Numerical Approach to the Optimal Control and Efficiency of the Copepod Swimmer

Jérémy Rouot, Bernard Bonnard, Monique Chyba, Daisuke Takagi

Numerical Approach to the Optimal Control and Efficiency of the Copepod Swimmer.

55th IEEE Conference on Decision and Control

Bernard Bonnard, Monique Chyba, Jérémy Rouot and Daisuke Takagi

- Geometric optimal control
  - . Maximum principle
  - . Second order optimality conditions (local, conjugate points)

- Swimming at low Reynolds number
  - . sub-Riemannian (SR) geometry framework
  - . Varational approach to find trajectoires of a Hamiltonian system with periodic projections
  - . Find optimal stroke (efficiency concept)
  - . micro-device implementation (D. Takagi, M. Chyba)

$$\begin{array}{lll} \min & c(\widetilde{q}(0),\widetilde{q}(T)) & \\ \text{subject to} & \dot{q} = F(q,u), & \dot{q}^0 = L(q,u) \\ & (\widetilde{q}(0),\widetilde{q}(T)) \in B & \end{array}$$

- final time T is fixed
- $\widetilde{q} = (q,q^0)$, q state, $q^0 \in \mathbb{R}$
- $u(.) \in L^\infty([0,T],U)$, $U \subset \mathbb{R}^m$
- B : smooth closed set of $\mathbb{R}^n \times \mathbb{R}^n$, $(\widetilde{q}(0),\widetilde{q}(T)) \in B$ stands for **mixed** boundary conditions.

Ex. : $B = \{q(0) = q(T), q^0(0) = 0\}$,
$B = \{q(0) = q_0, q(T) = q_f, q^0(0) = 0\}\ldots$

Necessary condition. $(\widetilde{q},u)$ optimal $\Longrightarrow \exists\ \widetilde{p}=(p,p^0):[0,T]\to(\mathbb{R}^n\times\mathbb{R})$ abs. continuous and $\lambda\geq 0$ cst with $(\widetilde{p},\lambda)\neq(0,0)$ and s.t.

$$\dot{\widetilde{q}}=\frac{\partial H}{\partial \widetilde{p}},\quad \dot{\widetilde{p}}=-\frac{\partial H}{\partial \widetilde{q}},\quad H(\widetilde{q},\widetilde{p},u)=\max_{v\in U}\mathscr{H}(\widetilde{q},\widetilde{p},v)$$

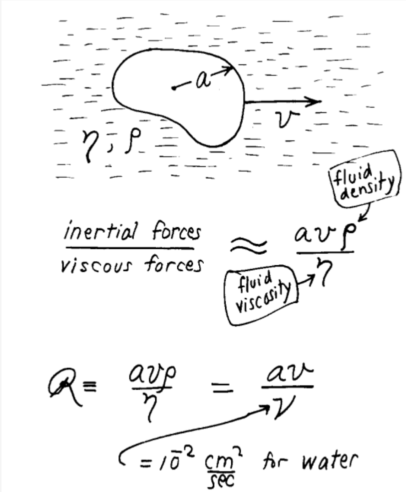
where $\mathscr{H}(\widetilde{q},\widetilde{p},u)=p\cdot F(q,u)+p^0L(q,u)$.

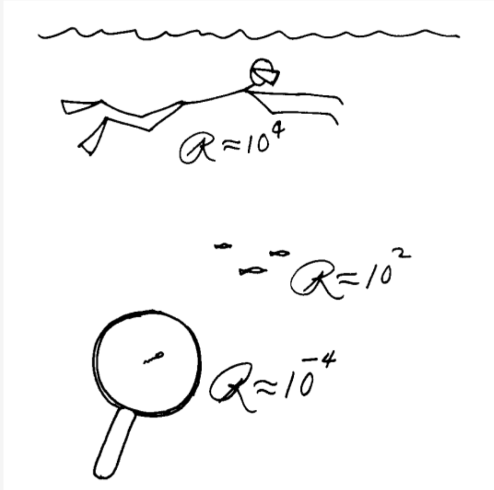
Transversality conditions:

$$(\widetilde{p}(0),-\widetilde{p}(T))-\lambda\left(\frac{\partial c}{\partial \widetilde{q}(0)}(\widetilde{q}(0),\widetilde{q}(T)),\frac{\partial c}{\partial \widetilde{q}(T)}(\widetilde{q}(0),\widetilde{q}(T))\right)\in(T_{(\widetilde{q}(0),\widetilde{q}(T))}B)^{\perp}$$

