^{\gamma t} \phi(t, x, \sigma)\| \leq M(R) e^{(\gamma - \lambda(R))t} \|x\|. \tag{50}$$

Let us check inequalities (21) and (22). The right-hand inequality in (21) follows directly from (50), with

$$\overline{c}_r = \frac{M}{\gamma}. \tag{51}$$

The left-hand inequality, with

$$\underline{c}_r = \frac{1}{\gamma}, \tag{52}$$

is a straightforward consequence of the definition of V_r. Concerning (22), remark that for all $x \in B_X(0, r)$ and $h \geq 0$, we have

$$\sup_{\sigma \in \mathcal{S}, t \in [\bar{t}, \bar{t}+h]} \|e^{\gamma t} \phi(t, x, \sigma)\| \leq \sup_{t \in [\bar{t}, \bar{t}+h]} M e^{(\gamma - \lambda)t} \|x\| \leq M e^{(\gamma - \lambda)\bar{t}} \|x\| = \|x\|,$$

and hence

$$\sup_{\sigma \in \mathcal{S}, t \in [0, \bar{t}+h]} \|e^{\gamma t} \phi(t, x, \sigma)\| = \sup_{\sigma \in \mathcal{S}, t \in [0,\bar{t}]} \|e^{\gamma t} \phi(t, x, \sigma)\|.$$

Then, for every $\tilde{\sigma} \in \mathcal{S}$, $x \in B_X(0, r)$, and $h \geq 0$, we have

$$\begin{aligned} V_r(\phi(h, x, \tilde{\sigma})) &= \frac{1}{\gamma} \sup_{\sigma \in \mathcal{S}, t \in [0,\bar{t}]} \|e^{\gamma t} \phi(t, \phi(h, x, \tilde{\sigma}), \sigma)\| \leq \frac{1}{\gamma} \sup_{\sigma \in \mathcal{S}, t \in [0,\bar{t}]} \|e^{\gamma t} \phi(t + h, x, \sigma)\| \\ &= \frac{e^{-\gamma h}}{\gamma} \sup_{\sigma \in \mathcal{S}, t \in [0,\bar{t}]} \|e^{\gamma(t+h)} \phi(t + h, x, \sigma)\| = \frac{e^{-\gamma h}}{\gamma} \sup_{\sigma \in \mathcal{S}, t \in [h, \bar{t}+h]} \|e^{\gamma t} \phi(t, x, \sigma)\| \\ &\leq \frac{e^{-\gamma h}}{\gamma} \sup_{\sigma \in \mathcal{S}, t \in [0, \bar{t}+h]} \|e^{\gamma t} \phi(t, x, \sigma)\| = e^{-\gamma h} V_r(x). \end{aligned}$$

Therefore, for all $x \in B_X(0, r)$ and any $\tilde{\sigma} \in \mathcal{S}$, it follows that

$$\overline{D}_{\tilde{\sigma}} V_r(x) = \limsup_{h \downarrow 0} \frac{V_r(\phi(h, x, \tilde{\sigma})) - V_r(x)}{h} \leq \limsup_{h \downarrow 0} \frac{e^{-\gamma h} - 1}{h} V_r(x) \leq \limsup_{h \downarrow 0} \frac{e^{-\gamma h} - 1}{h} \frac{\|x\|}{\gamma} \leq -\|x\|,$$

which implies that inequality (22) holds true.

Let us prove that the functional $V_r : X \to \mathbb{R}_+$ is continuous. For $x, y \in X$, we have