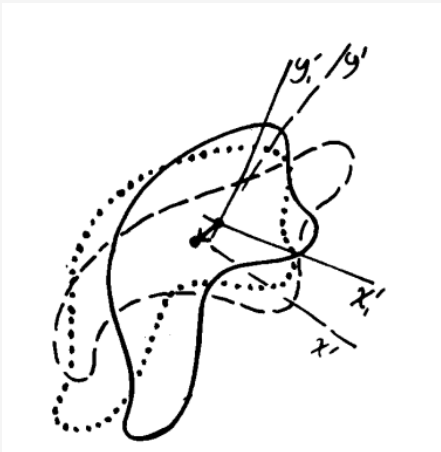
Definition 1. $(\widetilde{q},\widetilde{p},\lambda,u)$ *is a normal extremal if* $\lambda>0$, *abnormal if* $\lambda=0$. *An abnormal trajectory* q *which is not a projection of normal extremal is said strict.*

LIFE AT LOW REYNOLDS NUMBER - PURCELL, 1977

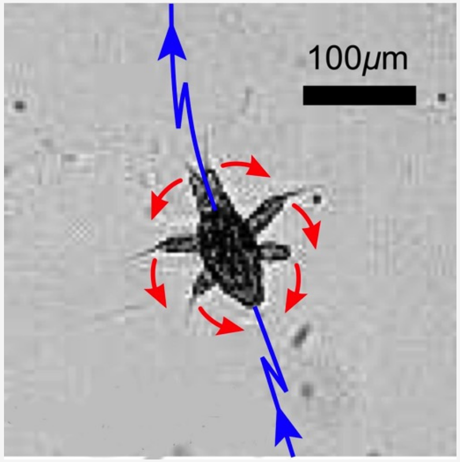




Reynolds number

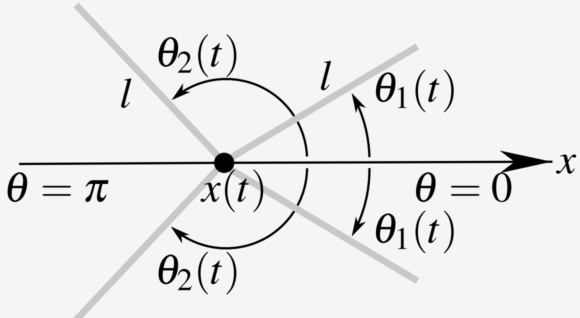


Swimmer deformations



Zooplankton

Symmetric 2-link swimmer

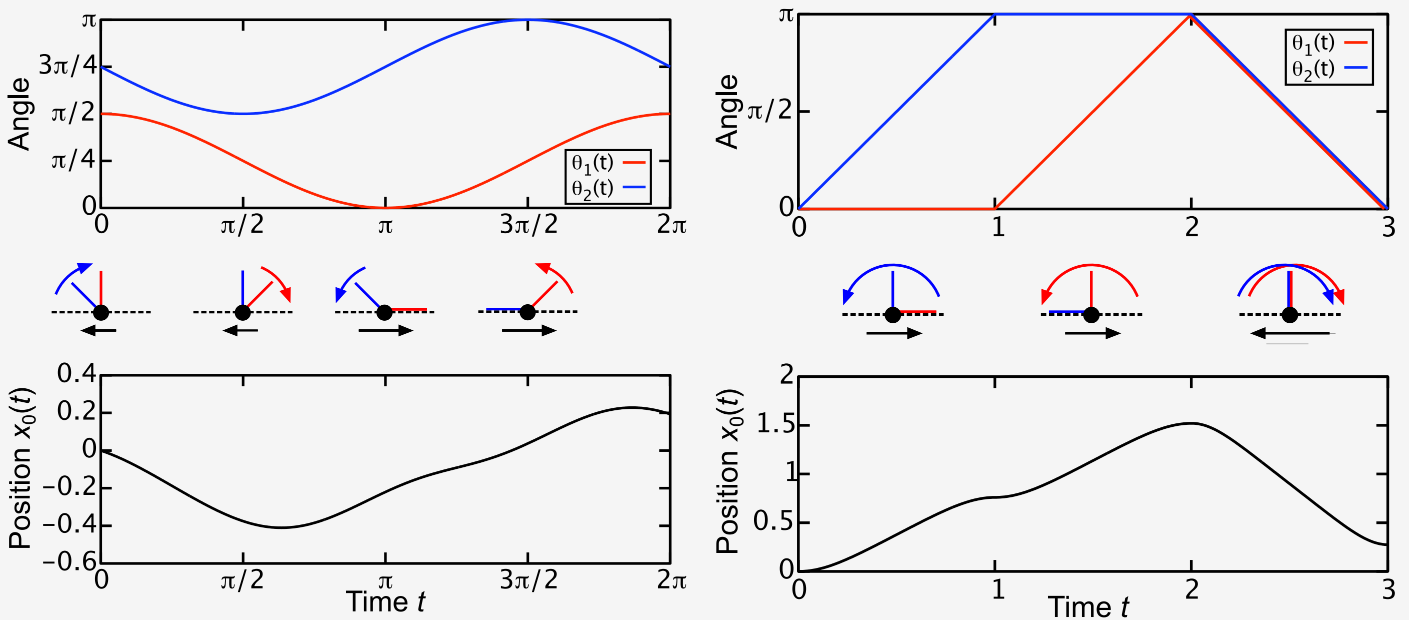


Controlled dynamic (Takagi, 2014)

$$\dot{x} = \frac{\sum_{i=1}^{2} l^2\, \dot{\theta}_i \sin(\theta_i)}{2 + \sum_{i=1}^{2} \sin^2(\theta_i)}, \qquad \dot{\theta}_i = u_i,\ i = 1, 2, \qquad \text{state constraints: } 0 \leq \theta_1 \leq \theta_2 \leq \pi.$$

Criterion : $\min$ of the dissipative energy: $\dot{q}S(q)\dot{q}^{\mathsf{T}}$ where $q = (x, \theta_1, \theta_2)$ and S is a positive-definite matrix $\Longrightarrow$ quadratic form in (u_1, u_2).

Definition 2. *An admissible stroke is a periodic motion of the shape variables* (θ_1, θ_2) *associated with a periodic control and producing a displacement of the position variable after one period* T *(we can fix* $T = 2\pi$*).*