$$\begin{aligned} |V_r(x) - V_r(y)| &= \left| \sup_{\sigma \in \mathcal{S}, t \in [0,\bar{t}]} \|e^{\gamma t} \phi(t, x, \sigma)\| - \sup_{\sigma \in \mathcal{S}, t \in [0,\bar{t}]} \|e^{\gamma t} \phi(t, y, \sigma)\| \right| \\ &\leq \left| \sup_{\sigma \in \mathcal{S}, t \in [0,\bar{t}]} \left(\|e^{\gamma t} \phi(t, x, \sigma)\| - \|e^{\gamma t} \phi(t, y, \sigma)\|\right) \right| \\ &\leq \sup_{\sigma \in \mathcal{S}, t \in [0,\bar{t}]} \left| \|e^{\gamma t} \phi(t, x, \sigma)\| - \|e^{\gamma t} \phi(t, y, \sigma)\| \right| \\ &\leq e^{\gamma \bar{t}} \sup_{\sigma \in \mathcal{S}, t \in [0,\bar{t}]} \left| \|\phi(t, x, \sigma)\| - \|\phi(t, y, \sigma)\| \right|. \end{aligned} \tag{53}$$

The continuity of V then follows from the $\mathcal{S}$-uniform continuity of ϕ. Moreover, if the transition map ϕ is $\mathcal{S}$-uniformly Lipschitz continuous, then we deduce from (53) that

$$|V_r(x) - V_r(y)| \quad \leq \quad e^{\gamma \bar{t}} l(\bar{t}) \|x - y\|,$$

where $l(\bar{t})$ is as in Definition 16, which implies the Lipschitz continuity of V_r. In the case where the transition map is Lipschitz continuous on bounded sets, we conclude similarly.

Remark 32. *The construction of the Lyapunov functional* (49) *is based on the classical construction given in [38] in the context of linear C_0-semigroups. This is also used in [33] in order to construct a coercive input-to-state Lyapunov functional for bilinear infinite-dimensional systems with bounded input operators. An alternative construction of a coercive common Lyapunov functional can be given by*

$$V(x) = \sup_{\sigma \in \mathcal{S}} \int_0^{+\infty} \|\phi(t, x, \sigma)\| dt + \sup_{t \geq 0, \sigma \in \mathcal{S}} \|\phi(t, x, \sigma)\|, \tag{54}$$

or also by

$$V(x) = \int_0^{+\infty} \sup_{\sigma \in \mathcal{S}} \|\phi(t, x, \sigma)\| dt + \sup_{t \geq 0, \sigma \in \mathcal{S}} \|\phi(t, x, \sigma)\|. \tag{55}$$

For both constructions V satisfies (21)*: the right-hand inequality follows directly from the uniform exponential stability assumption with $\overline{c} = M(1 + 1/\lambda)$, and the left-hand one holds with $\underline{c} = 1$. Indeed, the second term appearing in* (54) *(respectively,* (55)*) guarantees the coercivity of the functional V. The first term appearing in* (54) *(respectively,* (55)*) is actually a (possibly non-coercive) Lyapunov functional which can be used to give a converse to Theorem 17.*

7.6 Proof of Theorem 28

Fix $r > 0$ and let $V_r : X \to \mathbb{R}_+$ be a L_r-Lipschitz continuous Lyapunov functional satisfying the conclusions of Lemma 26. Since V_r is decreasing along the trajectories of Σ_0 then, setting $\rho := r\underline{c}_r/\overline{c}_r$, we have $\phi^{\Sigma_0}(t, x, \sigma) \in B_X(0, r)$ for every $x \in B_X(0, \rho)$, $t \geq 0$, and $\sigma \in \mathcal{S} = \text{PC}$.

Let $x_0 \in B_X(0, \rho)$ and $\sigma \in \mathcal{S}$. Let $t^* > 0$ be such that the trajectory $x^\Sigma(\cdot) = \phi^\Sigma(\cdot, x_0, \sigma)$ of system Σ stays in $B_X(0, r)$ for $t \in [0, t^*]$. Computing the upper Dini derivative of V_r along $x^\Sigma(\cdot)$ gives

$$\begin{aligned}
\overline{D}_\sigma V_r(x^\Sigma(t)) &= \limsup_{h \downarrow 0} \frac{V_r\left(x^\Sigma(t+h)\right) - V_r\left(x^\Sigma(t)\right)}{h} \\
&= \limsup_{h \downarrow 0} \Big[\frac{V_r\left(x^\Sigma(t+h)\right) - V_r\left(x^{\Sigma_0}(x^\Sigma(t), h)\right)}{h} + \frac{V_r\left(x^{\Sigma_0}(x^\Sigma(t), h)\right) - V_r\left(x^\Sigma(t)\right)}{h} \Big] \\
&\leq \limsup_{h \downarrow 0} \frac{V_r\left(x^\Sigma(t+h)\right) - V_r\left(x^{\Sigma_0}(x^\Sigma(t), h)\right)}{h} + \limsup_{h \downarrow 0} \frac{V_r\left(x^{\Sigma_0}(x^\Sigma(t), h)\right) - V_r\left(x^\Sigma(t)\right)}{h} \\
&\leq L_r \limsup_{h \downarrow 0} \frac{\|x^\Sigma(t+h) - x^{\Sigma_0}(x^\Sigma(t), h)\|}{h} - \|x^\Sigma(t)\|.
\end{aligned} \tag{56}$$

Observe that for $k \geq 0$, $t \in [s_k, s_{k+1}) \cap [0, t^*]$, and $h > 0$ small enough we have

$$x^\Sigma(t+h) = T_h x^\Sigma(t) + \int_0^h T_{h-s} f_{\sigma(t+s)}\left(x^\Sigma(t+s), K(T_{t+s-s_k} x^\Sigma(s_k))\right) ds,$$

and

$$x^{\Sigma_0}(x^\Sigma(t), h) = T_h x^\Sigma(t) + \int_0^h T_{h-s} f_{\sigma(t+s)}\left(x^{\Sigma_0}(x^\Sigma(t), s), K(x^{\Sigma_0}(x^\Sigma(t), s))\right) ds.$$

Using the fact that f and K are, respectively, L_f- and L_K-Lipschitz continuous, one gets

$$\limsup_{h\downarrow 0}\frac{\|x^{\Sigma}(t+h)-x^{\Sigma_0}(x^{\Sigma}(t),h)\|}{h}\leq \Gamma L_f L_K\|x^{\Sigma}(t)-T_{t-s_k}x^{\Sigma}(s_k)\|.$$

Going back to (56), we obtain that

$$\overline{D}_{\sigma}V_r(x^{\Sigma}(t))\leq \Gamma L_r L_f L_K\|x^{\Sigma}(t)-T_{t-s_k}x^{\Sigma}(s_k)\|-\|x^{\Sigma}(t)\|. \tag{57}$$

We now need to estimate

$$\varepsilon(t-s_k):=\|x^{\Sigma}(t)-T_{t-s_k}x^{\Sigma}(s_k)\|,\quad t\in[s_k,s_{k+1}).$$

By adding and subtracting

$$\int_0^{t-s_k}T_{t-s_k-s}f_{\sigma(s+s_k)}\left(T_s x^{\Sigma}(s_k),K\left(T_s x^{\Sigma}(s_k)\right)\right)ds$$

in (34), and using the identity $f_q(0,0)=0$, we obtain

$$\begin{aligned}\varepsilon(t-s_k)\leq&\Gamma e^{\omega\delta}L_f\int_0^{t-s_k}\varepsilon(s)ds+\Gamma e^{\omega\delta}\int_0^{t-s_k}\|f_{\sigma(s+s_k)}\left(T_s x^{\Sigma}(s_k),K\left(T_s x^{\Sigma}(s_k)\right)\right)\|ds\\ \leq&\Gamma e^{\omega\delta}L_f\int_0^{t-s_k}\varepsilon(s)ds+\Gamma^2 e^{2\omega\delta}L_f(1+L_K)\int_0^{t-s_k}\|x^{\Sigma}(s_k)\|ds\\ \leq&\Gamma e^{\omega\delta}L_f\int_0^{t-s_k}\varepsilon(s)ds+\Gamma^2 e^{2\omega\delta}L_f(1+L_K)\delta\|x^{\Sigma}(s_k)\|.\end{aligned}$$