(Left) Trigonometric controls. (Right) Piecewise controls.

$$\mathscr{E} = \textbf{Geometric Efficiency} = x(T)/l_{SR}(q)$$

SR length:

$$l_{SR}(q) = \int_0^T \sqrt{L(q,u)}\,\mathrm{d}t, \qquad L(q,u) = a(q)\,u_1^2 + 2b(q)\,u_1u_2 + c(q)\,u_2^2$$

$$\min\, l_{SR}(q) \Longleftrightarrow \min \int_0^T |\dot{q}|_L^2\,\mathrm{d}t = \min \int_0^T L(q,u)\mathrm{d}t$$

With $c(\widetilde{q}(0),\widetilde{q}(T)) = q^0(T)$:

$$\dot{q} = \sum_{i=1}^{2} u_iF_i \leftarrow \min \int_0^T L(q,u)\mathrm{d}t,$$

$$\max\, \mathscr{E} \Longleftrightarrow \left(\min \int_0^T L(q,u)\mathrm{d}t \quad \text{with} \quad \mathbf{x(T)\ fixed}, \text{ then } \underbrace{\text{select} \max_{x(T)} \mathscr{E}}_{\text{transversality cond.}} \right)$$

Compute points on the SR-sphere $\rightarrow$
provide candidates to maximize the efficiency.

$$\dot{q} = \sum_{i=1}^{2} u_i F_i(q), \qquad q = (x, \theta_1, \theta_2)$$

$$F_i = \frac{\sin(\theta_i)}{\Delta} \frac{\partial}{\partial x} + \frac{\partial}{\partial \theta_i}, \qquad \Delta = 2 + \sin^2(\theta_1) + \sin^2(\theta_2)$$

PMP:

$$\mathscr{H}(z,u) = u_1 H_1(z) + u_2 H_2(z) + p^0 L(q,u), \qquad z = (q,p)$$

with $H_i = p \cdot F_i$, $i = 1,2$.

- normal case: $p^0 = -1/2$ and u_i, $i = 1,2$ given by $\partial \mathscr{H} / \partial u = 0$

$$H_n = 1/2\,(a(q)\,H_1^2 + 2b(q)\,H_1 H_2 + c(q)\,H_2^2)$$

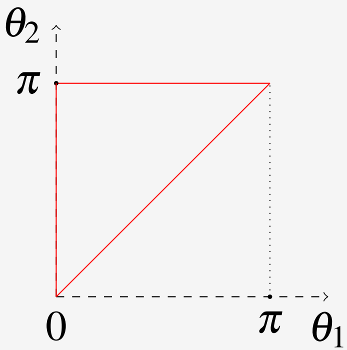
- abnormal case: $p^0 = 0$ hence $H_1 = H_2 = \{H_1, H_2\} = 0$, then

$$u_1 \{\{H_1, H_2\}, H_1\} + u_2 \{\{H_1, H_2\}, H_2\} = 0.$$

Proposition 3. *The surface* $\Sigma : \{ q ; \det(F_1(q), F_2(q), [F_1, F_2](q)) = 0 \}$ *contains abnormal trajectories and is characterized by*

- $\theta_i = 0$ *or* π, *for* $i = 1, 2$,
- $\theta_1 = \theta_2$.

$\rightarrow$ **physical boundary of the state domain.**: $\theta_i \in [0, \pi]$, $i = 1, 2, \theta_1 \leq \theta_2$.



: abnormal trajectories

in the plane (θ_1, θ_2)

Boundary conditions defined by

$$x(0) = 0, \quad x(T) = x_f, \quad \theta_i(0) = \theta_i(T),\ i = 1,2$$

Transversality conditions

$$p_{\theta_i}(0) = p_{\theta_i}(T),\ i = 1,2$$

Shooting problem

$$\begin{cases} \dot{z} = \overrightarrow{H}_n(z) \\ x(0) = 0, \quad x(T) = x_f \ \textbf{(fixed)} \\ \theta_i(0) = \theta_i(T),\ i = 1,2 \\ p_{\theta_i}(0) = p_{\theta_i}(T),\ i = 1,2 \end{cases},$$

- `HamPath` *indirect method*: based on the PMP, *Newton* type algorithm and compute **necessary second order optimality conditions**, not robust w.r.t. initialization.

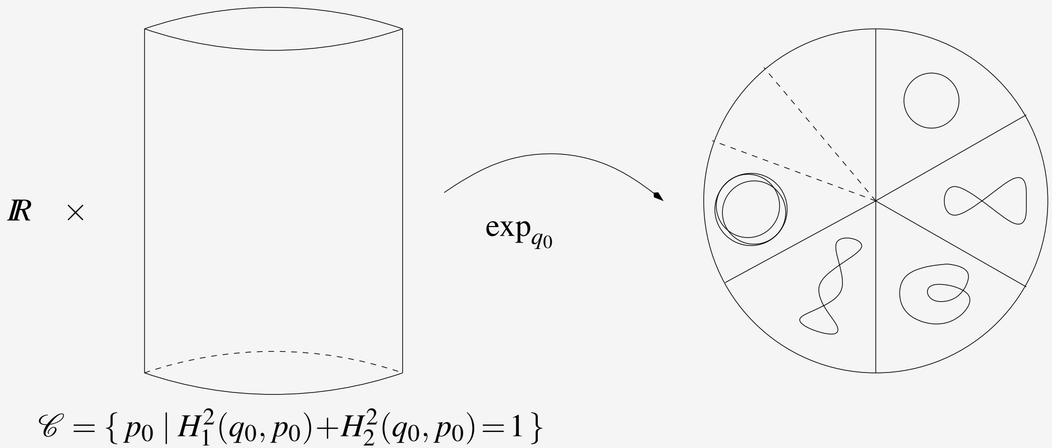
- `Bocop` *direct method*: give an initialization of the shooting algorithm of the `HamPath` code.