By Gronwall's lemma, we have

$$\varepsilon(t-s_k)\leq c\delta\|x^{\Sigma}(s_k)\|, \tag{58}$$

where $c=\Gamma^2e^{2\omega\delta}L_f(1+L_K)e^{L_f\delta\Gamma e^{\omega\delta}}$. Observe that, by (31),

$$\|x^{\Sigma}(s_k)\|\leq \Gamma e^{\omega\delta}\|T_{t-s_k}x^{\Sigma}(s_k)\|. \tag{59}$$

Hence, from (58), (59), and the triangular inequality, we get

$$\varepsilon(t-s_k)\leq c\delta\Gamma e^{\omega\delta}\|T_{t-s_k}x^{\Sigma}(s_k)\|\leq c\delta\Gamma e^{\omega\delta}\Big(\varepsilon(t-s_k)+\|x^{\Sigma}(t)\|\Big),$$

that is, for sufficiently small δ,

$$\varepsilon(t-s_k)\leq\frac{c\delta\Gamma e^{\omega\delta}}{1-c\delta\Gamma e^{\omega\delta}}\|x^{\Sigma}(t)\|.$$

Let $\delta^{\star}>0$ be such that

$$c^{\star}:=\Gamma L_r L_f L_K\frac{c\delta^{\star}\Gamma e^{\omega\delta^{\star}}}{1-c\delta^{\star}\Gamma e^{\omega\delta^{\star}}}<1.$$

It follows from (57) that for every $\delta\in(0,\delta^{\star})$, we have

$$\overline{D}_{\sigma}V_r(x^{\Sigma}(t))\leq\Gamma L_r L_f L_K\varepsilon(t-t_k)-\|x^{\Sigma}(t)\|\leq(c^{\star}-1)\|x^{\Sigma}(t)\|\leq\frac{c^{\star}-1}{\overline{c}_r}V_r(x^{\Sigma}(t)).$$

In particular, V_r decreases exponentially along x^{Σ}, which implies that t^* can be taken arbitrarily large and, since V_r is coercive, we conclude that Σ is UES in $B_X(0,\rho)$.

To conclude the proof, we need to verify that ρ can be taken arbitrarily large (as $r\to\infty$) if (35) holds true. In order to do so, notice that if V_r is constructed as in the proof of Theorem 18, then $\overline{c}_r$ and $\underline{c}_r$ can be chosen as in (51) and (52). In this case $\rho=r\underline{c}_r/\overline{c}_r=r/M(r)$, concluding the proof of the theorem.

7.7 Proof of Theorem 30

Applying Theorem 18, there exist a L_V-Lipschitz continuous functional $V : X \to \mathbb{R}_+$ and $\underline{c}, \overline{c} > 0$ such that

$$\underline{c}\|x\| \leq V(x) \leq \overline{c}\|x\|, \quad \forall\ x \in X, \tag{60}$$

and

$$\limsup_{h\downarrow 0} \frac{V(\phi_0(h,x,\sigma)) - V(x)}{h} \leq -\|x\|, \tag{61}$$

for every $x \in X$ and $\sigma \in \mathcal{S}$. Let $t \geq 0$, $\sigma \in \mathcal{S}$, and $u \in L^p(U)$, $1 \leq p \leq +\infty$. Set $x = \phi_u(t,0,\sigma)$. For $h > 0$ small enough, one has

$$\begin{aligned} \frac{V(\phi_u(h,x,\sigma)) - V(x)}{h} &= \frac{V(\phi_0(h,x,\sigma)) - V(x)}{h} + \frac{V(\phi_u(h,x,\sigma)) - V(\phi_0(h,x,\sigma))}{h} \\ &\leq \frac{V(\phi_0(h,x,\sigma)) - V(x)}{h} + L_V \frac{\|\phi_u(h,x,\sigma) - \phi_0(h,x,\sigma)\|}{h}. \end{aligned} \tag{62}$$