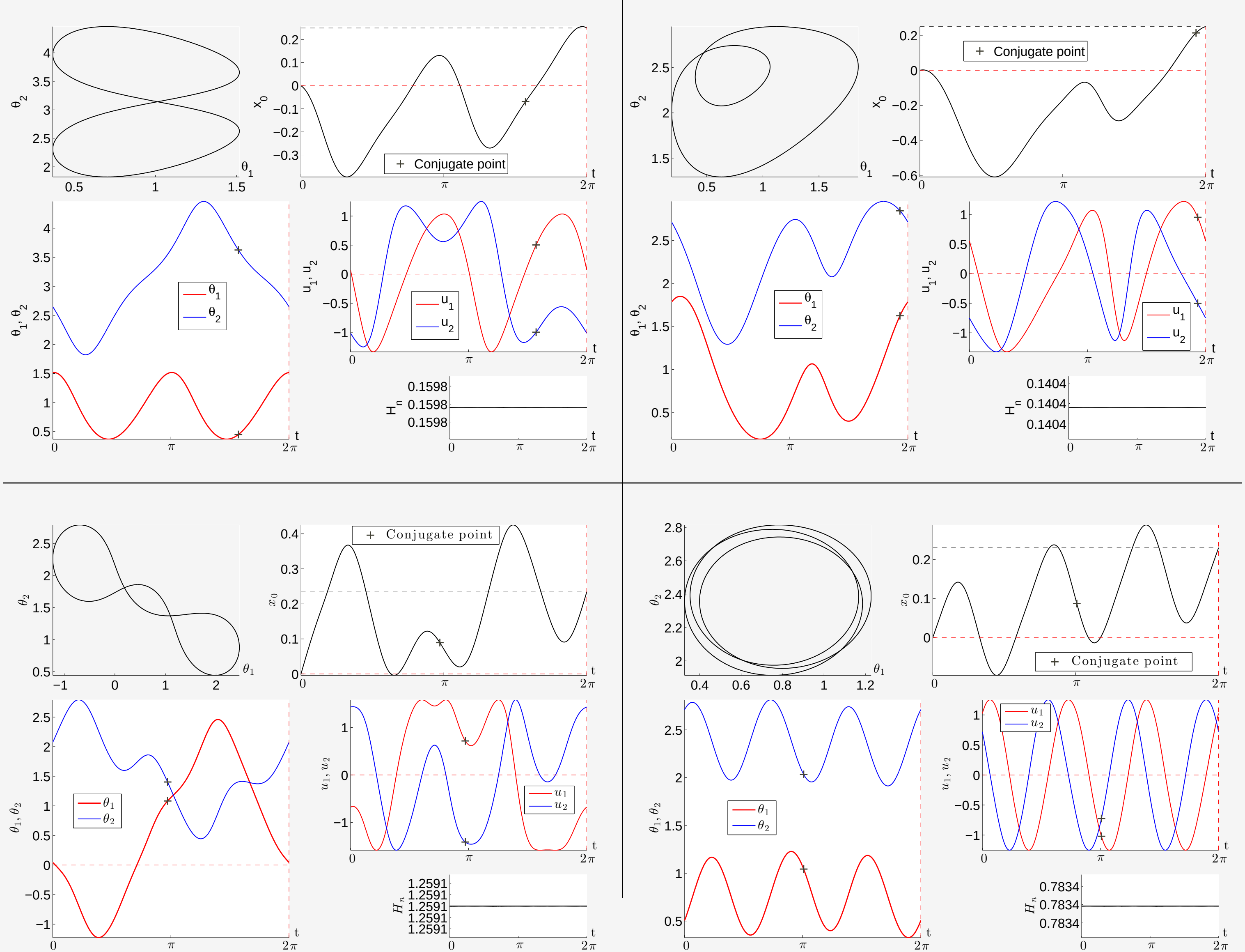
First conjugate time t_c: the exponential mapping

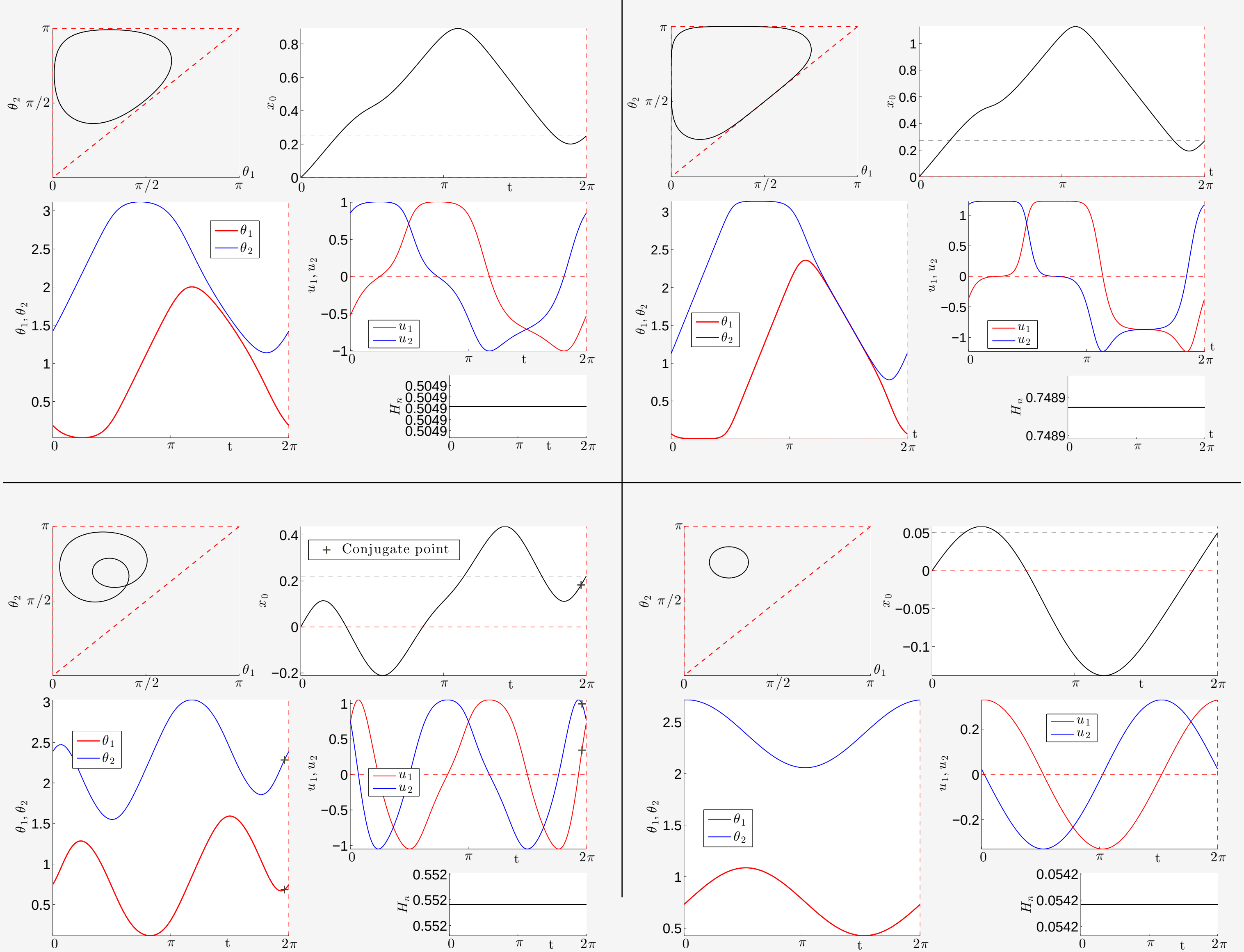
$$\exp_{q_0} : \mathbb{R} \times \mathscr{C} \to M, \quad (t, p_0) \mapsto q(t, q_0, p_0)$$

is not an immersion at (t_c, p_0).

Theorem 4. *Let $q : [0, T] \longrightarrow \mathbb{R}^n$ a strict normal stroke. If $q(.)$ has at least one conjugate point on $]0, T[$, then q* **is not a local minimizer** *for the L^∞ topology on controls, considering the fixed extremities problem.*