Define $e(s) = \phi_u(s,x,\sigma) - \phi_0(s,x,\sigma)$, for $s \in [0,h]$. By the variation of constant formula, one has

$$e(s) = \int_0^s T_{s-\tau}\Upsilon(t,\tau,x,\sigma)d\tau, \quad \forall\ s \in [0,h],$$

where

$$\Upsilon(t,\tau,x,\sigma) = f_{\sigma(\tau)}(\phi_0(\tau,x,\sigma) + e(\tau), u(t+\tau)) - f_{\sigma(\tau)}(\phi_0(\tau,x,\sigma),0).$$

Hence, one deduces that

$$\|e(h)\| \leq \Gamma L_f e^{\omega h} \int_0^h \left(\|e(\tau)\| + \|u(t+\tau)\|\right) d\tau. \tag{63}$$

For $p = +\infty$, one deduces that

$$\limsup_{h\downarrow 0} \frac{\|e(h)\|}{h} \leq \Gamma L_f \|u\|_{L^\infty(U)}.$$

Letting $h \downarrow 0$ in (62) and using (60) one gets

$$\begin{aligned} \limsup_{h\downarrow 0} \frac{V(\phi_u(t+h,0,\sigma)) - V(\phi_u(t,0,\sigma))}{h} &\leq -\|\phi_u(t,0,\sigma)\| + \Gamma L_f L_V \|u\|_{L^\infty(U)} \\ &\leq -\frac{1}{\overline{c}} V(\phi_u(t,0,\sigma)) + c\|u\|_{L^\infty(U)}, \end{aligned}$$

with $c = \Gamma L_f L_V$. Using [8, Theorem 9] for the function $t \mapsto e^{\frac{t}{\overline{c}}} V(\phi_u(t,0,\sigma))$, one concludes that

$$V(\phi_u(t,0,\sigma)) \leq c\overline{c}\|u\|_{L^\infty(U)}, \quad t \geq 0.$$

The theorem follows for $p = +\infty$.

We next assume that $1 \leq p < +\infty$. By a standard density argument, it is enough to prove (36) for those $u \in L^p(U)$ which are, in addition, continuous, since $u \mapsto \phi_u(t,0,\sigma)$ is continuous in $L^p(U)$. For u continuous, we deduce, from (63), that

$$\limsup_{h\downarrow 0} \frac{\|e(h)\|}{h} \leq L_f \|u(t)\|. \tag{64}$$

By using (61), (62), and (64) one gets

$$\limsup_{h\downarrow 0} \frac{V(\phi_u(t+h,0,\sigma)) - V(\phi_u(t,0,\sigma))}{h} \leq -\|\phi_u(t,0,\sigma)\| + c\|u(t)\|, \tag{65}$$

with $c = L_f L_V$. Let $\varphi_p(r) = \frac{r^p}{p}$, $r \geq 0$, and $W_p(x) = \varphi_p(x)$, $x \in X$. Since φ_p is increasing and differentiable, one deduce from (65) that

$$\limsup_{h \downarrow 0} \frac{W_p(\phi_u(t+h, 0, \sigma)) - W_p(\phi_u(t, 0, \sigma))}{h} \leq -V^{p-1}(\phi_u(t, 0, \sigma))\|\phi_u(t, 0, \sigma)\| + cV^{p-1}(\phi_u(t, 0, \sigma))\|u(t)\|.$$

According to [8, Theorem 9], it follows that

$$0 \leq W_p(\phi_u(T, 0, \sigma)) \leq -\int_0^T V^{p-1}(\phi_u(t, 0, \sigma))\|\phi_u(t, 0, \sigma)\|dt + c\int_0^T V^{p-1}(\phi_u(t, 0, \sigma))\|u(t)\|dt, \quad \forall\, T > 0.$$

By using (60) one gets that

$$\underline{c}^{p-1} \int_0^T \|\phi_u(t, 0, \sigma)\|^p dt \leq c\overline{c}^{p-1} \int_0^T \|\phi_u(t, 0, \sigma)\|^{p-1}\|u(t)\|dt, \quad \forall\, T > 0.$$

The theorem is proved for $p = 1$, by letting $T \to +\infty$. For $p > 1$, one first applies Hölder's inequality and then lets $T \to +\infty$ to get the conclusion.

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