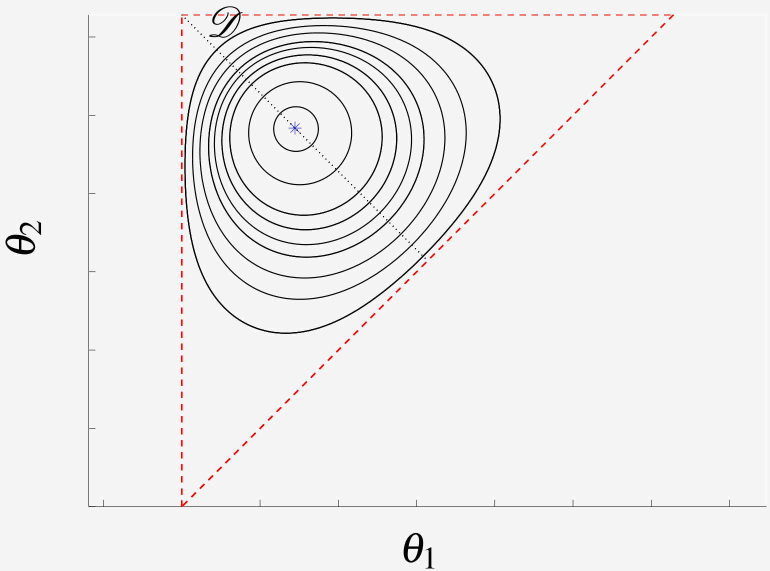



θ2
θ1
x0
t
Conjugate point
θ1, θ2
u1, u2
u1
u2
Hn
0.1598
0.1404
1.2591
0.7834
0
π
2π


θ_1
θ_2
x_0
u_1, u_2
θ_1, θ_2
H_n
t
u_1
u_2
+ Conjugate point
0.5049
0.7489
0.552
0.0542

Numerical fact:
Only simple loops doesn't have a conjugate point on $]0, T[$: they are the only candidates for optimality.

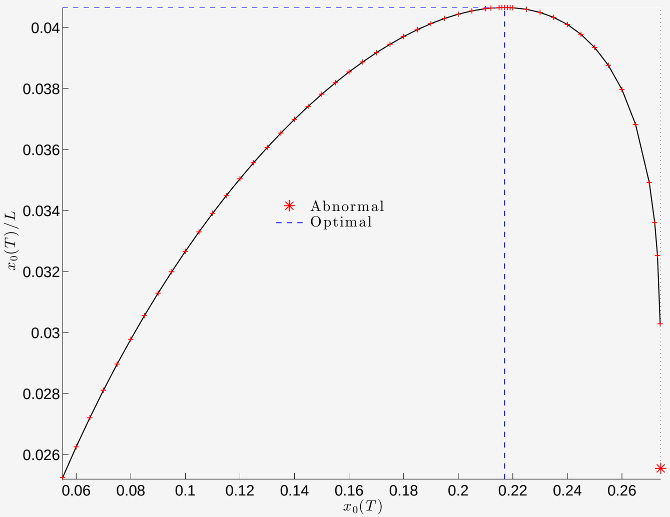
Theorem 5 (Numerical result)**.** *For each fixed displacement $x(T)$, one can find a simple loop associated with an energy level $\Longrightarrow$ one parameter family of strokes.*

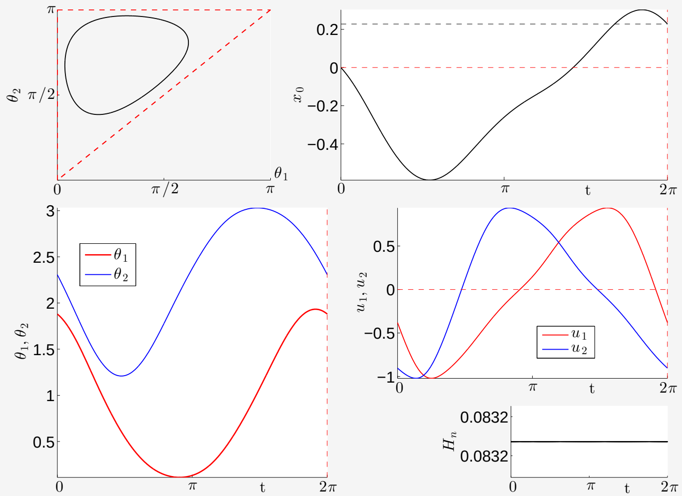


Geometric efficiency, $c(\widetilde{q}(0), \widetilde{q}(T)) = -x(T)\,/\,l_{SR}(\widetilde{q})$

Transversality conditions.

$$p_x(T) = q^0(T)/x(T), \qquad p^0(T) = -1/2$$





Efficiency curve with continuation method on $x(T)$. Corresponding optimal stroke

Theorem 6. *The abnormal stroke is not minimizing for the two costs (length with fixed displacement and geometric efficiency).*

- More general models,
- Local geometric study

Nilpotent model

$$\dot{x} = u_1 \hat{F}_1 + u_2 \hat{F}_2, \qquad \min \int_0^T (u_1^2 + u_2^2)\, \mathrm{d}t$$

. **Interior point of the triangle:** weigths $x_1, x_2 \leftarrow 1$, $x_3 \leftarrow 2$

$$\hat{F}_1 = \frac{\partial}{\partial x_1} + x_2 \left(1 + Q(x_1, x_2)\right) \frac{\partial}{\partial x_3}, \qquad \hat{F}_2 = \frac{\partial}{\partial x_2} - x_1 \left(1 + Q(x_1, x_2)\right) \frac{\partial}{\partial x_3}$$

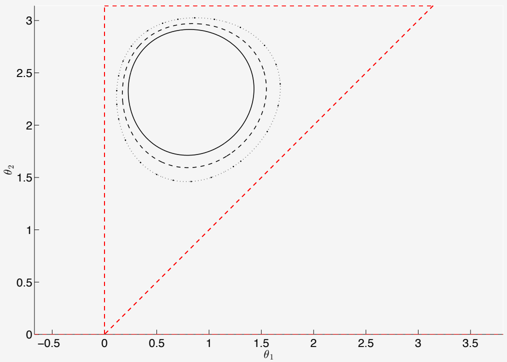
$\Longrightarrow$ simple loops and limaçons families

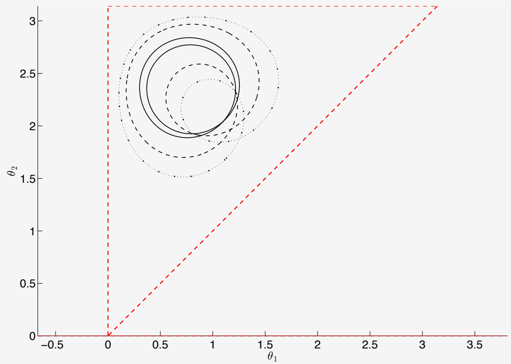
. **Point on the edges and $\neq$ vertices of the triangle:** weghts $x_1, x_2 \leftarrow 1$, $x_3 \leftarrow 3$

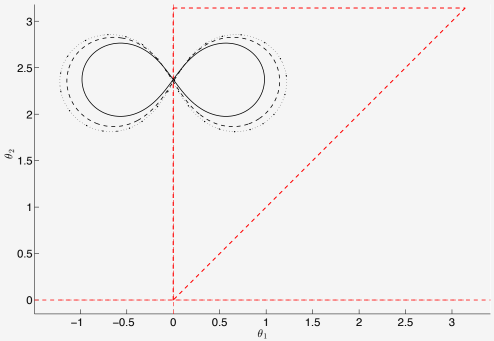
$$\hat{F}_1 = \frac{\partial}{\partial x_1} + \frac{x_2^2}{2} \frac{\partial}{\partial x_3}, \quad \hat{F}_2 = \frac{\partial}{\partial x_2}.$$

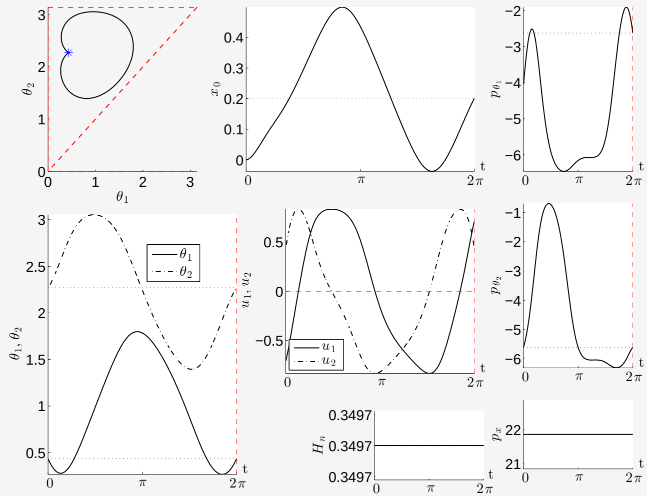
metric : $g = (1 + \alpha x_2)^2 dx_1 + (1 + \beta x_1 + \gamma x_2)^2 dx_2.$

$\Longrightarrow$ eight strokes


θ_2
θ_1


θ_2
θ_1


θ_2
θ_1


θ_2
θ_1
x_0
p_{θ_1}
θ_1, θ_2
u_1, u_2
p_{θ_2}
H_n
p_x
t